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British Society for Research into Learning Mathematics

Proceedings of the Day Conference held at Loughborough
MAY 1996

These proceedings consist of short papers which were written for the BSRLM conference in May 1996. The aim of the proceedings is to communicate to the research community the collective research represented at BSRLM conferences, as quickly as possible. For this reason the papers have not been edited.

We hope that members will use the proceedings to give feedback to the authors and that through discussion and debate we will develop an energetic and critical research community. We particularly welcome presentations and papers from new researchers.

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ANALYSING THE METACOGNITIVE BEHAVIOUR OF UNDERGRADUATES IN CALCULUS

Stephen Hegedus. University of Southampton.

This paper will i) comment on the confusing term metacognition and control, *ii) describe the analysis of some data obtained from verbal reports and retrospection with some undergraduates and, iii) propose some metacognitive skills for a certain area of the Calculus. It discusses how Calculus students use metacognitive strategies but inefficiently. The research is a continuing study for the purposes of a Ph.D in mathematics education.*

Research in mathematics education in the domain of Calculus has been sparse if not almost non-existent especially if one attempts to address particular topics. Throughout the 80's and 90's some relevant literature has started to emerge, particularly out of the Mathematics Association of America, in their MAA Notes Series, and out of the proceedings of the Psychology of Mathematics Education conferences, held annually.

The work has tended to fall under a few categories specified in Kairan (1992), an edited collection of reports, as part of the MAA Notes Series:

i) Questions for the future;
ii) the use of computers and symbolic computing in Calculus;
iii) learning theories and pedagogical utilities (including computer packages such as Mathematica, Maple);
iv) mathematical processes (symbols, functions, axioms, etc.) and symbols (notation).

Many recent publications in Calculus have been interested primarily in the last category and in the effects of using computers packages in instruction.

A recent article by White and Mitchelmore (1996) in the JRME add that:

> Changes in technology, the qualifications, and the mathematical competence of students have led many to question the role of traditional calculus courses in the curriculum.

This report and many others concentrate on the decreasing levels of conceptual knowledge in students work.

White & Mitchelmore's work with undergraduate students led to conclusions that they obtain a conceptual understanding only after extensive successful experiences using variables in the operational mode. Such a mode is where a mathematical process or symbol becomes an object in its own right. For example, the use of differentiation algorithms without having a sound conceptual understanding of rates of change.

Orton (1983a, b) makes the point in his rigorous analysis of the errors made in differentiation and integration that most are due to a failure to grasp conceptual principles.

More specific reports have come from Vinner (1982; 1991), David Tall (1987), and Seldon & Mason (1989; 1994).

There is a problem. One area of mathematics educational research which has attempted to answer such problems is the literature concerning problem-solving. Within this field one has seen a growing body of research in metacognition, in the past decade.

Schoenfeld has dedicated a lot of time to developing control strategies, including ones for Calculus students, and implementing them into courses. He highlights the need for control in his book *Mathematical Problem-Solving*(1985):

> Selecting and pursuing the right approaches, recovering from inappropriate choices, and in general monitoring and overseeing the entire problem-solving process, is equally important. One needs to be efficient as well as resourceful. In broadest terms, this is the issue of control. (p. 99)

This is one type of metacognition. Metacognition was introduced in the literature on metamemory by Flavell et al.(1970). He defines it as:

> Metacognition refer's to one's knowledge concerning one's own cognitive processes or anything related to them, e.g. the learning-relevant properties of information or data ... Metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of those processes in relation to the cognitive objects or data on which they bear, usually in the service of some concrete [problem-solving] goal or objective.

Since then mathematics education and psychology have mixed up various definitions of

metacognition over the past two decades, and it can now be seen as an umbrella term for various historical approaches to the topic.

Schoenfeld (1987) summarises the research on metacognition as focusing on three topics:

- Knowledge about your own thought processes. How aware are you in describing your own thinking ?
- Control, or self-regulation. How well do you keep track of what you're doing when (for example) you're solving problems, and how well (if at all) do you use the input from those observations to guide your problem-solving actions?
- Beliefs & Intuitions. What ideas do you bring with you ? How does this shape the way you do Mathematics?

Ann Brown (1987) explains how differences in the metacognition literature comes from four historical roots:

1. Research involved with using verbal reports as data
2. Executive control
3. Self-Regulation
4. Other-regulation.

A Pilot Study was held last summer with two 1st year undergraduates doing an advanced Calculus course for students with double A-Level in Maths and who were in the tutorial I took. They were asked to speak out aloud everything they were thinking whilst completing a homework problem for the week and not to ask me for much help. The data was then transcribed and analysed along with other work they had done and the work that had been taught in lectures. The method was seen as fruitful, though, slightly contrived, and more intervention occurred than originally planned.

There was an apparent lack of control in the management of resources which they had recently acquired in lectures. A conceptual understanding about the construction of double integral was lacking, and so various methods of approach to the problem were explored but none where implemented until 20 minutes have passed.

From this study three main items were highlighted which can be said to be managed in the process of solving a Calculus problem. These are allocated and employed with varying degrees. They include: managing algebraic expressions (symbols, functions, etc.), managing algorithms (formula, methods of attack, etc,) and managing the concepts involved. From observing time-line models of problem-solving behaviour (See Schoenfeld, 1985) one could see points where a re-allocation of resources was completed either through metacognitive reflection or through a less ordered way. At these points extracts of the protocols obtained were analysed and discussed in the light of how much control of the above items they were using and how this affected them in taking wild goose chases or sound solution pathways.

The study is now concerned with the topics from the second half of a 1st year's 2nd semester Calculus course. These are all concerned with Calculus topics of a more visual nature including volumes of solids of revolution, and double integration.

METHOD

A method of think-aloud/interviewing was used for the main empirical work combined with retrospection during and after the interview. This came about from relaxing the restrictions of the cognitive science methodology.

The aim is to analyse and log aspects of control with respect to the questions highlighted above, in order to devise specific tasks whose intention is to explore some significant aspects of control at a later date.

Control-skills are concerned in improving the efficiency with which undergraduates apply algorithms and formulas, and manipulate their algebra.

Over the past year, work with undergraduates has led to a growing concern with control being defined in the context of cognition, ie. metacognition, and seeks to break free of the restraints of the information-processing model, which have repercussions in the choice of methodology, ie. a more clinical approach to think-aloud. It seeks to redefine the research topic in terms of meta-mathematics problem-solving rather than meta-cognition, as it sees mathematics as a discipline in itself rather than as a function of psychological models, where students engage in constructing their own modes of control from social interaction.

This piece of analysis highlighted several areas in solving problems of the sort below where students use control and were it possibly needs developing.

> **Find the volume of the solid generated by rotating the region, $0 \leq y \leq 1 - x^2$ about the line y=1.**
>
> *From Adams (1995) Ex. 8.1 Q. 12.*

In a lengthier paper the verbal data could be included and a time-line diagram using Schoenfeld's method (1985) could be shown to illustrate their problem-solving process.

Times when Control is used and where it needs developing:

1. In the use of formulas and their applicability to many situations. This comes from a more conceptual understanding of the reasoning behind their algebraic structure.
2. A more visual understanding of the problems concerned in order that 2D diagrams are effectively used in directing problem-solving.
3. In the early stages of doing problems on solids of revolution draw a diagram and a strip. This strip is easier to spin around the required axis. With this simple piece of visualisation in mind one might manage the resources which the problem offers more effectively.
4. Maybe use a more flexible form of notation - in the student's mind - in order for them to manage the resources they have with a deeper conceptual understanding.

This is a primary piece of analysis and it does seek further refinement. It has been brought into criticism through the levels of rigour and justification it uses to qualify the types of metacognitive activity being highlighted. There are many forms which metacognitive activity can take in such involved problems. As the analysis develops I might be able to identify trends in such skills in order that they can be generalised over several Calculus topics. One might also see such skills as a function of the user as well as the specific domain of mathematics being addressed. The analysis will seek to convince the reader in the future by including control strategy accounts of experts solving the same problems in order to provide a contrast.

This piece of analysis suggests that control-skills or meta-skills, are a function of conceptual understanding when dealing with visual problems. It seems easy, in an exploration phase, to embark upon various 'mathematical wild goose chases'. Without sound conceptual understanding of such problems one can find oneself in such a position. The control-skills developed might seek to eliminate certain mistakes, before one has explored the proposed method of attack too far, and so, too inefficiently.

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ALGEBRA WITH STUDENT TEACHERS

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Abstract

How should we be educating teachers to teach algebra effectively? This apparently 'easy' question is so large and so problematic there can be no easy answers, if indeed there can be any answers at all. However, much criticism is being levied at the algebraic performance of children and much research has been conducted into the learning of algebra.

This paper outlines some of the major current criticisms, some of the major strands of research into the learning of algebra and an analysis of the algebra curriculum for ITE students within one institution. It concludes with a discussion of the issues facing those working with student teachers on the learning and teaching of algebra.

Current criticisms and recommendations

Over the past two years many concerns have been expressed about the school mathematics curriculum in general and algebra in particular. The joint LMS, IMA, RSS report "Tackling the Mathematics Problem" (1995) identifies three major concerns :

> *' a serious lack of essential technical facility - in particular, a lack of fluency and reliability in numerical and algebraic manipulation and simplification '*
> *' a marked decline in students' analytical powers when faced with simple two step or multistep problems '*
> *' most students.....no longer understand that mathematics is a precise discipline... '*

Such concerns are echoed by other reports (See for example IMA,1995, Sutherland & Pozzi,1995,).

The annual analysis of SATs at Key Stages 2 and 3 for Algebra suggests that at KS2 children show

> *' a more secure understanding of patterns when they are expressed as a number sequence rather than shown as a spatial arrangement '*
> *' a generally secure understanding of number patterns and the ability to interpret relationships when expressed in words '* but
> *'a general weakness in the ability to write such relationships in a generalised form '*

At KS3, students found difficulty with literal formulae, in particular:

> *'forming literal expressions which include a number '*
> *'substituting into formulae, particularly where brackets were used '*
> *'finding the value of a variable which was not the subject of a formula '*
> *'using calculators to evaluate formulae which involved a series of operations and brackets '*

A working group set up by the Joint Mathematics Council with support from the Royal Society has identified from an analysis of assessment papers that GCSE examinations appear to include few questions which make demands of students requiring a structural understanding of algebra (also few multistep problems) This feature is which is continued at A level (although not at Further Mathematics level) whilst the algebraic demand of all GNVQ routes, with the exception of engineering, is virtually non existent

Neil (1996) reports on the conclusion of a working group on the potential use of computer algebra systems (CAS) within the 16-19 curriculum that

CAS should be used, but their use should be planned
students still need a thorough grasp of the process they want CAS to perform
important principles and techniques should still be taught thoroughly
it will be important for students to develop a good symbol sense (see Arcavi, 1995)

Research into the learning of algebra

There is a considerable body of research on the learning of algebra. Grouws (1992) and Wagner,S.& Kieran,C.(1989) provide useful summaries. Kieran(1989), Sfard & Linchevski (1991) and others point to the difficulties created through students' lack of structural understanding of algebra. They distinguish between operational or procedural and structural conceptions of abstract mathematical notation.
So, for example

3 (x + 5) + 1 may be conceived operationally as " add 5 to the number at hand, multiply the result by 3 and add 1" or may be conceived structurally as " a certain number" or " a function" or "one of a family of functions".

