Designing for invariance: a tool-fading geometric construction task supporting conceptual reasoning for beginning teachers.

Peter Kwamina Awortwe

University of Nottingham

Dynamic geometry environments (DGEs) enable learners to explore geometric relationships through construction and manipulation, particularly by identifying invariant properties under dragging. However, learners may rely on visual and technological affordances without developing deep conceptual understanding. This paper examines the design and implementation of a tool-fading geometric construction task for beginning teachers working collaboratively in a DGE. The task invites participants to construct and explore a configuration in which an angle bisector emerges as an invariant under dragging. The task design integrates pedagogical principles (scaffolding, collaboration, guided reflection, instrumental orchestration) with epistemic principles (invariance, structural generativity, and intentional tool fading). Through guided exploration and dialogic interaction, participants generate and test conjectures about invariant relationships. In a subsequent tool-fading phase, they reconstruct the angle bisector using only a straightedge, prompting a shift from tool-mediated exploration to structurally grounded reasoning. Analysis of on-screen video recordings, dialogue, and written reflections shows that dragging supports conjecture generation, while tool fading encourages justification and reconstruction, leading to deeper conceptual understanding. The study highlights how carefully sequenced task design can support transitions from exploratory interaction to conceptual reasoning in technology-enhanced geometry learning, with implications for teacher education and professional knowledge development.

Keywords: dynamic geometry; invariance; tool fading; conceptual reasoning

Introduction and theoretical orientation

Dynamic geometry environments, including GeoGebra, provide powerful opportunities for exploring geometric relationships through construction and manipulation (Banu & Naveena, 2024). When learners drag elements while maintaining constraints, they can observe invariant properties, relationships that remain unchanged under variation. Such interaction makes mathematical structure visible and supports meaningful engagement with geometry (Clark-Wilson & Timotheus, 2013). Dragging plays a key role in conjecture generation, enabling learners to test and refine emerging ideas (Baccaglini-Frank, 2019; Anwar et al., 2024; Quaresma et al., 2024). This resonates with Hershkowitz's (2020) notion of *thinking in pictures*, where visual representations serve as a medium for reasoning. However, DGEs may also encourage reliance on visual evidence, with learners recognising stable patterns without understanding the underlying structure. This raises

a central question for task design: how can learners be supported in moving from tool-mediated exploration towards conceptually grounded geometric reasoning?

A key epistemic feature of DGEs is invariance under dragging, where stable behaviour provides a basis for conjecture and justification (Laborde, 2002; Leung & Lee, 2013). While such invariance can function as empirical evidence (Baccaglini-Frank, 2019), conceptual understanding depends on learners' ability to interpret and explain these relationships. Tasks that foreground mathematical structure and provide purposeful and mathematically meaningful goals are, therefore, essential for supporting this transition (Clark-Wilson & Timotheus, 2013).

From an instrumental perspective, digital tools mediate mathematical activity and shape the ways in which learners engage with mathematical ideas (Rabardel, 1995). The organisation of these tools within a task, referred to as instrumental orchestration (Trouche, 2004), influences how learners interact with representations and develop understanding. In this sense, task design plays a fundamental role in structuring learners' engagement with technology and guiding their progression from tool use towards conceptual reasoning. At the same time, learning in DGEs is socially mediated. Dialogic interaction can enable learners to articulate, challenge, and refine ideas, supporting the co-construction of knowledge (Mercer & Littleton, 2007; Alexander, 2018) within a sociocultural framework (Vygotsky, 1978).

Despite recognition of the epistemic role of dragging, less attention has been given to how task design can support the transition from instrumented interaction to independent reasoning. This study addresses this gap through a task design that integrates invariance exploration with intentional tool fading. Tool fading is conceptualised as a pedagogical mechanism that progressively withdraws technological support, prompting learners to reconstruct underlying geometric relationships. Accordingly, this study conceptualises task design as a sequence in which (1) invariance is encountered through tool-mediated interaction, (2) reasoning is developed through dialogue, and (3) conceptual understanding is consolidated through the withdrawal of technological support and subsequent tool-independent reconstruction.

Methodology

Research Design

This study adopts a design-informed qualitative approach, drawing on design-based research principles to examine how a carefully designed task supports conceptual reasoning in a dynamic geometry environment. Rather than iterative design cycles, the study focuses on how theoretically informed principles, specifically invariance under dragging and tool fading, shape participants' mathematical activity and reasoning. The task is treated as a designed intervention, analysed in relation to its intended epistemic and pedagogical functions.

