The British Society
For
Research into Learning Mathematics

Proceedings of the Day Conference
at
Brighton Polytechnic
6th May 1989

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Contents

RADICAL CONSTRUCTIVISM, MATHEMATICS AND MATHEMATICS EDUCATION

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Through the two sessions that we have organised, at Nottingham in January and Brighton in May, it has been our intention to open up discussion in the mathematics education world, about the nature of "Radical Constructivism". We chose as our focus the paper delivered by Kilpatrick at PME XII in Montreal 1987, and published in the proceedings. In that paper he criticised strongly and, we felt, unfairly, the idea that radical constructivism had anything to offer either to mathematics or to mathematics education, and even the idea that it made any sense. He based his argument around the claim that the first hypothesis of the radical constructivists was accepted by mathematics educators, and the second was patently absurd, these hypotheses being:

> *(1) Knowledge is actively constructed by the cognizing subject, not passively received from the environment.*
> *(2) Coming to know is an adaptive process that organises one's experiential world; it does not discover an independent, pre-existing world outside the mind of the knower.*

We suggested that Kilpatrick misunderstands, or chooses to misinterpret, the second hypothesis, and claims that it implies the denial of the existence of the real world. It clearly does not do this, and we offered a re-statement of this hypothesis as follows:

> *It seems to me that the radical constructive hypothesis about the way in which we 'come to know' places ontological questions (within the context of human thought) in a similar position to that of undecidable statements (within the context of mathematical logic). I can't prove it though!*
>
> *Scott-Hodgetts 1988*

In the first session, a number of extracts from the writings of Frege, Hilbert and others were presented, in an attempt to demonstrate the power of a philosophical position that has no ontological commitments, for productive and creative mathematical work. A heated and stimulating discussion ensued, aimed at both clarifying the terms used, and at whether the participants agreed with this claim. A particular issue was the nature of mathematical truth, and the role that plays for mathematicians. It appeared to be clear to all that hypothesis (2) cannot be dismissed, and that mathematicians and mathematics educators need to engage with the deep ideas involved, despite their difficulty.

In the second session, we attempted to bring the discussion into the realm of mathematics education. We claimed that, contrary to the picture presented by Kilpatrick, hypothesis (2) does not leave one powerless to engage with the world around, and to form concepts and theories about that world, rather it **empowers**, through an

acceptance that we cannot know whether those theories are ultimate truths, either of realism or idealism, but are social constructions, inter-subjectively negotiated and developed. We proposed the following thought:

> *What would an implication of an alternative paradigm, in this case radical constructivism, look like?*
> *Certainly NOT a Popperian critical experiment, rather a Kuhnian paradigm shift, another pair of glasses viewing the same picture, a subtle change of emphasis.*
> *It's result would not be a conversion through an instant falsification of one view. It would rather be an awareness of a <u>resonance</u> with experience.*
>
> *Lerman 1989*

Inititially, the focus of debate concerned the nature of a world where knowledge is in the social domain, not a domain of timeless and absolute truth. It then moved on to trying to illustrate these alternative 'pairs of glasses' with experiences in teaching mathematics. Many people seemed to feel that the situations described were methods of teaching, that were not the prerogative of one view or another, and they wished to have some clear differences. Others emphasised the subtle nature of the implications of accepting the radical constructivist position, and described the distinction as one of a greater tendency on the part of mathematics educators working within a radical constructivist framework to offer equally valid alternatives in the classroom, or accept as equally valid the offerings of students. This raised the question of the role of 'well-established conjectures', as accepted mathematical skills were termed, in the classroom. It was suggested that without acquisition and 'understanding' of this 'basic knowledge' students are not empowered in society, to criticise given information, particularly statistical justifications for political decisions. This was countered with the suggestion that rather than 'basic knowledge', what students require is best characterised as confidence, being able to guess, check etc., in fact, processes. A short discussion followed on the meaning of 'understanding', and a quotation from Wittgenstein's Philosophical Grammar was shown, to offer his essentially social interpretation of the term.

> *Do I understand the word 'perhaps'? - And how do I judge whether I do? Well, something like this: I know how it's used, I can explain its use to somebody, say by describing it in made-up cases. I can describe the occasions of its use, its position in sentences, the intonation it has in speech. - Of course this only means that 'I understand the word "perhaps"' comes to the same as : 'I know how it is used etc.'; not that I try to call to mind its entire application in order to answer the question whether I understand the word.*
>
> *Wittgenstein 1974*

The session ended with a reaffirmation of the importance of discussions such as these, and an invitation to others to initiate further discussions at future meetings of BSRLM.

Pure and Applied Mathematics
This session followed the discussion "Pure and Applied Mathematics: is there a distinction?" held at Nottingham in November and was stimulated by three letters of response I received from Derek Peasey, Keith Harcourt and Jeff Evans respectively.

Derek Peasey, Beatrice Shire and Ann Dowker joined me and I introduced the session with the report of the previous one and the comment that I still cannot adequately answer an employer who asks why a 16 year old, after 11 years of mathematics education, cannot say how much wood is left from a 5' length after 2'3" has been cut off.