Such researchers postulate that for most people the operational conception comes first in the acquisition of new notions and that the transition from a process conception to an object conception is neither accomplished quickly nor without great difficulty. Sfard in particular proposes that this is a three stage process involving interiorisation, condensation and finally reification with the first two being gradual lengthy processes and the last a qualitative leap. Within such a framework, procedural operations are those such as add, subtract, multiply, divide... where the objects which are operated upon are numbers. By contrast, structural operations are those such as simplifying, factorising, solving, differentiating.... where the objects being operated upon are algebraic expressions. Kieran and Herscovics(1994) support Sfard's view that there is a cognitive gap between arithmetic and algebra which many people never jump.

The difficulties children experience over the " = sign" and the concept of equivalence are also discussed by writers such as Kieran who point out that for many children, " = " is unidirectional, meaning " and the answer is " rather than reflexive and meaning "is equivalent to" and that this is a major stumbling block for children moving towards a structural conception of abstract notation.

Sutherland (1995) discusses the current concerns in terms of the reduced focus upon algebra as generalised arithmetic and calls for more emphasis to be placed on young children working with arithmetic structure as a pre-algebraic activity. Suggestions are made both by her and others that spreadsheets and LOGO have a role to place in mediating the transition from operational use of algebraic notation to structural appreciation of algebraic entities.

Pimm(1995), takes a different perspective upon the learning of algebra and focuses upon language and learning, examining the role of symbols and their meanings. He discusses, for example, the development of mathematical registers and points to the tensions between the confusion arising from different meanings being attached to particular symbolic forms and the precision within the use of those forms in given contexts. He is also interested in the

connections between abstract thinking and physical actions and the role that for example, physical by hand forming of abstract notational script has in the formation of abstract mathematical meanings.

Within a socio-cultural framework, the focus on socially shared meaning within the context of algebraic knowledge has led some researchers (See, for example, Azarello et al,1995, Lins, 1994) to explore the learning of algebra within a linguistic framework. Arcavi (1994), Fey(1990), and others draw on such work when describing behaviour of students working with algebra and defining the notion of "symbol sense".

Many writers over the years have discussed the anxiety which mathematics in general causes for students. The learning of algebra does not escape such influences and indeed, my own work (Johnson, 1996) indicates that many undergraduates studying advanced mathematics suffer considerable anxiety over quite elementary algebraic ideas and techniques. Such anxiety is also evident in those following initial teacher education courses and is a key issue for staff working with students, irrespective of the nature of that work itself.

There is relatively little research on teaching of algebra despite the fact that there is much on learning of algebra. Slovin (1990) suggests that most teachers push quickly for a structural approach to algebra, despite the fact that apparently from the research most children adopt a procedural approach.

Algebra within ITE courses.

An analysis of the algebra related content of the ITE courses at my own institution suggests that this falls into one of five categories and forms a body of work with students on :

- ***their own algebra learning/knowledge, including:***
  - *pattern and structure through number and shape*
  - *generalisation and formula generation*
  - *functions and graphs*
  - *proof*
  - *formal alegbraic systems, Boolean algebra, group theory, rings, fields, vector spaces*
  - *historical development of algebra*
  - *use of spreadsheets*
  - *LOGO*
  - *school algebra topics through placement activity*
- ***knowledge of the issues surrounding algebra itself and the learning/teaching of algebra,including :***
  - *research findings on the learning of algebra*
  - *consideration of the nature of mathematics*
  - *observation of algebra focused teaching/learning*
  - *text analysis*
  - *assessment analysis*
  - *task creation*
- ***knowledge of activities/tasks/materials available for learning algebra, including :***
  - *texts*

schemes
individual tasks
specialist materials, eg RLDU, ATM etc
IT focused materials and tools, including graphical calculators
video/TV programmes
specialist and generic software

- ***pedagogical skills within the context of teaching algebra, including :***
  lesson planning/scheme or longer term planning
  questioning techniques
  observation techniques
  function of tasks
  management of tasks
  sequencing and selection of tasks
  assessment and evaluation of learning
- ***reflection and self awareness on own confidence/security of algebraic knowledge and the issues surrounding the learning/teaching of algebra, including:***
  self assessment in algebra and attitudes to algebra
  reading and discussion
  assessment tasks based in school
  individual small scale research projects
  discussion within mathematics units
  evaluation of own learning within the context of algebra
  analysis of materials, presentations of algebra knowledge, childrens' work, teachers in action

Discussion
The extent to which students engage with these categories of activity varies considerably, primarily depending upon whether the students are Primary or Secondary focused. It also depends upon a number of factors including; the choices individual students make within individual units, the schools they are placed within, their previous mathematical background, their level of personal confidence with mathematics generally, the nature of their prior learning experiences and the extent to which they are sufficiently free from concerns over classroom management competence to engage in reflective activity over learning.

Colleagues working with intending Primary teachers find themselves spending considerable amounts of time working with relatively low levels of confidence in mathematics generally, and often panic with regard to algebra. In that context, they inevitable focus upon the aspects of algebra and pre-algebra emphasised by the national curriculum, notably pattern and structure , particularly in relation to number and functional relationships arising from a variety of problem solving and investigative work. For many students this in itself challenges the limits of their confidence and ability to work with abstract mathematical ideas - not surprisingly given that most enter the course with the minimum GCSE grade C or equivalent in mathematics.

My own colleagues do draw some research findings on the learning of algebra to the attention of students and engage them in reading and discussing these and other professional writings about the learning/teaching of algebra. This is limited by their own

personal knowledge of the field and it is one which is almost unknown to most of the practising teachers in whose classrooms they are placed.

At secondary level, whilst we might assume that students are in general more confident with their own personal algebraic skills, the depth and nature of students' knowledge of algebra varies considerably depending on their previous background in mathematics. By the time students begin to engage with the practical realities of teaching algebra, we work with the assumption - rightly or wrongly- that the more extensive subject base of secondary students provides most with an appropriately sound base for the teaching of algebra. It is clear from our own small scale research (Johnson & Elliott, 1995) that many in fact operate largely procedurally, supporting Slovin's research cited earlier.

An underlying constraint, particularly on one year courses is the amount of time which students have to engage with any of these activities at any depth whatsoever. This leaves them significantly dependent upon the prevailing practice within the schools to which they are attached and the presentation of algebra and its learning through the National Curriculum documentation, through texts they are asked to use and through discussion with existing teachers.

Within these materials are hidden assumptions about key algebraic ideas such as the crucial importance of the notion of equivalence, the qualitative differences which Sfard refers to in procedural and structural conceptions of algebra, the variety of potential meanings which could be ascribed to particular symbolic forms, the ideas of algebra sense discussed by Arcavi and Fey. Such considerations are clearly important for the successful teaching of algebra. I suggest that they are not necessarily conveyed through the current descriptions that we have for the school algebra curriculum, at least to beginning teachers.

The connections between and implications of the statements
> '*consider how relationships between number operations underpin the techniques for manipulating algebraic expressions*' and
> '*manipulate algebraic expressions; form and manipulate equations or inequalities in order to solve problems*'
> '*solve a range of linear equations, simple linear simultaneous equations....* '
> (National Curriculum: Algebra 1b and 3c,3d)

do not emphasise a need for pupils to discuss meanings, to be able to classify statements or equations, the crucial sense of '=', the domain of existence of solutions, the form and nature of expressions, symmetry etc.

How then might beginning teachers, or even experienced teachers to infer what is important in the learning and hence teaching of algebra ? Given the constraints, what should be the aim for those of us working with beginning teachers concerning the teaching and learning of algebra? Can we agree ? Do we all fulfill such an aim ?

I do not think that we have yet formulated an appropriate model for working with students on algebra learning which integrates my list of five categories of learning successfully. So, even though some, or even all of the elements listed are part of the student teacher curriculum, the connections are not sufficiently clear. Observed pedagogy of students with children in classrooms confirms that. We are working towards greater coherence, greater transparency of connection between elements and stronger partnership with schools in collectively improving pedagogy. There are no easy solutions and progress feels slow at times.

We would welcome further debate on this whole issue and sense that it would be useful for us to share both information about what the algebra curriculum is for students in other institutions and examples of practice which is successful.

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TEACHER ASSESSMENT - TENSIONS AND CONTRADICTIONS

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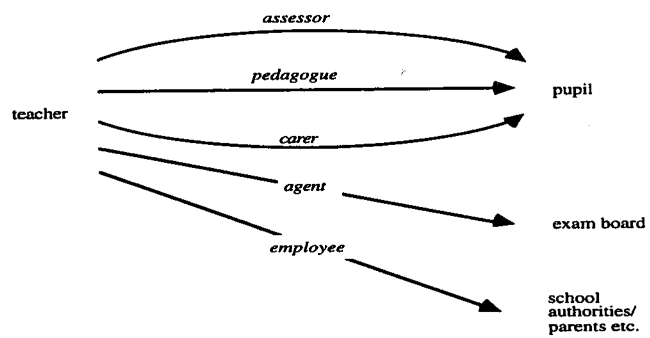
The involvement of teachers in formal assessment of their pupils creates tensions between their complex roles and problems in reconciling the assessment process with curricular aims. In this paper some of the results of a study of the practices of teachers engaged in reading and assessing GCSE coursework are presented and issues arising from it. are discussed Variations within and between these practices raise serious questions about the validity of the assessment process in general as well as in the particular context of coursework.

Whereas it may be argued that formative assessment has always been an integral part of teachers' everyday activity, recent moves to involve teachers in providing summative assessments of their pupils in high-stakes settings (e.g. National Curriculum Teacher Assessment and GCSE coursework) create problems for teachers as they attempt to reconcile possibly contradictory roles (Radnor & Shaw, 1995; Paechter, 1995; Morgan, 1993; in press). As I argued in Morgan (1993), teacher-assessors are in complex relationships with their pupils and with various other interested parties. Figure 1 indicates some of these and a range of concerns that may affect teachers' assessment practices

Figure 1: Teacher-assessor relationships



The examples and findings discussed in this paper arise from a study of the discourse of written reports of investigative work for GCSE coursework (Morgan, 1995). As part of this study, eleven experienced secondary mathematics teachers were interviewed while reading and assessing a small number of coursework texts. In the course of these interviews, the positions constructed by the various relationships outlined in Figure 1 and the responsibilities and perspectives associated with them were observed to give rise to a range of ways of making sense of students' work. Thus, at various points during the reading and assessment process, a teacher might be observed to adopt several of the positions outlined in Figure 2, sometimes switching repeatedly between them.

