# Developing student teachers' structural understanding of negative number arithmetic through the use of representations

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This paper describes three negative number representations explored by student teachers in a taught university session, examining how these representations can foreground curriculum-wide structures. Secondary mathematics student teachers were asked to script imagined classroom conversations before and after discussing a range of representations. The scripts showed variation in the extent to which the student teachers critically evaluated these representations and the connections they made to the underlying mathematical structures within, and beyond, negative number arithmetic.

**Keywords: representations; negative numbers; structure; initial teacher education (ITE)**

## Introduction

The study reported here investigates the conceptions of negative numbers drawn on by student teachers, the representations they use to convey these conceptions, and the connections they make between these representations and underlying mathematical structures. The research is situated in ongoing concerns about the difficulties student teachers experience in relating theoretical perspectives to classroom practice and seeks to examine how taught university sessions may influence their pedagogical choices. Student teachers were asked to script imagined whole-class conversations, articulating and justifying their choice of representations both before and after a taught session in which a range of representations for negative numbers was explored. The session was designed to foreground representations not merely as supports for arithmetic procedures, but as tools for making mathematical structures visible. This paper reports on three of the representations that were investigated in the taught session.

It can be challenging for student teachers to connect the theory they are studying with the practice they are observing in schools, and even more challenging to integrate it into their own teaching. In the UK, the majority of teachers become qualified to teach by gaining a Post Graduate Certificate in Education (PGCE) (Department for Education, 2024) which can be undertaken at a Higher Education (HE) establishment or through a school-based SCITT (School Centred Initial Teaching Training). Both routes combine practical experience in the classroom with taught theory. In the model of 'practical theorising' (McIntyre, 1995), theory and practice are integrated from the outset, with school-based learning from practitioners running in parallel with taught research-informed theory (Burn et al., 2022a). Student teachers are encouraged to bring these two sources of knowledge together, asking "critical questions of each in light of the other" (Burn et al., 2022b, p.24). Investigating the effect of a taught university session on student teachers' pedagogical decisions contributes to better understanding of how to support students to connect theory and practice.

Considering the university session as an intervention of short duration (Stylianides & Stylianides, 2014) and using scripting as an approximation for practice

(Zazkis & Herbst, 2018), student teachers' pedagogical choices were examined before and after they spent time investigating a range of representations for negative numbers. I report here on the implementation of three of the representations considered during the university session, outlining how a sense of structure can be developed in different ways depending on the representation. This report was written mid-way during the research project; results will be published elsewhere when the study is complete.

## The intervention

The university session was titled 'Building a sense of structure'. The term *structure* is used in the literature, and by mathematicians, to describe "substantively different phenomena", including to describe relationships, generality/generalisation and properties (Venkat et al., 2019, p.13). In this session, it was used to encourage student teachers to think about the generalities that underpin negative number arithmetic. For example, the calculation $3 + -2$ has some *structural* features and some *incidental* features. The 3 and 2 are incidental; we are not interested in whether students *know* that $3 + -2 = 1$, but instead that they are able to evaluate expressions of the type $a + -b$ with $b > 0$. Some generalities relate to *procedures*, like that just described, whilst others describe a particular *conception*, for example, that numbers could be considered as having a magnitude and a direction. There are a huge number of procedures and concepts underpinning negative number arithmetic, some of which, like those just described, are specific to this topic, and others which are relevant across maths, for example commutativity of addition which underpins negative number arithmetic, but is also used, for example, when adding fractions, decimals, algebraic terms, vectors, or matrices.

The term *representations* refers to physical objects, images, symbols and words (written or spoken) that symbolise an abstract idea (Jitendra et al., 2016). The use of representations has been extensively researched and has been shown to be effective in supporting students to engage with mathematical ideas (Hodgen et al., 2018). Bruner (1966) introduced the idea of an enactive-iconic-symbolic progression of representations: starting with physical manipulation of concrete objects, learners progress to the use of images, then language, numbers and abstract symbols. The National Centre for Excellence in the Teaching of Mathematics (NCETM) uses the terminology of concrete-pictorial-abstract to describe this progression (NCETM, 2024) and promotes the use of representations to expose the mathematical structure being taught (Mattock, 2019). Others argue that CPA should not be considered a progression as the types of representation are not hierarchical and students should move backwards and forwards between them (Drury, 2018). The overlap between pictorial and abstract is also not distinct, particularly with regards to spatial reasoning: how students are using mental imagery, or pictorial images in their heads, when working abstractly.