Task Design

The task comprised three phases: construction, exploration, and tool fading, completed using GeoGebra in a remote setting. Its primary aim was for participants to encounter an angle bisector as an emergent invariant rather than a predefined figure. Participants were not explicitly informed about the angle bisector or related geometric figures, including kites, congruent triangles, similar triangles, isosceles triangles, isosceles trapeziums, or arrowheads. Instead, they were invited to explore the

configuration through construction and manipulation in order to identify invariant relationships. The design structurally embedded the angle bisector using intersecting loci, while deliberately obscuring it from immediate visual perception.

Phase 1: Construction

Participants first constructed an initial dynamic geometric configuration in GeoGebra, following the task instructions. The construction began with angle *ABC* and an auxiliary line segment *DE*, which served as a slider. Points *F* and *G* were placed on *DE*, and circles centred at *B* (the vertex of angle *ABC*) were drawn using radii *DF* and *DG* defined by the auxiliary line segment. The intersections of these circles with the sides of the angle produced points *H*, *I*, *J*, and *K*. Participants then constructed line segments *HK* and *IJ* and identified their intersection point *L*. Finally, they drew line *BL*, which became the focus of investigation. The construction structurally embedded the angle bisector without explicitly revealing it. The resulting configuration provides the basis for subsequent exploration of invariant relationships under dragging (see Figure 1). This configuration then becomes the focus of systematic investigation in the next phase, where participants examine how geometric relationships change or remain invariant under manipulation.

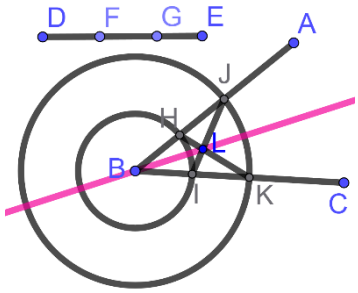


Figure 1. Initial dynamic geometric configuration for exploring invariant relationships under dragging.

Phase 2: Exploring invariance through dragging

Participants explored the configuration by dragging key points and responding to prompts, including: *what do you observe? what changes when you drag the points? what remains the same? and why might this be happening?*

Through this exploration, participants observed that line *BL* consistently bisected angle *ABC*. As points defining the angle were moved, the line *BL* continued to divide the angle into two equal parts. This stability under dragging provoked discussion about why the relationship remained invariant. Participants also began to examine other geometric relationships embedded in the construction, including symmetrical configurations and sets of congruent segments. Having identified stable invariant relationships through dragging, the task then progresses to a tool-fading phase in which technological support is deliberately withdrawn, prompting participants to reconstruct these relationships through conceptually grounded reasoning.

Phase 3: Tool-fading challenge

In the final phase, participants were asked to construct the angle bisector using only a straightedge. This removed access to digital construction tools and required participants to rely on the relationships identified during the previous exploration phase (see Figure 2). The task, therefore, shifted from tool-mediated interaction to tool-independent reasoning. The design intentionally combined epistemic principles (invariance, structural generativity) with pedagogical principles (scaffolding, collaboration, reflection, and tool

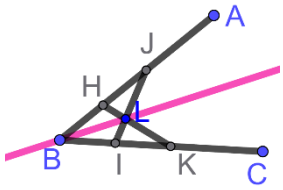


Figure 2. Straightedge reconstruction of the angle bisector based on relationships identified during exploration.

fading), thereby creating a trajectory from exploration to conceptual reconstruction.

Participants and context

Eight beginning secondary mathematics teachers from four UK universities participated. All were enrolled in postgraduate teacher education programmes and had prior exposure to dynamic geometry software, though with varying levels of confidence. Participants worked in pairs online, thereby supporting collaborative dialogue and making their reasoning processes more explicit and observable.

Data collection and analysis

Data were collected via screen recordings of GeoGebra interactions, audio recordings of dialogue, and written reflections, enabling analysis of both actions and articulated reasoning. Analysis followed a deductive-interpretive approach, informed by dialogic learning and technological-pedagogical-content knowledge (Mishra & Koehler, 2009). Video data were segmented into episodes based on activity shifts, including construction, dragging, phase transitions, and conjecture or explanation. Episodes were coded across three dimensions: engagement with dragging (exploratory, testing, confirmatory), nature of reasoning (descriptive, relational, explanatory), and forms of dialogue (cumulative, exploratory, evaluative). Cross-phase analysis examined the development of reasoning from perceptual observation to conceptual explanation and from tool-mediated to tool-independent reasoning, focusing on invariance and justification. Triangulation across multiple sources increased the trustworthiness of the analysis. Limitations include the small sample size, variable prior experience, and reliance on a single task, which constrain generalisability.