Figure 2: Teacher-assessor positions observed during coursework assessment

examiner	*using externally determined criteria*
autonomous examiner	*setting and using the teacher's own criteria*
teacher/advocate	*looking for opportunities to give credit to the student*
teacher/adviser	*suggesting ways in which the student might meet the assessment criteria*
teacher/pedagogue	*suggesting ways in which a student might improve her mathematical understanding*
interested reader	*trying to make sense of the mathematics*
imaginary naive reader	*hypothesising about the clarity of the text for a non-expert reader*

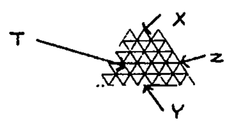
In comparing different teachers assessing the same pieces of student's work, it became apparent that the adoption of different dominant positions and the use of different reading strategies could give rise to different interpretations of the student's text and hence to different evaluations of the work. These contrasts were particularly great when the student's text deviated from the 'norm' in any way. For example, two teachers from different schools, Joan and Fiona, read and assessed Richard's work on the investigation 'Inner Triangles', an extract of which is shown in Figure 3.

Figure 3: Richard's 'Inner Triangles' text (extract)

Results

Top length	Bottom length	Slant length	Unit triangles
1	3	2	8
2	4	2	12
3	5	2	16
1	4	3	15
2	5	3	21
3	6	3	27
1	5	4	20
2	6	4	32
3	7	4	40
1	6	5	30
2	7	5	45
3	8	5	55

Formula



$$Z(X+Y) = T$$

This text is aberrant (although not unusual) in that, while correct, it is very succinct, containing very few words and no extended passages of writing. While both teachers appeared to experience discomfort with Richard's text, they nevertheless, like other teachers who read it, recognised it as belonging within their experience, relating it to similar work produced in the past by their own students. There were, however, substantial differences between the positions that these two teachers tended to adopt in relation to the text and its author and the strategies they used in attempting to make sense of and to evaluate the text.

Joan's initial response to the page of text shown in Figure 3 was to attempt to make mathematical sense of the formula, interpreting the reference of the variables and checking the formula with the values in the table. While she comments on the perceived lack of 'lead up' to the formula, she excuses this by providing a story about how Richard might have been working in class.

> Gosh . . . he's leapt into a formula . . . so let's just see if it works . . . . . . so he's given the top, the bottom, the slant height and number of triangles, top plus bottom times the slant height . . . let's try one of these *[referring to the data in Richard's table]* . . 3 and 5 is 8 . . it seems to . . yeah, so that seems to work just as a quick check. What he hasn't done is given any sort of lead up as to how he arrived at it. Which he probably did orally in the group but, you know, it would be nice to have it written down . .
> *(Joan)*

In contrast, Fiona's initial response was evaluative: expressing uncertainty about Richard's 'ownership' of the work and commenting on the absence of diagrams (apparently ignoring the diagram on this page because it does not fulfil her wish to see evidence of data gathering). Fiona appears relatively unconcerned to make sense of the mathematics.

> This is a major problem because he's got these results but unless one is there in the class and you're a teacher you don't know whether this is his results or somebody else's. He hasn't shown any diagrams of where these results have come from. He hasn't done any drawings as far as I can see. He's come up with a formula which is Z equals. Z must be the slant height. Is equal to X plus Y equals T. I assume that's right, I don't know.
> *(Fiona)*

It is not possible in the space available here to provide further examples or detailed analysis of the rest of Joan's and Fiona's assessment of Richard's work or to discuss further differences that were observed between the two teachers' attitudes to the nature of investigative tasks (see Morgan, 1995; in press). The differences that have already been identified were, however, displayed repeatedly throughout the interviews with these two teachers. Figure 4 summarises the main characteristics of the two teachers' reading and assessment strategies.

The contrasting assessment strategies identified in Figure 4 were also found to characterise other teachers in the sample interviewed. Thus, while some evaluated students' texts by comparing them with some imaginary 'ideal' response to the given task, others appeared to construct pictures of the personal characteristics of each student (e.g. 'clever', 'lazy', 'a typical girl') and stories about how the student might have worked on the problem. In the case of 'difficult' texts such as Richard's, these different strategies gave rise to very different evaluations: while Joan considered this text the best of the three texts she read, Fiona ranked it last. Whereas in less extreme cases teachers using different strategies might agree on rankings and grades for a piece of work, it is clear that the rankings and grades assigned would have very different meanings.

Figure 4: Contrasting Teacher-Assessors

	dominant reading positions	*reading strategies*	*assessment strategies*
Joan	interested reader teacher/advocate (teacher/adviser)	seeking to understand the mathematics	hypothesising about student-author's actions and intentions
Fiona	examiner (imaginary naive reader)	seeking for signs of fulfilment of criteria	comparison with 'ideal text'

(The reading positions enclosed in parentheses are not displayed in the passages discussed above but were found elsewhere in the interviews with these two teachers.)

The discussion so far in this paper has focused on general differences in teachers' orientations and strategies. A number of other bases for difference in teachers' evaluations also emerged from analysis of the teacher interviews, including:

- *teacher understanding of the mathematical content*

For example, a non-standard (yet correct) generalisation was judged by one teacher to be mathematically correct and to demonstrate understanding on the part of the student:

> There's also found some sort of linearity in the results whereby he can just multiply up the numbers which again shows quite a good understanding of the problem I think. *(Charles)*

Another teacher, apparently with a less secure mathematical interpretation of the same text, judged the student to have used haphazard methods of guesswork and intuition rather than understanding:

> It's interesting that the next part works, I don't know if it works for everything or it just works for this but he's spotted it and again he hasn't really looked into it any further. He's done it for one case but whether it would work for any other case is er I don't know, he hasn't looked into it... And he's used it in the next part er used the this multiplying section in the next part and it's just a knowledge of number that's got him there I think intuition whatever. He may have guessed at a few and found one that works for it *(Grant)*

It might be argued that a non-standard answer needs greater explanation and that this might have helped Grant to appreciate it. On the other hand, how is the student to know that his generalisation is non-standard? In constructing any text, knowledge of the audience's state of knowledge is vital to including the right level of detail. Either too much or too little detail is likely to lower the reader's evaluation of the text. In an education context, students are likely to expect that their teacher knows 'everything'; distinguishing those non-standard parts that require elaboration is therefore problematic.

- *teacher assessment of the student's 'ability'*

As has already been noted, although the teachers did not know the students whose work they were assessing, some of them constructed a picture of the 'ability' and other characteristics of each student. This picture appeared to influence the ways in which they interpreted some sections of

students' work, in particular parts which were identified as 'errors'. Where an error occurred in the work of a student perceived to be 'high ability', it could be interpreted as a 'careless slip' and appeared not to affect the overall judgement of the work substantially. Errors made by 'low ability' students, however, were sometimes seen to invalidate large parts of the rest of their work.

- *teacher expectations of an 'ideal' coursework text.*

Some teachers appeared to have very strong expectations of the forms which students might use in their texts, some of which were rigidly interpreted as 'evidence' of particular ways of working or thinking. The ubiquitous table has clearly become an important indicator of systematic data gathering - one of the criteria by which investigative work is assessed. A two-way table, however, does not display evidence of this sort of system (although it may display evidence of other kinds of achievement). Such a table may thus be a source both of tension for individual teachers and of difference between teachers. For example, responding to a two-way table in Clive's work, Joan focused on the value of the mathematics she saw in it:

> ... his tables are probably the best because he's put, you know, he's actually correlated two different things (...) I actually like these tables. (...) So he has actually looked for a relationship that way, you know, joining things together rather than just that it builds up in 2's or it builds up in 3's. *(Joan)*

Fiona, on the other hand, while recognising 'some organisation', ascribed little value to the table because it failed to provide her with the evidence of systematic data gathering that she associated with tables.

> at least he's tabulated his results and made some effort to, so that.. so there's some organisation there. Um, he doesn't seem, he's gone one two three four five on the slant, one two three four five on the top. He hasn't kept anything constant, you know, at any point, and certainly there's no evidence of it here. *(Fiona)*

In searching for evidence of systematic control of the variables in the form of a linear table, Fiona's positioning as examiner appears to prevent her from making any other kind of sense of Clive's table. The table is read as (in this case inadequate) evidence of process rather than as a problem solving or communication tool.

Conclusions

I would like to stress that in attempting to critique the assessment of coursework I am not criticising individual teachers. Rather, I am suggesting that their multiple positionings within the discourse of coursework assessment and the tensions within their situation make it inevitable that these contradictory behaviours should occur. It is necessary, therefore, to ask: is such coursework (or other modes of teacher-assessed, performance-based assessment) a 'good' means of assessment? There are a number of criteria for 'good' assessment that appear not to be met by the summative assessment of investigative work.

- *"Assessment tasks should reflect the values of the intellectual community from which the tasks are derived" (Eisner, 1993)*

One of the values that mathematics educators ascribe to investigative work is that of allowing or encouraging students to be creative and valuing divergent responses. As has been seen, teacher-assessors may have expectations about both the form and the content of a piece of work that make it likely that student responses diverging from these expectations will not be valued.

- *"Assessment should promote valid inferences about mathematics" (NCTM, 1995)*

The stereotyping of 'investigations' (probably itself a response to the need for reliable assessment) has led the development of more or less rigid associations between particular forms of text (such as tables) and particular desired types of mathematical activity. The search for the anticipated features of an 'ideal' text makes the false assumption that there is a one-to-one mapping of textual form to mathematical thinking and may lead a teacher to mis-interpret or to ignore mathematical thinking that is non-standard or is displayed in unconventional forms.

- *Reliability - different teacher-assessors should agree*

There is considerable anecdotal evidence suggesting a large degree of consensus among teacher-assessors (Wiliam, 1994). This study raises questions about the nature of the agreement. While within a small sample of teachers there was substantial (but not complete) agreement on rankings, there was substantial divergence in the meanings assigned to texts.

- *Beneficial (or at least benign) influence on the curriculum*

In spite of the problems with coursework assessment I am certainly not advocating a return to reliance on more 'objective' forms of assessment - their damaging effects on the curriculum have been critiqued adequately elsewhere and they often fail to satisfy the criteria of intellectual value and validity discussed above. It is nevertheless important to examine seriously the ways in which the assessment of investigative work may distort the nature of what happens in classrooms in the name of 'investigation'. In particular, the ways in which creativity and divergence may be valued and the effects of the use of textual indicators of processes must be considered.

