Mathematical reasoning in problem-solving: Analysis of undergraduate mathematics students' approaches to a problem

Irene Biza and Tim Rowland

School of Education and Lifelong Learning, University of East Anglia

The work presented in this paper was inspired by a *flowerbed problem* and Rowland's (2009) reflection on how systematic exploration of particular cases led to the identification of common features, and the formulation of conjectures about the properties of all valid solutions. A similar problem was used with final-year undergraduate mathematics students enrolled in a mathematics education module led by Biza. Students were asked to solve the problem and reflect on their problem-solving approaches as part of a summative assessment. In this paper, we draw on literature on mathematical reasoning and Rowland's reflections to discuss two students' approaches to the problem. Preliminary analysis indicates points of interest in relation to students' use (or non-use) of explorations, variations in their reasoning approaches (with attention to deductive/inductive reasoning), coherence (or lack of) in their reasoning, and, variations in what students perceive to be a solution to the flowerbed problem.

Keywords: Mathematical problem-solving; exploratory routines; conjecturing; inductive and deductive reasoning

Introduction

Problem-solving is related to tasks that solvers do not know how to approach in advance. What makes a task a 'problem' depends on the context in which this task is encountered and the solver's precedent and ongoing experiences. Especially in mathematics, problems might be pure mathematical problems within a mathematical theory (e.g., proving a conjecture that will lead to a new theorem); application problems that are related to real-life situations (e.g., calculating the volume of a 3D shape); or, modelling problems that are application problems, in which a transformation of a real-life situation to the mathematical structure is required (e.g., modelling the spread of a virus) (Verschaffel et al., 2014). Very often, problem-solving activity has been seen through performing mathematical explorations, planning actions, building conjectures, verifying (proving) those conjectures, etc. – see, for example, the four problem-solving steps proposed by George Pólya (1945). In this paper, we focus mostly on those actions with attention on the mathematical reasoning that warrants those actions. We contextualise our work within a mathematical problem, the *flowerbed problem* (Rowland, 2009), and its use in a problem-solving activity for undergraduate mathematics students who attend a mathematics education module. Our preliminary analysis is guided by two questions (a) what reasoning approaches do students employ? and, (b) what do they perceive to be a solution to the flowerbed problem? In the next section, we frame how we see mathematical reasoning in a problem-solving context with attention to deductive and inductive reasoning approaches and mathematical structure.

Reasoning in a problem-solving context and attention to mathematical structure

We draw on the distinction between deductive and inductive reasoning (e.g., Heit & Rotello, 2010). Essentially, deductive reasoning begins from *known* (or *agreed*) truths and applies endorsed laws of inference to them. *Modus ponens* is one such law: if $P \rightarrow Q$ and P, then Q. The conclusions of this process are themselves agreed to be true. Deductive reasoning is a valid and confident mode of reasoning. On the other hand, inductive reasoning involves making predictions – conjectures – about unknown or untested situations based only on knowledge of situations that bear some similarity with the as-yet untested ones. These predictions are necessarily uncertain, but plausible. Although it seems that deduction judgments are more affected by validity, while induction judgments are more affected by similarity, a fast deduction judgment might also be affected by similarity, similarly to the induction ones (Heit & Rotello, 2010). In the context of a mathematical problem-solving, building a conjecture is essential towards the identification of a solution, but not the ending point of the process.

It seems that *similarity* has an important role in reasoning approaches. However, what triggers a *similar course of action* can be better seen within an educational context, which legitimatises expectations, and in students' precedent experiences. For example, students' overreliance on algebraic approaches may guide them towards those approaches as the most valid and acceptable by teachers and the mathematical community (e.g., Healy & Hoyles, 2000). Students' perspectives on reasoning that justifies a mathematical proof, for example, are often characterised by tension between an argument which satisfies the teacher (mostly an algebraic one) and an empirical argument that students may find personally convincing (ibid). It seems that reasoning, in what justifies a mathematical proof, "constitutes ascertaining and persuading" (Harel & Sowder, 2007, p. 809) of individuals or communities, with "ascertaining" being the removal of their own doubts and "persuading" being the removal of the doubts of others.