Findings, Discussion, and Implications

Dragging as a mechanism for conjecture testing and refinement

Analysis of screen recordings and dialogue shows that participants used dragging both to observe changes and to systematically test emerging conjectures. Repeated manipulation of points defining the angle allowed monitoring whether line *BL* continued to bisect the angle:

> Participant 1: *When I move A, the line through B and L still splits the angle.*
> Participant 2: *Yes, it's always cutting it into two equal parts.*
> Participant 1: *So that must mean BL is the angle bisector.*
> Participant 2: *But why does that stay the same even when we move the points?*

This episode illustrates the progression from perceptual confirmation to conjecture formulation, where invariance under dragging served as provisional empirical evidence, strengthening confidence in the conjecture. This aligns with Baccaglini-Frank (2019), who frames dragging as a form of instrumented abduction, inducing generalised geometric claims. The dialogue also shows a shift from noticing what remains invariant to questioning why, marking the onset of explanatory thinking.

Dynamic interaction alone is insufficient; its epistemic potential is improved when combined with reflective prompts and dialogic interaction (Clark-Wilson & Timotheus, 2013; Hershkowitz, 2020). Through collaborative dialogue, participants transformed individual observations into shared reasoning, highlighting the significance of social interaction in conceptual development (Alexander, 2018).

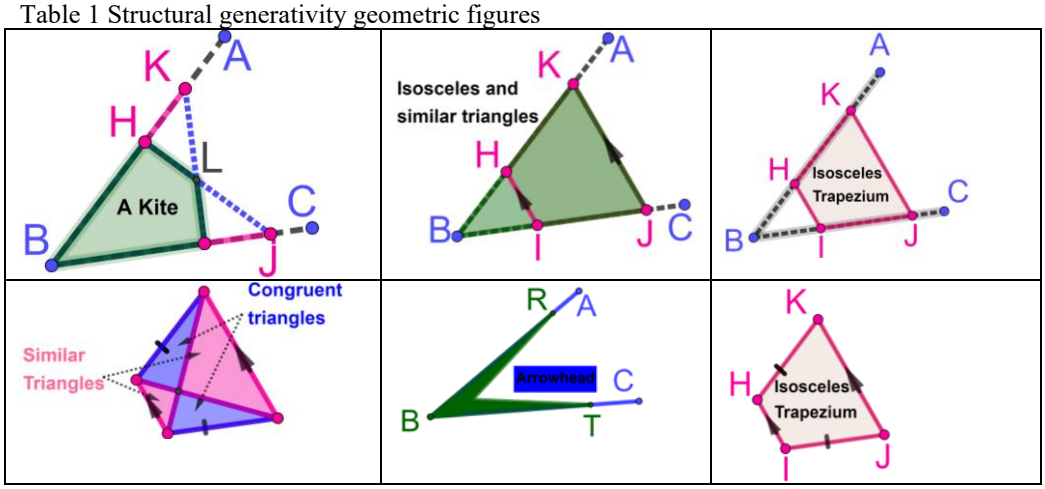
From perceptual recognition to explanatory reasoning

Participants initially described invariant relationships in perceptual terms ('*it stays the same*') before attempting explanations. Collaborative discussion stimulated consideration of structural properties, including symmetry and equal distances. This reflects Hershkowitz's (2020) '*thinking in pictures*,' where visual interaction supports pattern recognition but requires dialogue to reach conceptual understanding. Across all pairs, participants progressed from noticing invariance to articulating reasoning, indicating that the task effectively scaffolded the transition from perceptual recognition to conceptual explanation.

Structural generativity and multiplicity of mathematical relationships

The task's design supported the emergence of multiple geometric relationships beyond the angle bisector, including kites, congruent triangles, isosceles triangles, and trapezium-like configurations (see Table 1). These incidental observations demonstrate the construction's richness and afford multiple pathways for reasoning, reinforcing the idea that well-designed tasks can facilitate both targeted conceptual goals and broader exploration of geometric structure.