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Exploring ways of Improving Practice

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This paper considers the role of school-based tasks in Initial Teacher Education. It relates how these 'course requirements' in the mathematics P.G.C.E. at the University of Sheffield have evolved in response to the views of students and school mentors. It suggests that, by involving school mentors in the setting (and possibly monitoring) of such school-based tasks, we may go some way towards improving the school practice of student teachers.

I would imagine that most Initial Teacher Education (ITE) course have some form of 'course requirements', i.e., tasks which students undertake over and above written assignments. From my own experience, the distinction between the two is often one of assessment: whereas the latter are graded, course requirements merely need to be completed. Some of these may take place in the university-based part of the course; others in the school-based component. My concern in this paper is with the latter, i.e., with how school-based tasks can help student teachers improve their practice.

The school partnership scheme at the University of Sheffield involves 35 secondary schools, and mathematics students are placed (in pairs) in a subset of ten of these schools. We have two block practices of 7 weeks and 11 weeks respectively, and student teachers change schools and partners for the second practice. Last Summer, at a meeting with the ten mathematics mentors, we discussed the role of course requirements in improving school practice. Prior to this discussion, the students themselves were given an opportunity to express their views on the school-based tasks at that time. The outcome of these two meetings was that only two out of our six previously used tasks survived. These were a display task and a writing task (details below).

The students felt that a school-based task should not be something that they might have done as a matter of course. Thus, 'Creating a Resource Folder' and 'Using a Visual Aid' (two of our earlier tasks) were rejected for this reason. They also felt that there should be an obvious usefulness to the task. 'Keeping a Reflective Journal' (another of our previous tasks) was rejected for this reason[1]. The mentors approached the question of school-based tasks from a different perspective. As one of them put it: "There was a need to do something for the school, and not merely in the school." Once we had

[1] I had mixed feelings on this. On the one hand, I share the belief that reflectiveness is the essence of learning to teach. On the other hand, I was beginning to question whether keeping a journal was the best way to promote this objective. In the university-based part of the course, I believe reflectiveness can best be encouraged through written assignments and seminars, both of which require students to consider the views of others; while in the school-based component, detailed lesson evaluations are a powerful way of facilitating reflection on practice.

overcome the hurdle of assuring them that, in raising the issue, we were not trying to impose an additional assessment burden on them, their responses was very positive. The end result was that four new school-based tasks were suggested by them. In what follows, the details of all six tasks are given along with some preliminary evaluations.

Display in the Classroom

This was one of the two surviving tasks. Its aim is to encourage student teachers and their pupils to experience the challenge, pleasure and pride of producing a quality display. In the end-of-course evaluation, few student teachers found this problematic, but several mentors expressed the view that there was a need to do more than simply mount a display. They felt that students should not see the display as an end in itself, but rather as a resource. Two observations stand out for me this year. Firstly, the most common displays were ones that students had seen in use earlier at the university, often as a result of a session given by one of their peers (see O'Reilly, 1996). Secondly, some of the best exhibitions of children's work were those which combined the display with the writing task. Several of these were mounted in school lobbies and it was common to see other children reading them.

Children's Writing in Mathematics

This was the second of the two surviving tasks. Its aim is to motivate student teachers to consider the role of writing in the learning of mathematics. Our original decision to incorporate this task into our course requirements stemmed from the belief that writing offers pupils a way of making their understandings public, and gives both themselves and their teachers windows into those understandings. Not all of our students were convinced. One student teacher remarked: "Many pupils look forward to mathematics because it does not involve writing, and now we are taking that away from them!" On the other hand, the mentors thought it worthwhile retaining the writing task because of the increasing importance of this aspect of mathematics in coursework assessment, a view backed up by recent research (Morgan, 1996).

Even if we agree that children should be encouraged to write in mathematics, there is still the question of how best to achieve this objective. Our guidelines made several loose suggestions, but often the most successful ideas were those that the students thought up. For example, Dave gave his class a set of questions, supposedly done by 'Puzzled Pete', most of which were incorrect. The task for the children was to play the role of the teacher, i.e., to give feedback to Pete on where he went wrong. Jean gave her group set of statements on card (e.g., "It doesn't matter which way round you do division."). The children then had to decide and justify whether these were always true, sometimes true or never true. There is not enough space here to do justice to

either the tasks or the responses. A preliminary analysis would suggest that creating mathematical tasks which evoke writing in a telling way is not simple, but may be well worth the effort as it can give insights into children's understandings in ways that the routine marking of exercises cannot.

Coursework Marking and Moderation

The aim of this task is to motivate student teachers to consider the special demands of school coursework marking and moderation. It is in two parts. Firstly, student teachers undertake a piece of coursework themselves, write a running commentary of the processes which they experience, and construct a mark scheme based on this. Having done so, the idea is that they then set the same task for a group of children (targeted at a year 9 class), using or adapting the mark scheme. Finally, the student teacher is asked to reflect on differences between the two, and whether the former influenced the latter.

On the positive side, students became aware of discrepancies in their perception of this task (almost inevitably an investigation) and children's views, and it encouraged them to undertake such a project early in the course. Typical comments were "The children approached the problem from a different direction to mine." (Adam). " I do not think my mark scheme was very appropriate to the children. ... More credit should be given to their constructions." (Nina). "The mark scheme was very harsh for a year 8 group. ... the mark scheme should be tailored to them, instead of being aimed at what I discovered." (Derek). On the other hand, several students suggested that, were they to do the investigation again, they would give clearer step-by-step guidance, which raises the concern whether an assessment-driven task will lead to a tendency to 'close down' the investigational activity even more than usual (Lerman, 1989).

Special Needs and Mathematics

The aim of this enquiry is to motivate student teacher to focus on one or two children who are experiencing difficulties in the learning of mathematics, to investigate and write a report on the nature of those difficulties and to suggest practical remedies which might be used by the teacher or department. The outcomes were mixed. Few doubted the value of the exercise, but the magnitude of the difficulties was greater than many had anticipated. Problems relating to memory or language recur frequently. Eric, for example, wrote of how he had taught a year 9 boy Len, who appeared to be understanding the work, only to discover that all was forgotten within a few days. Sandra relates how Martin, a year 9 boy, was very reluctant to put pen to paper, partly she says because he could not write his numbers small enough to fit into the small squares of his maths book. She also notes how this was overcome through the simple expedient of using square centimetre paper. She reveals how this same child showed her how to use a spreadsheet to draw a pie chart!

At the beginning of the university course, it was very noticeable how these well-qualified graduates stressed the cognitive side of mathematics. What stood out from their observations of these special needs children was the use of descriptions such as 'He is lacking in confidence', 'He needs reassurance', 'Her behaviour and attitude are related to her fear of failure', 'He cannot concentrate because he is tired', 'She burst into tears when I spoke to her'. It seems that, in working with students with learning difficulties, the 'affective filter' (Larcombe, 1985) through which all learning is mediated had become especially visible.

Mathematics Carousel

The aim of this task is to encourage student teachers (perhaps with their teaching partner) to set up a carousel of between 5 and 10 activities which are targeted at a year 7 or 8 class. These can be indoors or outdoors, or a mixture of each, and can take place over two or more lessons. There seemed to be two reactions to this task from students: that it took a lot of work to set up, and that it was worthwhile. Several students adopted a theme approach. Simon, for example, constructed his carousel of 5 activities around the theme of Christmas. This took place over four lessons with a year 7 class. Norma and Eric spent three lessons with a year 8 class on a carousel of 6 activities, based on finding the areas of rectangles. Adam used an angle theme with his year 7 spread over two lessons. Two notable side effects of the carousel were the need to organise the children into groups, and the need to reconsider the layout of the room. Several students drew up grids to help them and the children keep track of what they had done and needed to do next. One final observation: the carousel proved very popular with the children.

Mathematics Trail

The aim of this task is for student teachers (perhaps with their teaching partner) to construct a mathematics trail inside the school boundaries in a way that is both safe and fun to use, and which involves children learning mathematics in a practical way. This had several features in common with the carousel: it was often targeted at year 7 or year 8 pupils. It involved lots of activities. It involved group work, and it was very different to the usual routine. But, whereas the carousel took place in the first school practice, the trail was meant to be constructed in the second. The mathematics trail also required more organisation, e.g., letting other staff know what was going on, and more personnel to monitor different areas of the school.

I saw several of these on my travels. A number of students again said that they had to put a lot of work into the preparation, but they were pleasantly surprised at how well the trail worked. In particular, they remarked how the self-discipline of the children was often very different to that which they displayed in their ordinary pen and paper lessons. It may simply be the novelty of the activity, but perhaps this form of mathematics is closer to normal child activities than sitting at a desk?

Assessment

What about assessment? There is a dilemma here. On the one hand, there is a need to get students to undertake school-based activities without too great an increase in the assessment burden on them. On the other hand, there is a need to ensure that the tasks are undertaken in a satisfactory manner. To reconcile these demands, we have asked students to provide various forms of evidence that they have completed the tasks, e.g., photographs of their display, examples of children's writing, but the last word resides with the mentor. We have said that it would be sufficient for the mentor to verify that the school-based task has been undertaken. Most do considerably more, e.g., writing a detailed description of what has been done in the student's professional profile. At the time of writing (late May), we have yet to fully evaluate the effectiveness of these assessment procedures. What I can report is that the students are beginning to realise that these school-based tasks are a powerful personal resource which they can draw upon at interviews.

Conclusions

The above account was entitled 'Exploring ways of Improving Practice'. There is no suggestion that we have got it right. The key word here is 'Exploring'. In a few weeks time, our end-of-course evaluations by the students, and by the mentors, may well suggest otherwise. In sharing our experiences with colleagues, the hope is to begin a dialogue about ways of improving practice.

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BEYOND THE OBVIOUS: ARITHMETIC IN THE MIND

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Abstract

This paper considers children's predisposition when communicating about imagery. Evidence obtained from interviews with children aged 8 to 12 who are at extremes of mathematical ability, associates imagery of arithmetical and non-arithmetical items with that used to solve basic number combinations. "Low achievers" concretise and personalise words and give colour and detail to icons, an emphasis associated with the mentally imitation of mathematical procedures. "High achievers" reject the superficial and focus on the abstract, qualities associated with the flexible interpretation of symbolism. The discussion suggests that imagery formed from qualitatively different perceptions of the salient features of arithmetical processes may inhibit the encapsulation of these processes as numerical concepts.

