Integrating multi-duo artefacts to develop spatial skills and geometrical reasoning

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Geometrical reasoning and spatial skills are essential for students' success in geometry and for their future lives. The study adopts a design-based research approach with three phases of classroom observation. Participants were students in grades 7 to 10 from three schools in China. This paper presents findings on how combining any two of three artefacts, namely tangible objects, virtual reality, and GeoGebra 3D, into what we call multi-duo artefacts creates a highly integrated learning environment. The results show that geometrical reasoning supports spatial skills by helping learners externalise their skills. In turn, spatial skills provide epistemic feedback that enriches geometrical understanding. Overall, these findings suggest that this integrated learning environment catalyses students' development of a deep understanding of geometry.

Keywords: multi-duo artefacts; embodied cognition; spatial skills; geometrical reasoning

Introduction

Geometry learning has long been recognised as challenging due to its inherently abstract nature. Many students struggle to develop a deep understanding, often relying on surface-level visual reasoning interpretations rather than engaging in analytic reasoning and conceptualisation. Seah and Horne (2021) suggest that higher-level reasoning is characterised by the use of multiple representations. As a result, learners may be able to recognise shapes or manipulate representations yet fail to construct coherent geometrical knowledge or justify their solutions without developed spatial skills. With the rapid development of educational technologies, a variety of tools such as Virtual Reality (VR), dynamic mathematics software like GeoGebra, and tangible manipulatives have been increasingly integrated into mathematics education (Huang & Fujita, 2025). These multi-artefact environments provide rich opportunities for visualisation, interaction, and exploration in geometry. However, while such tools enhance perceptual engagement, they do not automatically lead to deeper mathematical understanding (Drijvers et al., 2010). This establishes the paper's aim: to examine how multi-duo artefacts can be effectively employed to support geometrical reasoning and spatial skills.

Literature review

Despite the increasing use of multi-artefact learning environments (e.g., Sinclair et al., 2016; Huang & Fujita, 2025), the mere use of multiple tools does not necessarily lead to deep learning. While technologies such as VR, GeoGebra, and tangible objects enhance visualisation and interaction, students often remain at the level of perceptual engagement, failing to develop meaningful reasoning or coherent conceptual understanding. This reveals a critical gap between spatial skills and geometrical reasoning. While tools such as VR and GeoGebra show potential for supporting learning, little is known about how they can be effectively combined to bridge this gap. In this study, we draw on the notion of epistemic feedback, defined as feedback that supports learners in interpreting, explaining, and justifying their observations (Nieminen et al., 2026). Its role will be further elaborated later in the paper.

Multi-duo artefact learning environment

A multi-duo artefact learning environment has been recognised as an effective approach for supporting mathematical understanding, as it enables learners to engage with concepts through multiple forms of representation, including visual, symbolic, and physical modalities (Soury-Lavergne, 2021, 2024). We extended this approach through the integration of multiple learning artefacts, such as Virtual Reality (VR), dynamic mathematics software (e.g., GeoGebra), and tangible objects, and our early evidence suggests that these multi-artefact environments provide rich opportunities for interaction, exploration, and representation (Huang & Fujita, 2025).

From the perspective of embodied cognition, learning is not solely a mental activity but is grounded in bodily interaction with the environment. Digital and tangible artefacts activate learners' perceptual and sensorimotor experiences, supporting meaning-making processes (Pittalis & Drijvers, 2023) and engagement with abstract geometrical concepts through concrete and experiential interactions, further enhanced by epistemic feedback.

Spatial skills in mathematics learning

Spatial skills are considered fundamental to mathematics learning, particularly in geometry. In this study, spatial skills are defined as a set of developable cognitive competencies that support the processing and transformation of spatial information in mathematical contexts. Building on prior literature, spatial skills are primarily defined through three core components: spatial visualisation (Vz), spatial orientation (SO), and spatial relations (SR) (Pittalis & Christou, 2010). Spatial visualisation (Vz) refers to the ability to perform complex, multi-step transformations of spatial representations; spatial orientation (SO) involves maintaining an understanding of spatial configurations under changing perspectives; and spatial relations (SR) refer to the ability to mentally manipulate objects, such as through rotation, quickly and accurately.

However, while spatial skills support cognitive perceptual engagement with geometrical representations, they do not necessarily lead to deeper mathematical reasoning. Learners may be able to visualise or manipulate shapes without fully understanding the underlying principles or justifying their observations. This distinction highlights that spatial competence alone is insufficient for the development of geometrical reasoning.