Especially in mathematical reasoning, where the relation of mathematical objects is essential, the attention to such relations beyond the mere manipulation of the objects is critical. To describe such relations, we endorse the term of *mathematical structure* "to mean the identification of general properties which are instantiated in particular situations as relationships between elements" (Mason et al., 2009, p. 10). We are wondering whether students are attentive to mathematical structures, whether they have been given opportunities to notice and appreciate those structures, and, whether they seek such structure in their problem-solving activities. We present now the context of our research and the flowerbed problem.

Context, module and participants

The study was conducted in the context of a Mathematics Education course (*The Learning and Teaching of Mathematics*) led by Irene since 2016. The course is for finalist students of Bachelor of Science programmes (BSc in Mathematics or Physics) in a research-intensive university in the UK. It introduces students to the study of the teaching and learning of mathematics, with a focus mostly on secondary and post-compulsory curriculum (Biza, 2023). The course is assessed through a *Portfolio of Learning Outcomes* that involves, amongst others, solving a mathematical problem (P1) and reflecting on problem-solving approaches (P2), see Figure 2. Here, we analyse responses from Hasan and Toby, two students studying a BSc in Mathematics (out of nine who consented to their work being used for research purposes).

The flowerbed problem: Tim's approach and reflection

Tim had encountered the problem a few years ago, in the context of a four-year undergraduate course for prospective primary school teachers. A subject specialism occupied about half of their taught course. The programme for mathematics specialists included 90 minutes per week for 16 weeks on mathematical 'processes' – mainly problem solving. His repertoire of 'starter' problems for the class sessions was enhanced one year by the 'Flowerbed Problem' in Figure 1, suggested by a colleague.

> A rectangular garden has a flowerbed in the middle. Between the boundary of the garden and the flowerbed is a path, the same width all the way round. The area of the flowerbed is half that of the garden. What are the possible dimensions of the flowerbed and the garden, if all the lengths (including the width of the path) are integers?

Figure 1: Flowerbed problem (Rowland, 2009, p. 52)

Rowland (2009) thought that he'd better try it himself, to see whether or not it was likely to be of interest. He sketched a rectangle within a rectangle, chose literal variables for the dimensions – garden L, W; flowerbed l, w; path-width x – then, represented the problem algebraically as $(l+2x)(w+2x)=2lw$. With some straightforward manipulation this became $(l-2x)(w-2x) = 8x^2$ and continued (p. 52):

> Most likely I would then have found values of l, w and x to 'fit' this equation. Start with x=1, so (l-2)(w-2)=8, giving the following possible solutions:

x=1	l-2	w-2	l	w	L	W
(l-2)(w-2)=8	8	1	10	3	12	5
	4	2	6	4	8	6

Rowland, who also taught a course on the Theory of Numbers, noticed that the (*L, W*) pairs (12, 5), (8, 6), are the smallest elements of Pythagorean triples. And what is more – in each case, $l+w$ is the third element! – the diagonal of the 'garden' (the outer rectangle).

This was a *mere* conjecture - an *inductive hypothesis***.** Could it possibly hold for all solutions to the flowerbed problem? Rowland looked for confirmation. Yes! – when x=2, (*L, W*) is (40, 9) and (24, 10), with $l+w$ equal to 41, 26, respectively. Well, at that point this observation was still a conjecture that generated excitement, as Rowland recorded his affective response to those four solutions, and to his conjecture (2009, p. 52):

> I firmly believed to be true, because it was too remarkable to be mere coincidence. Because it was beautiful, it had to be true.
>
> Beauty is truth, truth beauty - that is all
>
> Ye know on earth, and all ye need to know
>
> [John Keats, Ode to a Grecian Urn]

Later, he proved the conjecture: "That belief [on beauty] did not lessen my determination to prove the conjecture!" (p. 52)

Flowerbed problem revised as a portfolio of learning outcomes item

The flowerbed problem in Figure 1 inspired the one that was used in the portfolio of learning outcomes of the Learning and Teaching of Mathematics course. Figure 2 presents the first two items of the portfolio that are related to the revised problem. In P1, students are asked to solve a mathematical problem (P1.1) and to provide their working on the problem (P1.2). Guidelines, especially on the use of digital resources are provided. In P2, students are asked to write their reflection on the actions they took by drawing on relevant research literature. The working on the problem (P1.2) is not marked but used as a reference for the contextualisation of students' reflection in P2.

P1 Solve the following problem:

> "A rectangular garden has a rectangular flowerbed in the middle. The space between the boundary of the garden and the flowerbed is a path, the same width all the way round. The area of the flowerbed is half that of the garden. Find all the possible dimensions of the flowerbed and the garden, if all the lengths (including the width of the path) are integers?"