## The representations

There is a vast range of representations to choose from when working with negative numbers. Different structures can be emphasised, or ignored, depending on the teacher's choice of representations. During the intervention, the student teachers investigated nine representations of negative numbers. These are outlined in Figure 1. Some representations were static (for example considering subtraction as the distance between two points on a number line, or multiplication grids), others were dynamic (for example subtracting by removing counters, or moving from one position to another on a number line) and some could be used both statically and dynamically (for example arrows on a number line, or the analogy of 'pits and piles').

Written expressions and verbal descriptions, classified as 'abstract representations', were used throughout the intervention, regardless of the other representation being focused on. The terms that can be used when verbalising '$-$' are numerous (for example negative, minus, subtract, take away, and difference between) and the specific choice of language can emphasise different structural elements (Macmillan & Ingram, Forthcoming). Which terms to choose were discussed, and modelled, throughout the session.

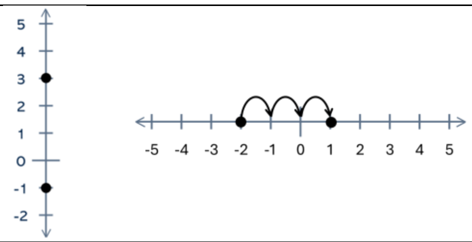
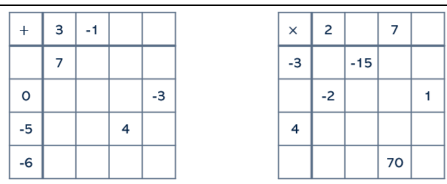
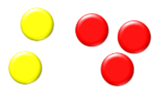
<table>
<tr><th>CPA</th><th>Representation</th><th>Example</th></tr>
<tr><td rowspan="6">Abstract</td><td>Written expressions</td><td>$2 + -3$</td></tr>
<tr><td>Verbal descriptions</td><td>'What is two subtract negative three?'</td></tr>
<tr><td>Verbal analogies</td><td>Pits and piles<br>Credit and debit<br>Temperature changes<br>Floors in a building<br>Filming a car moving forwards or reversing and playing the film forwards or in reverse<br>'Not not' statements (e.g. I'm not doing nothing)</td></tr>
<tr><td>Physical analogies</td><td>Asking students to walk across the classroom forwards and backwards, and stand or move on a number line</td></tr>
<tr><td>Algebraic representations</td><td>$a + -b = a - b$</td></tr>
<tr><td colspan="2"></td></tr>
<tr><td rowspan="3">Pictorial</td><td>Positions on number lines</td><td></td></tr>
<tr><td>Arrows on number lines</td><td></td></tr>
<tr><td>Multiplication grids</td><td></td></tr>
<tr><td>Concrete</td><td>Double-sided counters</td><td></td></tr>
</table>

Figure 1: Representations considered in the intervention session 'Building a sense of structure'

### *Representation 1: Double-sided counters*

Double-sided counters are so named because the sides of the counters are different colours. Typically, they are red and yellow; when the yellow side is face up, the counter represents positive one, and when the red side is face up, negative one. Three red counters, therefore, represent negative three and subtracting negative one, for example, would mean taking one of those red counters away. Using counters emphasises addition as grouping, subtraction as removal, multiplication as repeated addition or duplication of groups, and division as sharing into groups, highlighting commutativity of addition and multiplication, and non-commutativity of subtraction and division.

One red counter and one yellow counter combine to make zero: a ‘zero pair’. Figure 2 shows two examples where zero pairs are used to support calculations with counters. Zero pairs emphasise inverse operations ($-a + a = 0$) and the additive identity ($a + 0 = a$). These features are fundamental to the structure of the number system and are used across the secondary mathematics curriculum. For example, when solving $2x + 3 = 7$, we might first identify an inverse operation ($3 - 3 = 0$), perform that operation on both sides of the equation ($2x + 3 - 3 = 7 - 3$) then use the additive identity ($2x + 0 = 2x$) to conclude $2x = 4$. Teachers who choose to use counters when teaching negative number arithmetic can decide whether to explain these structural features or to present them as ‘rules’ or ‘tricks’ for using the counters. One student teacher who chose not to use counters explained they made this choice because they did not see these structural features as *necessary* for performing negative arithmetic. Another explained that they thought introducing zero pairs as rules for manipulation adds unnecessary complexity.