Introduction

> I find it easier not to do it [simple addition] with my fingers because sometimes I get into a big muddle with them [and] I find it much harder to add up because I am concentrating on the sum. I am concentrating on getting my fingers right... which takes a while. It can take longer to work out the sum than it does to work out the sum in my head. (Emily, 9)

Emily is one of many children who appear to wish to do things 'mentally', or has been described by children so frequently "*in my brain*". Many do so because they know things and engage in a form of automatic processing. Others have to make a conscious effort to do so, not so much because they realise that such effort may gradually become automatic, but because of the social climate of the classroom. "*We are not allowed to use fingers*", "*I am to old for counters*" and perhaps the saddest from a boy of ten who "*wanted to do things like the clever children*". However, the recognition that some do things mentally does not give others insight into how things are done.

The focus of this paper is to give some insight into "doing it in my brain"–to go beyond the obvious and consider arithmetic in the mind. It does this by considering children's predispositions when communicating about their imagery of arithmetical and non-arithmetical objects and icons. To establish the latter it assumes that an image is mediated by a description (Kosslyn, 1980; Pylyshyn, 1973). It builds upon the notion that different connotations of the language and concrete items associated with objects of thought have implications for the quality of children's imagery and their processing ability.

A considerable amount of primary school teaching involves metaphors as mediators between actions with actual objects and concept formation. Given the appropriate instructional representation the hope is that children may eventually construct the "expert" representation by internalising or encapsulating a mathematical process as a concept. In such a way actions on concrete objects may become mathematical abstractions. There is a problem though–pedagogical activity may not be taking into account what it is children internalise. The metaphor may seem obvious to the teacher but it may not be so obvious to the child. Such an issue is one of importance–"What perceptions do children have of such representations?". "What are the important things [for them] to focus on?" We suggest that a partial answer to our questions may be determined from the qualitatively different ways in which children interpret the objects of their world, its words, pictures and icons and its symbols. It is our fundamental thesis that different qualities of mathematical abstraction are a function of the perceived reality associated with mathematical processes.

In attempting to address the relationship between imagery and achievement in simple arithmetic we recognise that there are difficulties. To a large extent our understanding of children's imagery relies upon their words and/or drawings. Both are sequential and therefore an initial image may be modified through verbal or visual clarification. Words may spark images and/or images words. It is only possible to make conjectures about images–we can make no precise claims about their exact nature. They may appear to be well wrapped possessions, covered in many fine layers, sometimes in discrete packages. We may believe it is possible to shake the package to find out what is inside, but by doing so we risk the possibility of breaking it. Such pitfalls, particularly in terms of operational definitions and interpretation, are clearly identified by Pylyshyn (1973).

Method

In an effort to uncover the layers and establish the relationship between children's qualitatively different approaches to simple arithmetic and the quality of their imagery we considered children's responses to a range of auditory and visual stimuli. The children–24 in all–are aged from 8 to 12 They cover the extremes of mathematical ability in two groups–"high achievers" and "low achievers". Drawn from a "typical" school in the English Midlands each child talked about their "images" of a series of stimuli over four separate interviews. Responses were obtained using semi-structured interviews recorded through field notes, audio and/or video tapes. At each interview children were asked to talk freely about their imagery and what came to mind with each item. The auditory items included common concrete nouns such as "ball" and "car", and concrete or abstract nouns such as "number", "fraction" and "five". On presentation of each auditory item children were asked to talk about their first image. Later they were asked to provide an "explanation" that would help a Martian understand it.

The visual components, included symbols such as "5", and icons such as (two quarters), (dancing man), ,(marbles) and (honeycomb). Later still, the children were presented a series of one and two digit addition and subtraction combinations, for example, 6+3, 9–5, 13+5, 15–9. Solution approaches to these combinations were classified similarly to that of Gray & Tall (1994).

Results

After repeated analysis the most powerful descriptive concepts and categories for discussion of the results of the auditory and visual items has proven to be as in Table 1. Not all classifications were used for each item as can be seen by shaded portions.

	Auditory Items	Symbolic /Visual Items
1. Not Known	Unable to give meaning or any sense of recognition	Unable to give meaning or any sense of recognition
2. Associations and contextual	Child unable to pinpoint meaning–child conjectures and provides an associative theme or context.	Child unable to pinpoint meaning–child conjectures and provides an associative theme or context.
3.1 Single example	Single example that does not include symbol	
3.2 Multi examples	Several examples of the item	
3.3 Symbolic examples	Symbolic references with general characteristics: prototypical example	
4.1 Visual concrete examples.	Details of visual characteristics given. Descriptive.	Details of visual characteristics given. Descriptive.
4.2 Imaginative Extensions	Item forms basis for imaginative and/or concrete extensions.	Item forms basis for imaginative and/or concrete extensions.
4.3 Insight to abstract qualities.	Descriptions of non-visual characteristics. Insight into meaning and relationships–tend to resemble definitions	Description of non-visual characteristics. Insight into meaning and relationships–tend to resemble definitions.
5.0 Proceptual Interpretation.	Emphasis on equivalence and interpretation.	Process and concept described. Examples of equivalence and interpretation

Table 1: Classification of responses to auditory and visual items

When "low achievers" talked freely of their first image of the auditory items responses identified as 'association' or 'single example' tended to dominate. "High achievers" provided responses similar to those of low achievers when talking about the concrete nouns but, in contrast, when discussing 'mathematical' items, they tended to evoke aspects of symbolism.

"Low achievers" 'explanatory' responses to the Martian continued to place an emphasis on 'association' and 'single example' but there were also tendencies to provide description which emphasised 'visual' characteristics. It may well be that the "lower achievers" provided better explanations than the high achievers; they focused on the objects and their explanations were related to visual attributes, properties or the use of items. To clarify their explanation they frequently indicated that they would show a picture or a model. In contrast, "High achievers" explanations were dominated by 'qualities with insight'. They frequently ignored the concrete and more fundamental characteristics through which the item may be recognised but their focus on deeper qualities would have required further explanation.

"Low achievers" responses to the symbolic and iconic items demonstrated less evidence of 'association'. Icons were discussed using 'visual description' and/or 'imaginary extensions', categories used by every "low achiever". Symbolic representations of whole numbers and fractions were associated with other objects or described through visial qualities. In general, "high achievers" responses to these items focused on identifying 'qualities with insight'. Responses, depended upon item difficulty. Numerical interpretations were expanded to include visual qualities, as in the case of 'two quarters' or grew out of an initial descrpition of the visual qualities, as in the more difficult 'honeycomb'.

The overall differences between the two groups of children's responses to the auditory and visual stimuli are summarised in Table 2.

	Low Achievers	High Achievers
Words	• Concretised • Unable to reject information • 'Horizontal' thinking–directed towards surface features. • Imitation • Excessive memory	• Focus on abstract qualities • Able to reject information • 'Vertical' thinking Direct attention towards core features or definitions. • Thought generator • economic memory
Icons	• Interpreted as a "picture out of focus" an incomplete concrete reality which needs focusing. • Given colour, detail and realism (through imagination). • Display "horizontal" thinking–'imaginary extensions' similar in quality. • Imitation • Excessive memory	• Concentrate on abstract qualities. • Able to ignore detail and concentrate on the interpretation • "Vertical" thinking associated with free movement between abstract and descriptive aspects. • Thought generator • Economic memory
Symbols	• Order to carry out an action • Concretised by either: (a) associating with a concrete item or (b) identifing as an icon. • "Horizontal thinking" demonstrated through procedural association. • Imitation • Excessive memory	• Recognised as both the holder of an idea and an action. • Detached from concrete qualities, associated with abstraction • "Vertical thinking" demonstrated by proceptual flexibility. • Thought generator • Economic memory

Table 2: Differences in children's interpretations of words, icons and symbols

Similarities within groups and differences betwen groups were remarkable. Attention was either directed towards the core aspects of the concept, as exemplified by the "high achievers", or it was directed towards identification features, as seen amongst the "low achievers".

The "low achievers" responses to the words were strongly associated with the adjectival aspects. Common nouns were described and abstract nouns were concretised by association with concrete nouns. These interpretations provided evidence of "horizontal" thought–the descriptive qualities always at a similar level. The children also tried to give real substance to the icons; they were pictures out of focus (Pitta, 1995). Attention was directed towards properties and characteristics that could be named and have meaning–"*it is a window*", "*it is a plate with spilt peas*". Mathematical notions, (whether whole numbers or fractions) appeared to possess no logic or importance until concretised. Children appeared unable to detach themselves from a search for substance and meaning–no information was rejected, no surface feature filtered out. The focus was on **visual** characteristics and **parts** of objects. The children did not talk about an image as a **skeleton** upon which they may pin ideas.

"High achievers" appeared to move straight to the heart of things and by doing so they rejected superficial information. It is hypothesised that this provides them with a mechanism for focusing on the generative properties which will allow them to do mathematics.

Classification of the children's responses to the numerical items was closely modelled on that used by Gray & Tall, thus distinguishing between procedural and proceptual thinkers. The interviews elicited some sense of the children's imagery in the absence of the use of figural items. Though the strategies of the "low achievers" extensively involved counting procedures there was substantial evidence of the use of images. However, such imagery was generally associated with easier items and appeared to be age related; it was more frequently described by older children. Images were usually analogues of physical representations–fingers, tally lines, number tracks and marbles. Imagery associated with the strategies of "high achievers" was always symbolic; symbolism had a distinctive role.

	Low Achievers	High Achievers
Arithmetical context	• Concretised • Unable to reject information • 'Horizontal' thinking–directed towards a procedural thinking with variations of the figaral/imaginary items • Imitation • Excessive memory, overload of WM	• Focus on abstract qualities • Able to reject information • 'Vertical' thinking Direct attention towards known facts or transformations. • Thought generator • Economic memory good use of LTM-use of symbols

Table 3:Childrens imagery in a numerical context

Table 3 shows the differences with which children faced the numerical items of the interviews. Comparison with Table 2 clearly shows the similarities and diffences between the two groups of children over the range of items that formed the basis for comparison. We once again see the tendency of "low achievers" to concretise and focus on all of the information. This leads to an emphasis on imagery in the numerical context that is strongly associated with procedural aspects of numerical processes. The children carry out procedures in the mind as if they were carrying out procedures with figural items on the desk in front of them.

"High achievers" appear to focus on those abstractions that enable them to make choices. Their ability to reject information is again apparent and by doing so they may direct their attention towards knowing or using what they know. This contrast leads us to suggest that the qualitative differences observed when children discuss the images of their environmental reality transfers to the quality of image generated for their mathematical reality.

Discussion

Initially it is necessary for the learner to focus on the specific and more evident action of counting. Eventually it is to be hoped that they focus on the subtle and more generative concepts that arise from such actions. However, it appears to be clear that though children may look at and experience the same thing, through their "minds eye' they may see, use and manipulate things differently, Such a distinction appears to lead to contrasting outcomes associated with children's imagery and its use in the context of elementary arithmetic:

- an image is formed of the collection, the collection is named and symbolised and associated with other collections
- an image which is an imitation of the action is formed.