Any correct and accurately justified response within mathematical theory will receive full marks. Please present your solution to the problem in detail.
*First, start your exploration of the problem by using only analogue resources and, if needed, search online only for definitions. Then, **and only if you wish**, in your exploration, you may consider drawing on digital resources (e.g., computer or scientific calculators), software (e.g., Geogebra or other Dynamic Geometry Software, Desmos, Excel, MATLAB, etc.), or other digital resources or online available materials (e.g., online search, blogs, etc.).*
Please keep an honest and detailed record of the actions you have taken in order to solve the problem. In addition to your solution to the problem, attach your draft record of the actions you have followed. This record will not be marked but it will be essential to the markers' understanding of your reflection on the steps you followed in the solution in P2. Your record does not need to be particularly polished or tidy; a scanned version of your handwriting suffices.

P1.1. My response to the problem

P1.2. Record of actions I have taken to solve the problem (*not to be marked)*

P2 Use Mathematics Education research literature to reflect on the actions you have taken to solve the problem in P1. Please note that an honest and detailed account of your problem-solving actions, with or without digital or analogue resources, will give you more opportunities to provide a realistic, rich, specific and consistent reflection.

Figure 2: Flowerbed problem in the Portfolio

Students' problem-solving approach (P1) and reflections (P2) are seen in our analysis as one task. We are mostly interested in students' approaches to the problem (e.g. algebraic approach, experimentation with specific cases, conjecturing, testing a hypothesis, etc.) with attention to the mathematical structure. Although digital resources had some influence on the students' responses, we keep this influence in the background in this paper. We now consider the responses of Hassan and Tobi.

Hassan and Tobi's responses

Hassan draws the garden first, and follows a deductive approach of modelling the problem algebraically with notation *G, F* (longer sides of the garden, flowerbed), *B, A*, (shorter sides) and *p* (path width). He notes that *G* and *B* must be even, in order that the area of the flowerbed be an integer. Algebraic manipulations lead him to a quadratic equation for *p*: $8p^2 - 4p(G+B) - GB = 0$. Then, he applies the quadratic formula to derive: $p = [(G+B) \pm \sqrt{(G^2+B^2)}]/4$ [EQ, our notation]. He argues that *p* is less than *B/*2 and eliminates the 'addition option', as it does not meet this condition, and continues:

> We know (G+B) - √(G2+B2) has to be multiple of 4 and √(G2+B2) represents the diagonal of the garden with this information we know the diagonal has to be an integer as G+B will be an integer, a p needs to be an integer.

Of course, he could have concluded that G^2+B^2 must be a perfect square directly from (EQ). Also, the digression concerning + or – is unnecessarily convoluted. From his statement that $\sqrt{(G^2+B^2)}$ must be an integer he proceeds directly to state that "using this we can use primitive Pythagorean triples – this is when $a^2+b^2+c^2$ [sic]". He then states that "this is satisfied using the formula generated by m and n where $m>n$ such that $a=m^2-n^2$, $b=2mn$, $c=m^2+n^2$". This is the classical form of primitive Pythagorean triples, which he then uses to elaborate equation (EQ).

The Pythagorean triples in the problem need not be primitive – they are not in solutions such as garden 8x6 with path width 1 and diagonal 10. Also, his instant recourse to the classical form of primitive numbers stayed unexplained to us. Such recourse may suggest some precedent encounter with Pythagorean triples in a Number Theory course (which is in the MSc Mathematics curriculum). It may also suggest an input from online search; in P1.2, he states: "I used online resources to find Pythagorean triples (such as 3-4-5) online and used this knowledge to find a suitable theorem for the triples". However, it is not clear whether the idea of the "triples" was motivated by his observation about the diagonal or by the online search itself.

We note that Hassan indicated how a solution could be found, over five handwritten sides of A4 pages, without stating any solution-values of *G, B* and *p.* Also, in P2 he did not report any steps of exploration apart from drawing on previous knowledge and visualising the problem through a diagram.

Toby's approach has much in common with Tim's. He draws a diagram and arrives at $(L-4p)(W-4p) = 8p^2$, where L, W are the flowerbed dimensions and p the path width. Like Tim, he then proceeds to look for factors of $8p^2$ when p=1 and 2, giving 5 distinct solutions. He remarks, "As the path width is not restricted […] there must be infinite values for the dimensions".