Table 1 Structural generativity geometric figures



Reasoning during the tool-fading phase

Tool fading required participants to reconstruct the angle bisector using only a straightedge, drawing on relationships identified during exploration (e.g., equal lengths and intersections of line segments, see Figure 2). This shift from verifying properties through dragging to justifying constructions conceptually resonates with Rabardel's (1995) view of instrumented action leading to conceptualisation. All participants articulated coherent justifications, although these were expressed in different ways, reflecting varied approaches to reasoning while demonstrating a shared transition from dynamic interaction to formal geometric reasoning. Tool fading functioned as a productive mechanism: it encouraged independent reconstruction, reinforced structural understanding, and highlighted the importance of sequencing tasks to support progression from exploration to tool-independent reasoning.

Implications for teacher education

For beginning teachers, tasks like this highlight the potential of technology to support conceptual reasoning when carefully structured. Effective task design sequences exploration, conjecture testing, and tool reduction to scaffold understanding of invariance and structural properties. Teacher education should emphasise the use of digital tools and the strategic design of tasks that foreground invariant relationships, encourage reflective dialogue, and incorporate tool fading as a means of deepening mathematical reasoning. Such approaches support reflective, conceptually oriented teaching practices and contribute to integrated professional knowledge.

Conclusion

This study demonstrates that a geometric construction task integrating invariance exploration with tool fading promotes a shift from perceptual engagement to structural understanding. Dragging supported conjecture generation, dialogue facilitated reasoning, and tool fading encouraged independent reconstruction. When combined with intentional task design, dynamic geometry environments can foster conceptual reasoning that extends beyond the technological context, providing a model for both learner engagement and teacher education.

References

Alexander, R. (2018). Developing dialogic teaching: Genesis, process, trial. *Research Papers in Education, 33*(5), 561–598. https://doi.org/10.1080/02671522.2018.1481140

Anwar, L., Malic, A., & Goedhart, M. (2024). Formulating a conjecture through an identification of robust invariants with a dynamic geometry system. *International Journal of Mathematical Education in Science and Technology, 55*(7), 1681–1703. https://doi.org/10.1080/0020739X.2022.2144517

Baccaglini-Frank, A. (2019). Dragging, instrumented abduction and evidence in processes of conjecture generation in a dynamic geometry environment. *ZDM Mathematics Education, 51*(5), 779–791. https://doi.org/10.1007/s11858-019-01055-0

Banu, S. P., & Naveena, S. (2024). Exploring GeoGebra: An interactive tool for mathematical visualization and analysis. In *Futuristic trends in contemporary mathematics & applications* (Vol. 3, Book 1, pp. 19–37). IIP Series. https://doi.org/10.58532/V3BJCM1P1CH3

Clark-Wilson, A., & Timotheus, J. (2013). Designing tasks within a multi-representational technological environment: An emerging rubric. In C. Margolinas (Ed.), *Task design in mathematics education: Proceedings of ICMI Study 22* (p. 47). ICMI.

Hershkowitz, R. (2020). Shape and space: Geometry teaching and learning. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 774–779). Springer.

Laborde, C. (2002). Integration of technology in the design of geometry tasks with Cabri-Geometry. *International Journal of Computers for Mathematical Learning, 6*(3), 283–317. https://doi.org/10.1023/A:1013309728825

Leung, A., & Lee, K. (2013). Dynamic geometry environments and mathematical thinking. *ZDM Mathematics Education, 45*(3), 377–389. https://doi.org/10.1007/s11858-012-0470-3

Mercer, N., & Littleton, K. (2007). *Dialogue and the development of children's thinking: A sociocultural approach*. Routledge.

Mishra, P., & Koehler, M. J. (2009). Technological pedagogical content knowledge: A framework for teacher knowledge. *Teachers College Record, 111*(9), 1017–1054.

Quaresma, P., Santos, V., & Telesc, J. (2024). Proof exploration using dynamic geometry systems with integrated automated deduction capabilities. *International Journal of Mathematical Education in Science and Technology*. Advance online publication. https://doi.org/10.1080/0020739X.2024.2377724

Rabardel, P. (1995). *Les hommes et les technologies: Approche cognitive des instruments contemporains*. Armand Colin.

Trouche, L. (2004). Managing the complexity of human–machine interactions in computerised learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning, 9*(3), 281–307. https://doi.org/10.1007/s10758-004-3468-5

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Harvard University Press.