Geometrical reasoning

The dual nature of geometry, namely that it is a field of theory and a field of practical experience, presents opportunities and dilemmas for mathematics educators. The opportunity is to link theoretical perspectives of geometrical properties to students' everyday knowledge, but the dilemma is that learners often find the dual nature of geometry a difficult chasm to bridge (Fujita & Jones, 2003). In this study, geometrical reasoning is defined as a form of mathematical thinking that involves the coordination of spatial perception, domain-specific geometric knowledge, and logical inference. Building upon this definition, Seah and Horne (2019) refined the progression of geometric reasoning by categorising it into a series of increasingly sophisticated levels: (1) visual reasoning, (2) descriptive reasoning, (3) analytic reasoning, and (4) relational-inferential property-based reasoning. Higher-level reasoning, as a more advanced form, requires learners to construct logical arguments and proofs based on established principles.

The development of geometrical reasoning represents a transition from "seeing" to "proving" where learners move from intuitive and perceptual understanding to formal and structured mathematical thinking. This transition is essential for achieving a deep understanding of geometry, but it is often difficult for learners to accomplish without appropriate support.

Epistemic feedback

In multi-duo artefact environments, we need to link effectively between spatial skills and reasoning, and the idea is epistemic feedback. Epistemic feedback focuses on learners' reasoning processes and knowledge construction rather than merely indicating correctness. It provides explanatory information that supports learners in interpreting, justifying, and refining their understanding (Nieminen et al., 2026), thereby emphasising the "why" behind answers and promoting deeper engagement with mathematical concepts. A key function of epistemic feedback is to bridge the gap between perception and reasoning, guiding learners from visual-level observation toward property-based interpretation and verification. In doing so, it supports processes such as justification, validation, and conceptual refinement. Within this study, epistemic feedback is conceptualised as a mediating mechanism connecting spatial skills and geometrical reasoning. It facilitates the transformation from perceptual engagement to deep mathematical understanding by supporting the externalisation of spatial skills and the development of geometrical reasoning.

Methodology

Data collection

This study adopts a design-based research (DBR) approach to investigate the role of epistemic feedback in multi-duo artefact learning environments. The study was conducted across three phases of classroom implementation involving students in Grades 7 to 10 at three secondary schools in China. The learning environments integrated multiple artefacts, including Virtual Reality (VR), GeoGebra 3D, and tangible manipulatives, with students engaging in at least two artefacts in each setting. Data were collected through classroom observations of students' engagement in spatial geometry tasks, including their interactions with artefacts, verbal and non-verbal responses, and task-based outputs. These data captured how learners engaged with representations, externalised their thinking, and responded to feedback during the learning process.

Data analysis

This study adopts a conceptual modelling approach to examine the role of epistemic feedback in multi-duo artefact learning environments. Rather than relying on empirical data, the analysis is grounded in a theoretical synthesis of existing literature on multi-representational learning, embodied cognition, spatial skills, and geometrical reasoning.

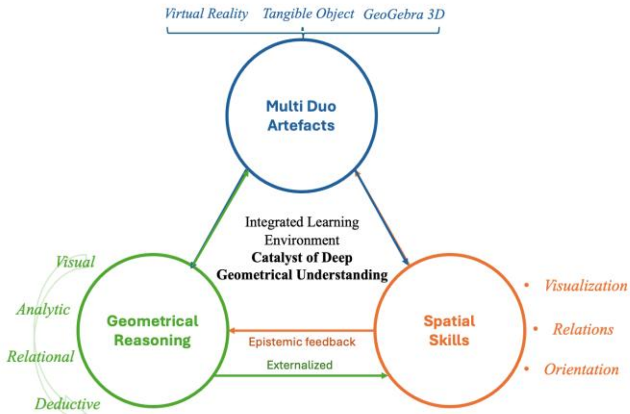

Figure 1: Multi-duo Artefacts to Develop Spatial Skills and Geometrical Reasoning Framework

An analytical framework (Figure 1) is developed to interpret the learning process as a transformation from perceptual engagement to structured reasoning, with epistemic feedback included as a supporting mechanism.

The analysis focuses on explicating the relationships among multi-duo artefact interaction, spatial skill development, and reasoning processes, and on identifying how feedback contributes to knowledge construction within this system. Data analysis was conducted using a qualitative coding approach, guided by the study's conceptual framework. Three main coding categories were developed to capture the key dimensions of the learning process:

1. Spatial Skills: including spatial visualisation, spatial relations, and spatial orientation, as reflected in students' interactions with representations and artefacts.
2. Epistemic Feedback: including instances where feedback prompted explanation, justification, or verification of understanding.
3. Geometrical Reasoning: including (1) visual reasoning, (2) descriptive reasoning, (3) analytic reasoning, and (4) relational-inferential property-based reasoning ; reasoning processes demonstrated in students ' problem-solving.