We suggest that such differences and the way in which the imagery is used is a contributor to the divergence in mathematical thinking exemplified by the proceptual divide (Gray and Tall, 1994). Action on objects of the environment forms the basis for arithmetical development. Excessive concentration on these objects in a visual or tactile way, or close concentration on the action, as opposed to concentration on the action and the result of the action, may be a reason for some children's inability to personalise numerical procepts. Images associated with the properties of objects seem to be assiociated with a need to concretise symbols, an interpretation leading to a procedural orientations Images that are essentially imitations of procedures that not only direct thinking but become indespensible to it. They make a dramatic strain on working memory.

Conclusion

"High achievers" appear to have images of collections which can be transformed, talked about and symbolised. The ability to focus on generative properties leads to the formation of qualities of abstraction that are essential to the flexible utilisation of the strength of mathematical symbolism. These 'symbolic images' are an economical way to present or hold an idea. They remind the child of what is being done and are used to refresh memory: they trigger the idea but do not need to accompany every thought thus enabling the child to oscillate between seeing an image and thinking.

To gain more insight into this difference we need to focus on the quality of abstraction associated with arithmetical processes. Children's interpretations of the salient features of the processes that form part of perception may seriously inhibit the cognitive shift associated with the formation of numerical concepts from these processes. For some, "encapsulation" may be an unattainable goal.

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DESIGNING MATHEMATICAL MICROWORLDS: AN ITERATIVE APPROACH

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This paper presents a case study of the iterative design and development of a computer-based mathematical microworld for non-euclidean geometry. It outlines a model for the microworld which was used to provide windows both on the processes of learning non-euclidean geometry and the design of a computational context in which the learning might take place.

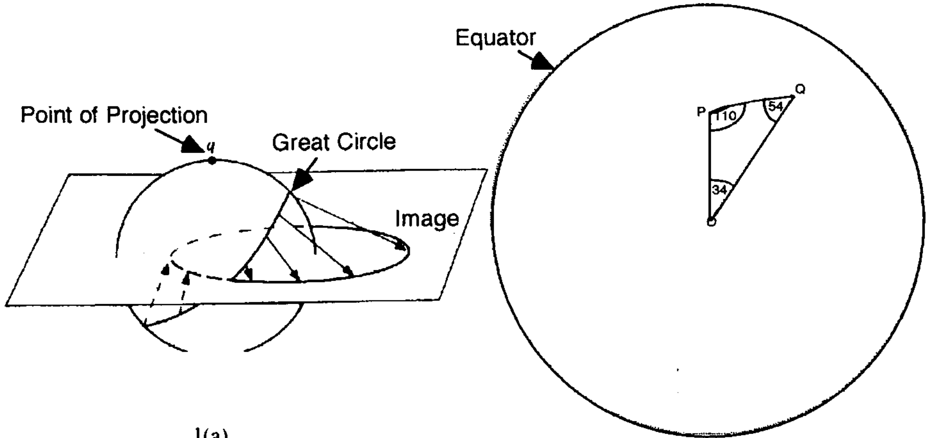
Introduction

Mathematical microworlds are concerned with providing two things. First, they give the learner simple entry points into mathematical domains so that he or she can connect with the domain's central concepts. Second, microworlds supply tools designed to enable the learner to develop their understanding of the underlying knowledge domain. It is not clear in advance, however, what aspects of a knowledge domain are cognitively significant for the learner, and consequently, what could act an entry point or support understanding. This paper presents an approach to building microworlds based on the notion of design as an iterative process. It takes the form of a case study that describes the design and development of a computer-based microworld for non-euclidean geometry. This process creates and then makes use of a model for the microworld to investigate cognitively significant aspects of the mathematical domain.

Why non-euclidean geometry ?

Historically, non-euclidean geometries emerged from a variety of attempts to clarify the logical status of Euclid's axioms. These attempts centred on investigations into the logical status of the so-called "Parallel Postulate" or "Euclid's Fifth Axiom" and its relationship to the other axioms of euclidean geometry. A central difficulty for investigators, both in formulating the problem and finding a solution, was the fact that their results ran contrary to common sense and accepted mathematical methods and facts. The resolution of the issue, demonstrating that non-euclidean geometries were possible and valid, came about by treating geometry in an abstract and logical way rather than relying on visual intuition.

Visualising abstract mathematical structures such as non-euclidean geometries is difficult, although not impossible. Standard euclidean models for elliptic and hyperbolic geometry, obtained by projecting surfaces that produce the geometries, have been in existence for over a century. Figure 1(a) shows how the sphere can be modelled by projection onto a flat plane through its "equator" from a point q at its "north pole". Figure 1(b) shows the image of a triangle projected from the sphere onto the equatorial plane. The triangle OPQ, in Figure 1(b), is formed from the diameters through P and Q and the arc PQ which lies on a circle that cuts the unit circle at opposite ends of the same diameter. The measurements shown are euclidean, obtained by measuring the angle made by the tangents to the circle at P and Q. Gray (1989 p.215) describes this model as the *Hot-Plate Universe* in which one imagines that the equatorial plane is a plate whose temperature increases as one moves out radially from its centre. Using a metal rule to measure the distance along an arc or a straight line out from the centre, one would be unaware of the ruler's expansion due to the increase in the plate temperature. Only by comparison with a ruler "at room temperature" would the change in length and the increase in "unit step" of measurement be revealed. This variation of distance measure according to position in the model is a key perceptual feature, and is contrary to our usual perception of distance measurement.



1(a) 1 (b)

Figure 1(a) The Projection of a sphere. 1(b) The image of a triangle projected from the sphere onto the equatorial plane

A similar model for hyperbolic space exists by projecting the positive branch of the hyperboloid $Z^2 - X^2 - Y^2 = 1$ through the point (0, 0, -1). This maps the hyperbolic surface to the inside of a unit Euclidean circle, whose circumference represents infinity. Space does not permit diagrams to illustrate this, but the model produced in the X-Y plane is more commonly known as the Poincaré Disc Model (Stillwell 1992). Again, straight lines on the hyperboloid are mapped either to the arcs of euclidean circles orthogonal to the unit circle or euclidean straight lines from the origin. This is the *Cool-Plate Universe* (Gray *ibid.*) in which the temperature decreases as one moves radially towards the circumference of the unit circle. Hence, a metal ruler used to mark out distances along the arcs of circles would contract as it was moved out from the centre and the "unit length" would decrease. However, living in the surface one would be unaware of the variation in length. The change in distance measure would only become apparent if one could compare the rule with that in a "constant temperature" euclidean world.

Both these models preserve angle measure between lines representing elliptic and hyperbolic geometry, but do not preserve congruence. Congruence is concerned with the fact that objects which coincide are equal and ensures that measuring rods mark-out the same lengths *irrespective of position.* Our usual sense of distance measure is that it is does not vary with position in space, since a ruler measures a metre wherever it is placed. However, in these models for non-euclidean geometry, this aspect is not preserved: distance measure varies with position in the model. From a perceptual point of view this variation in congruence represents a difficulty. How one comes to

recognise and adapt to this change in the perception of distance is an interesting and significant question from both a pedagogical and cognitive point of view. Two sorts of issues must be considered. First there is the process by which individuals make sense of the models and use them in developing an understanding of non-euclidean geometry. The second relates to how this process can be charted so that the cognitively significant aspects of the knowledge domain can be identified and developed using a mathematical microworld. Some sort of framework is needed, therefore, which simultaneously provides windows on how individuals learn non-euclidean geometry *and* on how the context in which the learning takes place, itself, develops.

Designing a Microworld for Non-euclidean Geometry

A suitable place to begin designing this microworld was to consider, in general, the changes in geometric domains necessary to enable the learner to use computer-based versions of the models. Initially, those using the microworld would have an understanding of geometry based on Euclid, and rooted in their everyday intuitions of space. In order to connect the microworld users' euclidean intuitions of geometry with the computer-based models, two other stages had to be passed through. The first was the introduction of the learner to non-euclidean geometries, and the conflict generated by changes in geometric intuition. Second, they would have to be introduced to the flat projection of the geometries which were to be implemented in the computer software. The overall structure of the microworld was, therefore, one which started with the introduction of the learner to surfaces of non-zero curvature and their non-euclidean geometries, then proceeded to the projection of these surfaces, and culminated in the learner's fluent use of the microworld's computer-based models. These three distinct phases provided the basic structure for the overall development of the microworld.

The next step was to design, in detail, a microworld which implemented a Turtle Graphics version of the models for non-euclidean geometry. Drawing on Hoyles and Noss (1987), three key elements of the microworld were identified: the technical, pedagogic and cognitive. The technical element consisted of computational and non-computational components required by the knowledge domain. In this case it was made up of physical surfaces, their projected images and the computer-based models, implemented in Object Logo[1]. The pedagogical element describes the aims of the microworld, the type of activity which learners engage in, and the pedagogical strategy adopted. Finally, the cognitive element consists of the learners' responses to the experience of working with the technical element within the pedagogical structure. The development of the microworld was described by analysing the interactions of these three elements over time.

The method chosen to develop the microworld was based on diSessa's notion of iterative microworld design (1989). Two factors were important in this process. The first factor may be described as "documented redundancy", in which a large number of activities were tried and their outcomes carefully observed and documented. The intention was to build up experience for the designer of what "worked" and what did not within the area of investigation, and to "calibrate" a range of possible cognitive responses from the participants. Activities, therefore, may fail, but from

[1]This is an object-oriented version of Logo. For details see Drescher, G. L. (1987) "Object-Oriented Logo". In Lawler, R., and Yazdan, M. (eds.) *Artificial Intelligence and Education vol.1* New Jersey. Ablex Publishing Co.

these failures emerged a set of viable opportunities for learning. The second element in this process of iterative development was a clear framework of pedagogical aims to guide its overall direction. From this point of view, the pedagogic structure of the microworld played an important role, both in guiding the development of activities and in assessing their outcomes.

The developmental process consisted of *phases* and *cycles*. *Phases* referred to the internal structure of the temporal progression from curved surfaces to computer images, which constituted the technical component of the microworld. Phase 1 was concerned with activities associated with physical surfaces; Phase 2 was concerned with those activities that dealt with the flat projections of the physical surfaces; Phase 3 related to computer-based activities. *Cycles* referred to the iterative development of activities which covered one or more phases and were related to the overall development of the microworld. Figure 2 illustrates the structure of a cycle.