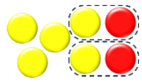
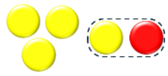
| $5 + -2 = 3$ | $3 - -1 = 4$ |
|---|---|
|  |  |
| Two yellows can be paired with two reds to create two zero pairs which can then be removed. | To subtract a red counter, we first need to add a zero pair. Then a red counter can be removed. |

Figure 2: Examples of how zero pairs can be used to support negative number calculations

Using different coloured counters to represent a number depending on whether it is positive or negative, suggests positive and negative numbers are concretely different; the number is treated as an *object*, where the number is a combination of the ‘sign’ and the digits. This representation, therefore, suggests that the dash symbol has two different roles: an instruction to ‘subtract’ and an assignment of the property ‘less than zero’. In the calculation $3 - -2$, then, we have a number (3 or $+3$), an operation ($-$), and another number ($-2$).

The different roles of ‘$-$’ may be a valuable distinction and teachers may choose to use counters to foreground this. But the property of ‘colour’ is incidental and does not obviously relate to the property of being ‘less than zero’. This can be problematic when comparing numbers. For example, it is not intuitive that three red counters represent a number less than three yellow counters, and the fact that five reds represent a number less than three reds is counterintuitive.

### *Representation 2: Position on a number line*

“Number lines are a particularly valuable representational tool for teaching number, calculation and multiplicative reasoning across the age range” (Hodgen et al., 2018, p. 101). Number lines are most usually drawn horizontally or vertically. On a vertical number line, moving up means getting bigger; a concept easily relatable to growth (Moeller et al., 2025). Yet horizontal lines are more prevalent, where numbers increase to the right. Using the (arbitrary) convention of ‘further to the right’ meaning ‘greater’, the *positions* of numbers on the line can be compared to ‘see’ that $-3 < 3$ and $-5 < -3$. Unlike the counter representation, the number line is not restricted to integers, allowing structural understanding of the number system to develop with the inclusion of fractions and decimals.

Arithmetic can be performed on the number line by moving to the right and to the left. For example, the calculation $-4 - -2$, could be considered as starting at the position $-4$, then moving 2 places, where the direction to be moved is determined by the symbols between the numbers $(- -)$. Some student teachers who chose this representation in their scripts, explained the process with descriptions of 'changing direction', for example one student teacher wrote 'Every time you see a negative sign you change direction', explaining that you always start facing in the positive direction then rotate $180°$ for each 'negative sign'. Here, the student teacher has disconnected the second dash from the number 2, treating it as an operator, not as an indicator that the 2 is less than zero. They have treated the first number $(-4)$ as an object, as in the counter representation, but not the second $(-2)$, thinking of the first as a position, and the second as a magnitude, or distance, to move, prioritising procedures over structural features. The student teacher is highlighting a structure of *directionality* with this representation, but not in a consistent manner.

### *Representation 3: Arrows on a number line*

In this representation, the number is represented as an arrow with length and direction. For example, $\mathbf{-4}$ is represented by an arrow of length $\mathbf{4}$ in the negative direction. This representation shares the same opportunities for highlighting structural features as 'position on a number line', whilst also bringing the feature of numbers having both *magnitude* and *direction* to the forefront. Figure 3 shows how the calculation $\mathbf{-4 - 1}$ can be represented in this way. In this example, $\mathbf{-4}$ instructs us to start at zero and move to the left four spaces before subtracting a movement of 'move one space to the left'. The resulting movement is a movement three spaces to the left from zero $(\mathbf{-3})$. As in the counter representation, both numbers $(\mathbf{-4}$ and $\mathbf{-1})$ are considered as objects and the '$-$' in the middle represents 'taking away', allowing the structures that counters occasion to be highlighted. Several student teachers chose this representation when attempting to respond to a student asking 'Why does a minus times a minus equal a plus?', drawing on an image of an arrow 'flipping' or rotating around zero.

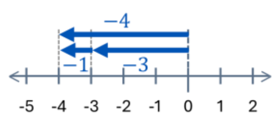



Figure 3: How $-4 - -1$ could be represented with arrow notation

### *Conclusions*

The choices teachers make when selecting representations depend on several factors of which 'developing a sense of structure' may be one. This intervention was designed to encourage student teachers to think about the 'bigger picture' of 'structure' alongside procedural fluency when planning to teach negative number arithmetic. It was also designed to give the student teachers the opportunity to rehearse their pedagogical choices in a 'safe space' with the support of their peers and the ability to draft and re-write. Early analysis of the data suggests that, whilst student teachers recognise the importance of developing structural understanding, when faced with the complexity of classroom discourse and the plethora of (potentially conflicting) goals in the classroom, they do not prioritise this in their choice of representations and explanations.

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