Although Toby arrives at several solutions – and could, it seems, find more – he appears to be unaware of what all those solutions have in common – that in each case, the garden dimensions are the least elements of a Pythagorean triple. So, there is no inductive hypothesis about something common to all solutions. Perhaps Toby had encountered relatively few Pythagorean triples beyond (3,4,5) in the past to notice a pattern. Also, it seems that Topy does not feel the need to elicit further generality in his response. As he mentions in P2: "This simply involved substituting the length values back into the original equation, reassuring myself that my solution was valid." It seems that for him, having a process that generates a numerical solution to the problem is a correct and accurately justified response within mathematical theory, as the P1 description requested.

Discussion

In this preliminary analysis, we draw on the distinction between deductive and inductive reasoning (e.g., Heit & Rotello, 2010) and mathematical structure (Mason et al., 2009) to investigate (a) what reasoning approaches students employ and (b) what they perceive to be a solution to the flowerbed problem.

Hassan's approach to the problem is deterministically deductive. Applying 'the formula' to the quadratic $8p^2$- $4p(G+B)$-GB=0 would seem at first to be unduly cumbersome – and yet, it presents Hassan with a sum of two integer squares, in need of an integer square root, which for him indicates the diagonal, and, in our eyes, immediately suggests Pythagorean triples. However, Hassan does not discuss how the diagonal is related to Pythagorean triples, either in his response (p1.2) or in his reflection (P2). We see this as a lack of attention to the mathematical structure, or at least a lack of interest in reporting such attention. We also remain surprised that he felt no need, no compulsion, to identify some solution-values of the three variables.

Toby, on the other hand, shows no inclination to 'follow his nose', to apply the formula. Instead, he tries some small values of *p,* the path width, and readily identifies some actual, numerical values of the variables that constitute verifiable solutions of the flowerbed problem. But sadly – for us, if not for Toby himself – he seems not to construe the Pythagorean connection between – and, indeed, within – his five solutions. Again, we do not see attention to the mathematical structure. Inevitably, then, there is no inductive hypothesis, because no pattern has been observed in his small solution sample. As we mentioned earlier, inductive predictions are necessarily uncertain. Nevertheless, the one who makes such a conjecture can be so overwhelmed with its comprehensive elegance that they never doubt its truth. "Beauty is truth, truth beauty". It seems, however, that both students did not seek, and most likely, did not experience this pleasure. This is a tentative comment on the grounds of students' written responses.

Reflecting on our and students' approaches to the problem, it seems that there is no common ground on what a *legitimate mathematical response* to the problem might be. We would like to investigate further and enhance students' ongoing experiences with problem-solving, conjecturing, attention to mathematical structure and noticing (and appreciation) of what we call 'mathematical beauty'.

References

Biza, I. (2023). Problem-solving in the digital era: Examples from the work of mathematics students with a problem on divisibility. In M. Trigueros, B. Barquero, R. Hochmuth, & J. Peters (Eds.), *Proceedings of the Fourth Conference of the International Network for Didactic Research in University Mathematics* (pp. 582–591). University of Hannover and INDRUM. https://hal.science/INDRUM2022/hal-04026630v1

Harel, G., & Sowder, L. (2007). Toward a comprehensive perspective on proof. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 805–842). Information Age Publications.

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education, 31*(4), 396–428. https://doi.org/10.2307/749651

Heit, E., & Rotello, C. M. (2010). Relations between inductive reasoning and deductive reasoning. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *36*(3), 805–812. https://doi.org/10.1037/a0018784

Mason, J., Stephens, M., & Watson, A. (2009). Appreciating Mathematical Structure for All. *Mathematics Education Research Journal*, *21*(2), 10–32. https://doi.org/10.1007/BF03217543

Pólya, G. (1945). How to solve it. Princeton University Press.

Rowland, T. (2009). Just enjoying the mathematics. In S. Lerman and B. Davis (Eds.) *Mathematical Action and Structures of Noticing: Studies Inspired by John Mason* (pp. 49–61). Rotterdam, the Netherlands: Sense Publishers.

Verschaffel, L., Depaepe, F., & Van Dooren, W. (2014). Mathematical problem solving. In P. Andrews and T. Rowland (Eds.), *MasterClass in Mathematics Education, International Perspectives on Teaching and Learning*, (pp. 113–124). Bloomsbury Academic.