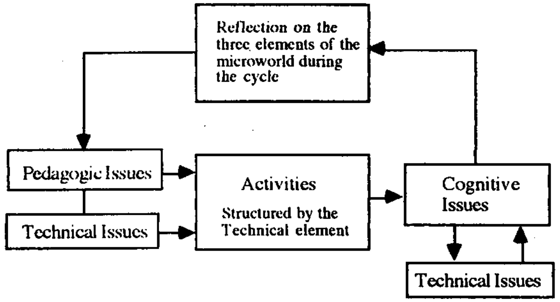


Figure 2. The Structure of the Developmental Cycles

Each cycle began with a set of pedagogical and technical issues which have either been carried over from a previous cycle or, in the case of the first cycle were part of the initial attempts to build the microworld. The activities were tried with the participants, and the development in their understanding as a result of their work with the microworld's technical components was reflected on. This generated a new set of pedagogical and technical issues which, together with new ideas, formed the basis for designing the next set of activities.

Modelling the Microworld

As figure 2 indicates, part of this iterative process was a systematic attempt to reflect on both "what seemed to work" for those learning non-euclidean geometry, and how the elements of the microworld mediated this learning. Analysing the dynamic interplay of the elements led to a recognition that it was possible to model the microworld in a way that provided both "local" windows on the process of learning non-euclidean geometry, and a "global" window on the development of the microworld as a whole. This model consisted of two views of the microworld: diachronic and synchronic. (Walkerdine 1988 p.184. Full details of the model, together with its theoretical framework can be found in Stevenson 1996).

The diachronic view of the microworld was concerned with the overall structure and temporal development of the microworld as a complete unit, made up of the phases mentioned above. The technical element was organised around the three phases of surface usage: objects, projections and

computer-based models. At a pedagogic level, this technical structure implied a period of induction in which the microworld's participants had their euclidean intuitions challenged as they were introduced to non-euclidean geometries. A process of instruction followed in which the microworld scaffolded the participant's understanding of the projective relationship between object and the computer-based models, leading to a "fading", in which the participants worked entirely with the computer images. Cognitively, there were three corresponding phases. First, a breakdown in understanding as the learners' euclidean intuitions were challenged by the microworld. Second, a period of re-structuring, in which the participants learned about non-euclidean geometries and their representations. Third, "fluency" in the new "language-game" (Wittgenstein 1953 23) of computer-based models of surfaces with non-zero curvature. Figure 3(a) illustrates the initial diachronic view. (For the final state, see Stevenson (*ibid*.)).

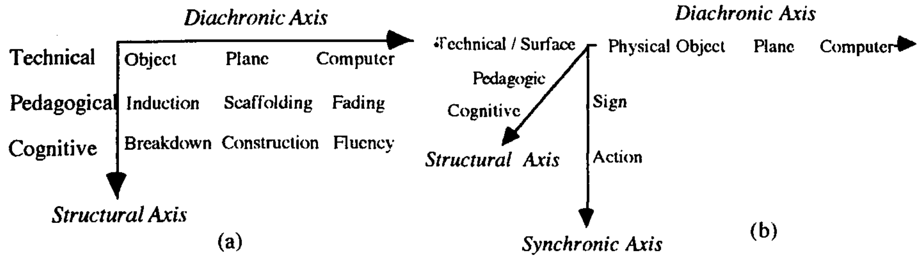


Figure 3.(a) The Diachronic View of the Microworld. (b) The Full model of the Microworld

The synchronic view of the microworld was concerned with analysing the construction of meaning, both pedagogically and cognitively. By examining the interplay between actions on physical surfaces and linguistic signs, an understanding of the "language-game(s)" associated with non-euclidean geometry could be developed. This view was used both to design the microworld and to analyse the way that the learners used the technical element of the microworld. The synchronic view gave "snapshots" of what was happening to the various elements of the microworld at any given time, and, hence, reflected their structure at different stages of development. The combined diachronic and synchronic views, shown in Figure 3(b), formed a "three-dimensional" model that was used to chart the interaction between the views across cycles, and hence, the development of the microworld itself.

"Significant Features", Models, and Microworlds

What, then, is meant by "significant features" of a knowledge domain? Two aspects emerged from the iterations of this microworld in relation to non-euclidean geometry. First, the learners developed their fluency with the software at the same time as they built up an understanding of the geometry. They made use of the physical objects, such as a sphere and a hyperboloid surface, both to support their understanding of the projective nature of the computer-based models, and to create their own investigations. Second, two features of the Turtle Graphics implementation of the computer-based models made a considerable difference to the learners' ability to work with the models. By "dashing" the tracks left by the Turtle, the notion of the hot and cold plate universes could be implemented, since the dashes varied in length according to the Turtle's position in the computer-based model. This provided both visual and dynamic support to the change in congruence implied by the models, and helped the learners to mobilise their internal resources in terms of visualisation and previous geometric experience. The second feature of the software was a screen button which, when pressed, drew the global path of the Turtle on the screen. This proved useful for investigating the models, but also helped the learners to connect the screen behaviour of the Turtle with large scale properties of the surfaces from which the models were obtained.

"Significant features", therefore, seemed to be those aspects of the microworld which *simultaneously* embedded the learner in the knowledge domain by mobilising their intuitions and prior experience, and separated them from it. This differential process (Ackermann 1991) both connects the learner with the knowledge domain and enables them to create "critical distance" from it. In this microworld, the object-screen combination and the features of the software seemed to mediate the dialectic of embedding and separation. The role of the microworld's model was to enable the construction of both the local and global windows. The global window was given by the changes to the model and its structure over the developmental cycles. The local windows, on the cognitive development of the participants and the pedagogical intentions of each phase, were given by the synchronic views on each of those elements in the microworld. Using the microworld's model, therefore, allowed a systematic investigation of the knowledge domain to be carried out which identified and supported important cognitive changes. The model could be used to *coarse-tune* for significant features by locating the search within a specific framework of known parameters. It could also be used to *fine-tune* the structures of the microworld to the responses of the participants, thereby aiding their understanding of the underlying knowledge domain.

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CHILDREN MAKING SENSE OF MATHEMATIC AND SCIENCE THROUGH CLASSROOM DIALOGUE

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Abstract

This paper presented and analysed a dialogue in which four children and the author discussed a problem arising from a practical in which the children's intuitive prediction about motion under gravity conflicted with the data collected. Reflective classroom dialogue is presented by the conceptual change literature as 'essential' but typically literature is vague about how this is to be constituted. The analysis looked at the evidence of children shifting their explanations for motion towards a more scientific framework through a process of inquiry. The characteristics of the inquiry were analysed from the point of view of a 'dialogue' in the sense of the Children's Philosophy Movement. The protocols themselves provide evidence of shifts in explanation and formulation by individuals, and in the apparent consensus of the group. Finally, the role of the author in the dialogue is analysed and tensions between two distinct role emerge.

Introduction

An Australian Research Council Project 'Practical Mechanics in Primary Mathematics' has been developing classroom research into the feasibility of shifting young children's explanation of force and motion towards a more sophisticated, scientific framework. The strategy involves mathematical modelling and data handling in practical mechanics situations. A key question for research is to investigate the extent to which an appropriate programme of practical mechanics activities, in which teachers engage children in discussions which support theory building, might result in a shift towards more formal Newtonian concepts amongst the children. The literature on conceptual change is somewhat vague about how the reflective discussion might be organised, and what the role of the teacher will be in this.

This paper explored the nature and function of one such discussion, held between the researcher and four upper primary pupils involved in one of the activities, which seemed to be productive. It explored the conditions for maintenance of an effective dialogue and the evidence that children could 'learn' from such a dialogue. Finally it explored the roles of the leader of the dialogue and the tensions experienced between conflicting demands related to leading the dialogue scientifically and supporting the children's own sense-making.

The timer ball activity and its problematic

Prior to this activity the children had done some work with modelling the motion of a rolling ball by marking it's position every second with a brick, cutting paper strips to the distances between bricks, and assembling the strips into a graph to represent the speed of the rolling ball. The idea is to discuss the graphs for different experiments, where the ball speeds up, slows down, stays the same speed etc. (See Doig et al, in press.) This experience proves an important background to the children's discussion of the timer ball activity.

The children were asked to drop a timer ball (a baseball which has a stop watch inside it and which gives a reading of the time taken for the ball to fall to the floor) a few times, and to try to find the height from which it takes exactly half a second to fall. In this episode the children for some reason homed in on a height of 95 cm which gave a consistent value of about 0.43 seconds. One boy, Daniel, has small hands and finds it difficult to operate the ball consistently.

I then asked the children to guess where they should drop the ball to get double the time, i.e. 0.86 seconds. They guess double the height, i.e. 190 cm. They try this and find it takes only 0.63 seconds for this height. They don't believe this and try it several times. The accuracy of the data gives most of them confidence that their prediction is false and they have a conflict to resolve. This is the problem which starts the discussion. Here is a summary of the 13 minute dialogue between Richard, Daniel, Kelly, Stephanie and the author in ten segments:

1. Richard suggested that the timer returned to zero, i.e. the drop took more than 10 seconds, he didn't really believe this but probably thought it might re-zero quicker than this...

2. Daniel expressed concern about the girls' measuring: suggesting they had made a mistake. Kelly supported Daniel's concern, suggesting that she and Stephanie had used the wrong side of the tape measure. She wanted to check. I asked for all explanations first.

3. Stephanie's first explanation was that there is more gravity pulling it down "up there" than lower down, and the ball had not got "as far" for gravity to pull it as lower down. Lots of hesitant use of language here. (See below.)

4. Daniel interrupted: "gravity is the same everywhere", and Kelly picked this up and supported this; she contradicted Stephanie's first explanation, and seemed to relish the change in Stephanie's position when she produced the second explanation (below).

5. Daniel was still questioning the reliability of the data, and wanted to rig up some clever way of taking human error from the dropping; Kelly had a shot at inventing such a device, but gave up. This drew the focus of the discussion from the science back to the data. I redirected the discussion to Stephanie, to hear more about gravity.

6. Stephanie's second explanation: she now accepted that gravity is the same everywhere, but she argued that it had more time, more "chance" or time to act in the top half of the motion because it was going slower then. Everyone liked this new explanation, especially Kelly, who really rubbed it in that this was a new explanation. Daniel agreed: it built on and validated his remark that gravity is the same everywhere. It seemed to be important to everyone that this was seen as a new and different explanation: the shared understanding made progress.

7. Richard produced an animist view: it took a while for a runner to "get going" in a race, and the ball was speeding up in a similar way. He had provided a similar point of view to describe the motion of a rolling ball which was accelerating in the previous rolling ball practical. Kelly listened to this and reinforced the explanation, paraphrasing it.

8. I asked Daniel why this explained the data. Daniel tried to explain why the 'speeding up' explanation gave a "less than 0.86" prediction, rather than more. He said "because it is moving faster". His expression suggested that this was patently obvious!

9. Stephanie made a connection with the accelerating rolling ball practical completed the previous week. She imagined the ball to be stopped in mid-air and the strips cut to the distances fallen. In both cases the strips get longer as the ball speeds up. (Though she referred to an experiment where the ball was slowing down due to friction, which gave the opposite.) She saw this as the explanation why the same lengths don't have equal times here, though the reasoning was not explicated. Kelly saw the connection: in both cases the ball speeded up, in the first as it rolled along, and here as it fell down, but said she didn't understand it.