The analysis focused on identifying patterns of transition from spatial engagement to geometrical reasoning, with particular attention to how epistemic feedback mediated this transformation.

Initial findings

We selected one episode (Table 1b,1c,1d) from our data in which students reasoned about the shape of a quadrilateral formed by connecting midpoints of edges in a cube (Table 1a). The analysis reveals that students' engagement with multi-duo artefacts (GeoGebra AR and VR environments) strongly activated a range of spatial skills, showing students' interactions with GeoGebra AR and VR during geometrical reasoning.

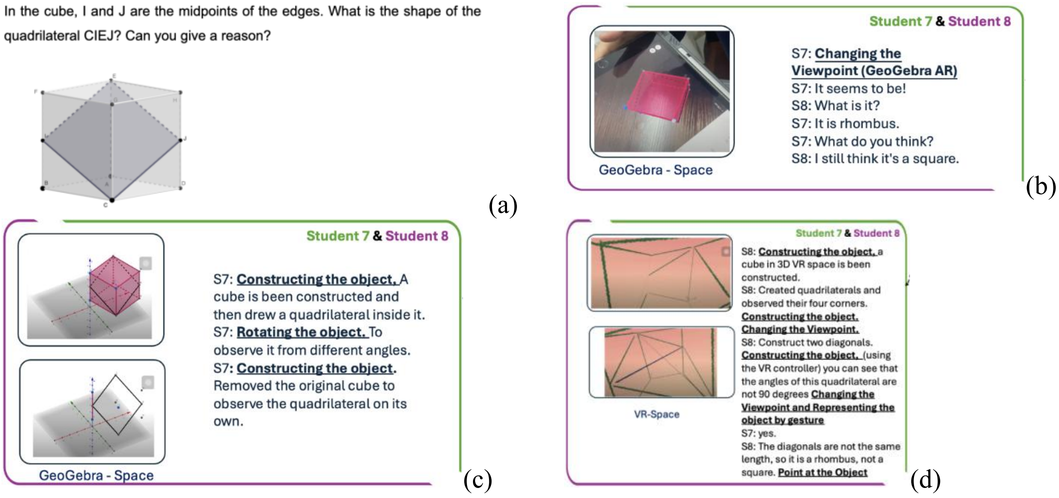

Table 1: (a) Spatial geometry task; (b) (c) and (d) duo GeoGebra and VR-Based Geometric Task for Interpreting Cube Cross-Section

At the initial stage (Table 1b), students relied primarily on visualisation (Vz) and spatial relations (SR) when interpreting the quadrilateral embedded within a cube (Table 1a). Through actions such as changing viewpoints in GeoGebra AR, they attempted to make sense of the figure based on its visual reasoning. However, this perceptual engagement led to inconsistent interpretations, as evidenced by conflicting claims ("It is a rhombus" vs. "I still think it's a square").

This indicates that while spatial skills enabled students to access and manipulate representations, they primarily supported visual reasoning (L1) rather than more stable, property-based conceptual understanding (L3 or L4), even when students engaged in object construction and spatial relations (SR).

The epistemic feedback emerged implicitly through students' interactions, actions, and dialogue. Three main types of epistemic feedback can be identified through this episode: (1) Students obtained explanation-based feedback through engaging in actions involving spatial relations (SR) and spatial visualisation (Vz), such as isolating and reconstructing the quadrilateral, which led them to move beyond surface observation and engage in informal visual reasoning (L2). (2) Students obtained justification-based feedback by attending to and reasoning about the properties of the constructed object, for example, through constructing diagonals, which enabled them to articulate analytic, property-based reasoning (L3). (3) Students obtained verification-based feedback through repeated use of spatial orientation skills, such as changing viewpoints and interacting in VR, which supported them in verifying and refining their assumptions. These forms of epistemic feedback were not externally imposed but emerged through interaction with artefacts, highlighting the role of the learning environment in generating feedback that supports reasoning.

This study contributes to the literature by proposing a mechanism that explains how learning occurs in multi-duo artefact environments: Multi-duo artefact → Spatial skills → (Epistemic feedback) → Geometrical reasoning. Within this framework, epistemic feedback is conceptualised as a mediating mechanism that transforms externalised spatial experiences into internalised geometrical reasoning. This extends existing research by positioning feedback not merely as evaluative information, but as a central component in the cognitive transformation process. In addition, the tools within this framework function as a catalyst, accelerating the interaction between spatial skills and geometrical reasoning by supporting processes of externalisation and feedback. While some learners may achieve such reasoning without the use of these tools, the integration of multi-duo artefacts appears to enhance and expedite this process. These findings highlight epistemic feedback as a critical mechanism that bridges spatial skills and geometrical reasoning, enabling the transformation from perceptual engagement to deep geometrical understanding.

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