10. The discussion was then distracted to new phenomena. Kelly described throwing a ball in rounders: they are taught to throw the ball "straight" to the catcher, rather then high up in the air. In the latter case the ball will take longer to get to the catcher. Daniel supported Kelly in making this connection and he suggested the higher the ball is thrown the longer it took to fall to the ground. Stephanie saw the possibility of interpolation to find a new position to drop the ball from so that it takes 0.5 seconds to reach the ground.

The weaving of this discussion in and out of the Physics (kinematical and dynamical) and the mathematics and mathematical modelling has been analysed elsewhere. (Doig et al, ICME-8 presentation, 1996: forthcoming pub.) To summarise briefly, the data is central in both providing the problematic for the dialogue, though the accuracy of the data and the children's feeling for this accuracy is essential, too, since otherwise they can avoid the challenge to their intuitive belief. (See Daniel's initial resistance.) An essential element of the challenge to Stephanie's first explanation is purely socio-cultural: the children have been told that gravity is the same everywhere and this is a key point! The graphical modelling is an effective mental model for Stephanie and provides a potential resolution at a deep level of explanation; this is because the graph represents speed, and its properties model the changes in speed, which is the central concept in Newtonian explanations.

However a key element in the children's relative success here is the implicit acceptance that the times found are explained by the ball speeding up. We have evidence that many younger children simply do not accept this, and that there is a subtle problem in understanding that data collected for drops from different heights relates to the data for a single motion. This was a surprise to us, but of course obvious after the event!

Example of a student's development of explanation in the dialogue: Stephanie

The evidence of progression in explanation is clearest in Stephanie's case. And her own development had a major impact on the other children, as we saw in the above summary. Here we look at the transcript and identify the three stages. First, Stephanie's explanation is elicited, then the group challenges it, and later she has the opportunity to patch up her explanation:

Stephanie's original position

Stephanie: *It's not really an idea it's more an explanation . ...'Cos there, it's not got a lot of gravity pulling it down; but up there, it's got quite a lot; so it's pulling it down a lot quicker. ... So, it's gone a bit slow ...*

Julian: *What's gravity doing?*

Stephanie: *It's pulling it down.*

Julian: *...and what's that got to do with it?*

Stephanie: *Like from 43, it hasn't got as much gravity as it has from ...* - interruption -..

Daniel: *Yes it has, it's got exactly the same!*

Stephanie: *... as from 95 to 190; because there, it's not got as far for the gravity to pull it; but up there, it has got; ... so I think that's just about right..*
...Time doesn't go quicker, the ball goes quicker.

The challenge to this position:

Kelly: *Stephanie said that its got more gravity than when it's up there , but I think that gravity is always the same ...*
It's not like you're lifting different weights, it's the same weight.

Julian: *The gravity on the ball is the same because its got the same weight.*
Right ......Stephanie's going to defend herself......

Stephanie's new position:

Stephanie: *The gravity's the same all the time, but from up there, the gravity's got more of a chance of pulling it; but from there, it's got less chance 'cos its falling down.*

Julian: *So because its moving down the gravity's different .. than when its just stationary?*

Stephanie: *No, like I say, the gravity isn't different. ... It's...'cos up there it's got more time to pull it. ... From there .. it's got hardly any time to pull it, 'cos it's going down..*
So it's not the gravity that's different.

Each of the other children can be similarly analysed. However, in other cases the children's appreciation of the sense of the dialogue has to be inferred from non-verbal clues and from the kind of interjections made as they attempt to make sense of the dialogue. My own remarks in the discussion were intended to summarise and reflect the shared sense, the consensus. But of course in many cases the very expression of this sense in the scientific language of a teacher changes the meaning of the dialogue. This role will be examined below.

What counts as 'dialogue'?

What makes an effective discussion? Why were the children apparently successfully engaged, generally listening to each other and building on each other's contributions, and apparently learning from the dialogue itself?

We draw on the Children's Philosophy movement for analysis of dialogue, which is contrasted with the more general notion of discussion. In a genuine community of inquiry, the dialogue plays the central role and has clear properties and must obey certain rules (See Splitter and Sharp, 1995, p35):

1. The focus question is problematic, or 'contestable'.
2. Self-regulation or self-correction: "participants are prepared both to question the views and reasons put forward by others, and to restate their own position in response to questions or counterexamples that come from the group."
3. An "egalitarian structure... participants value themselves and one another equally for the purposes of the dialogue, irrespective of where they stand in relation to a particular viewpoint."
4. There is a "mutual interest...... it is the participants (of whom the teacher is but one) who set the agenda and determine the procedures for dealing with the issues at hand."

Clearly such a dialogue requires that all involved have some shared understanding of the procedures and purposes, and agree to engage in the community as such. In practice, a teacher who wishes to develop a classroom community of inquiry has a lot of teaching to do. (Some elements of this are evident in the transcript where I have to teach individuals about the rules of discussion, and this even in a small group of four children.)

The leader's contribution to the dialogue

An analysis of all the author's contributions to this dialogue identified two classes. In the first I appear to behave as a leader of the dialogue as such, in accordance with the principles outlines by Splitter et al (1995). These involve managing the dialogue itself and eliciting or seeking clarification from children:

e.g.1: Julian to group: *Right. I think Daniel's got something to say first, and then Stephanie's going to defend herself and then we're going to ask Richard what he thinks in a minute*

e.g.2: Julian to Stephanie: *You said a lot of interesting things there, but I need to take it a bit slower. ... What's gravity doing?*

On the other hand there are contributions which seem to be quite in contradiction to those expected of a neutral leader of philosophical dialogue. These involve the use of the teacher's expertise as a scientist and mathematician, they focus the dialogue on apparently productive lines of enquiry and summarise the consensus of the dialogue in ways which may be consistent with scientific progress:

e.g.3: Julian to Stephanie: *Very interesting, you mean the ball experiment we did the week before last.. and you've done it since then ,again?*

e.g.4: Julian to Richard: *Say that again, what's speeding up?*

This second role has a major place in any Mathematical or Scientific dialogue, where the teacher is both the leader of the dialogue (in the sense of the chair who enforces rules) and the most expert person present. Even in a 'community of inquiry' in the Philosophical sense, any individual in the community who has relevant expertise has a duty to the community to share it in the pursuit after truth. But in the scientific inquiry the imbalance of knowledge is so great that this may distort the whole nature of the dialogue as a genuinely communal or collective effort. The teacher who plays both roles therefore has a special and difficult tension to manage.

The procedures of the dialogue allow the children to contribute to, build on and learn from the dialogue. The content of the dialogue is seen to make progress: we see the community itself in a sense make progress towards the solution of a problem and in this sense it learns. The leader of such a dialogue has this dangerous and dual role: to try to facilitate the forward progress of the dialogue (focusing on key ideas, summarising the consensus in productive terms etc.) and yet ensuring that the children have adequate opportunity to make sense of this progressive dialogue (through eliciting and seeking clarification of the children's understandings.)

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Working Groups

WORKING GROUP
INTERVIEWING - A SUPPORT GROUP
CONVENOR: LAURINDA BROWN
UNIVERSITY OF BRISTOL, SCHOOL OF EDUCATION

At this meeting, Stephen Hegedus, a PhD research graduate from Southampton University, shared with us his methodology used in research into the problem-solving behaviour of first year mathematics undergraduates in the domain of Calculus. What follows are his writings after the meeting:

My research has focused particularly on the types of control/metacognitive skills used in topics concerning integration using verbal reports as data amongst other sources.

The pilot study of last summer, and following sessions, looked to test the usefulness of think-aloud as a method in collecting data.

The working group at BSRLM met to discuss a piece of verbal data which describes a think-aloud session with a first year undergraduate Calculus student at Southampton University.

The piece of data covered the first ten minutes of the student's work solving a Maclaurin Series problem which had been given as a homework question the previous December. This able student obtained his primary answer in this time, although it was incorrect. He had differentiated $f(x)$ incorrectly and obtained an incorrect expression for the kth derivative of $f(x)$. This had serious repercussions in obtaining a correct Maclaurin Series approximation for $f(x)$.

The question was:

Find a kth derivative expression of $f(x) = ln(1 - 3x)$ and hence an expression for the Maclaurin Series of $f(x)$ including its nth term.

What came out of that work was a methodology which relaxed the restrictions of the Cognitive Psychology method. The main data collection period of this summer's semester (1996) used think-aloud/open-ended interviewing as its main source, triangulated with retrospection, on the part of the student both during and after the interview.

The Pilot Study highlighted three main areas where students use control: managing algebraic expressions (symbols, function, etc), managing algorithms (formulas, methods of attack) and managing concepts involved.

These aspects of control were logged this semester with respect to topics involving integration. These were used to devise specific tasks with the intention of exploring and justifying some significant aspects of control.

An agenda for each of the main data periods was created based upon the three topics highlighted above. It also listed possible times when they would occur and the sorts of inquiries which would be made in their eventuality. So my interventions were purposeful; they sought to answer questions. In the protocol analysis, my intervention times will be analysed to see if they do highlight how students use control.

Over the past year of data collection, there has been a slow transition from what were originally broad-based methods, into a methodology based upon a strict agenda.

To conclude, I will list two sets of notes which summarise the issues raised in the Working Group. They seek only to summarise some issues concerning control in Calculus students and issues concerning interviewing of this type:

Control-Skill Issues

Skills are context dependent and/or are a function of problem-solving.

Experience effects and shapes our problem-solving. It seemed clear that the student in the transcripts had had a lot of experience with algebraic manipulation (at A-Level) not necessarily Maclaurin Series.

"Think it possible that thou mid'st be mistaken" - a Quaker phrase.
Almost a pre-requisite for metacognitive behaviour. One must eliminate the arrogance which imbues maths and admit that one can make mistakes and know where one usually makes mistakes.

Control could be a function of belief structures which lie outside the context of the question.

Interviewing

Not quite an interview more like real-time mathematics.

Role changes between teacher and researcher have important epistemological and methodological implications - note.

How aware am I when I am interviewing about the types of interventions I make and the points I chose to intervene? How much do they match with my model of control?

I have a metric of how far they are from the end solution. This might also have methodological implications and structure my interventions.

Interventions are not just for qualifying what they're thinking. They also invite students to 'finely focus their microscope on their work' - not necessarily leading.

A final note from Laurinda:

Many thanks to all those who have participated in what has been a useful forum for discussion and exchange about methodological issues related to interviewing techniques. The group will now lie fallow for the time-being. There was a suggestion of interest in interviewing techniques with special needs students if there is someone with some thoughts on this who might like to convene a meeting.