We began by noting that although "relevance" is now taken seriously in maths education, maths education itself is ignored by those who now have power over the content and teaching of school leaver "numeracy" programmes. Their skill-based attitude is filtering down schools through vocational schemes and there is danger that if mathematics education itself does not take responsibility for numeracy as such, they will lose it to people with little knowledge of mathematics education. They could themselves become an elite body catering for "pure" mathematicians only. They could lose the "bottom end" for which they should be responsible.

Points made by Keith Harcourt's letter were read out: mathematicians do not face the facts, their fascination and enjoyment is not shared by most people, workers only use applied maths and attempt to deny that it's maths at all; once, when as a professional designer of 15 years KH was designing a box he was told that he was developing the net of a cuboid. He had never used such language in order to produce a box; children become confident in problem solving through enjoying their learning experience, its mathematical base is unimportant to them whatever we feel to be philosophically desirable. Their "right answer" is more satifying than recognising that they are doing maths.

KH did not discuss difference between practical and mathematical activity: if you want to develop a cuboid you can make a box or a fish tank but just making a box might not be so generalisable to other contexts; maths is context free. But in reality the cuboid as a common feature of a box and a tank might be displaced by a bigger generalisation eg. in terms of jointing down the sides.

DP's points were 1: that a lot of maths happens in other parts of the curriculum. He quoted the work of Betty Mastrantoni who as a HOD worked out her curriculum to complement the work of other departments, not as a skills-based approach because the other departments practiced the skills in their contexts. She thus had more time to spend on developing mathematics and its rationale. The children learned to recognise mathematics in other lessons and outside school and their maths lessons helped them to relate what they saw as well as developing more maths. 2: that a lot of maths occurs across the curriculum in the use of software, for example Jill Black's work on spread sheets in Herts. Users not only saw common maths but also became more efficient in its usage. Mathematics departments should be responsible for the mathematics curriculum rather than for staff and lessons.

But the definers of numeracy don't see this. Their definition of maths as

the four rules, fractions and percentages means they see nothing else. Actual problem solving on the shop floor is not seen because problem solving is a management skill. Numeracy is defined for the purpose of minimal skills of a labour force and industry seems to recognise three quite different kinds of maths, minimal numeracy skills, A level for "bright school leavers who will become M&S Management trainees" and graduate maths for those who will be recruited not so much to use their maths but because they have evidence of having had their brains maximally exercised. This is how they define maths and no other view matters. Most of industry is not interested in our experience, has never heard of the Cockcroft Report and, since our profession is now so thoroughly denigrated will not listen to anything we say. We have a marketing problem. How do you explain mathematical conceptualising to people who see computing solely as keyboard skills?

Do they say everyone is at the basic level to start with? They apparently have no interest in a connection between the three levels. It is irrelevant to recruiting and career. They still want the "bright lad" of the industrial revolution. They want what we know as graduate material operating at basic skills level. Many still use irrelevant entrance tests.

The skills attitude has also affected much research including Cockroft. The work of Lave and Walkerdine is a hopeful escape from this. Of course "Basic Skills" are used but always as only part of a context. One of the original questions of this discussion group was to ask why more people don't teach maths by analytical processes as contrasted with teaching isolated skills and then trying to transplant them.

One problem is that too often people try to generalise the symbols of maths rather than the forms. This is particularly so in the use of statistics packages where people try to follow the rules rather than relating to the problem under study ie: they follow the Basic Skills way. Understanding arises not from the practice of the skill but from the conceptualising that produced the package. Here the traditional view has been transferred to the computer. In fact packages offer opportunity to play around with "dirty data" - they don't force you to get out some figures or a standard deviation.

Are we talking about what Cockcroft calls a feel for number? Industry too wants a "feel for number" but they define it in terms of the application of Basic Skills. Should we learn maths as by-product of activity? Experience eg; of a packaging activity with a whole SMILE conference, has shown that it can be very fruitful. But often if you look at a problem in a theoretical way you can abstract it too early and too simplistically. If you want to do it more thoroughly you should start with the practical problem and treat each bit as another bit of maths. Mechanics is a good introduction to maths because of the constant interplay between physical activity and mathematical abstraction - but it is too often taught as an abstract maths subject.

But "practical" maths can be taught in primary school so that it ends up as neither practical or maths. Concrete materials per se do not necessarily have practical relevance. Multiplication table patterns can come across as lessons in neat colouring in. Inspiring teachers are those who encourage children to choose their own problems.

This has been happening in at least two schools who have used textile activities for teaching maths. Both reported that the children "took over their own learning" and one group spent a weekend on the sort of best-buy activity that is so often so drearily presented in text books.

Some children do have a good feel for abstract number but may not be so good at manipulating figures. They end up "bad at maths". If they were allowed to be conceptual and analytical, then perhaps they could take on employers' tests and challenge their contents.

DP quoted another teacher who taught the whole of maths through problem solving. Two terms off O level she presented the problem of getting the highest possible marks and all her students scored As.

Then followed some more anecdotes all on the theme of the effect of context and the language of that context on mathematical learning. This was an appropriate time to quote from the third response letter, that of Jeff Evans. He made the following points. 1: There are two senses in which "Basic skills" appear in the literature, one that we used in our discussion and "functional maths" as used eg. by Brigid Sewell. 2: The correctness of employers in castigating teachers depends on the correctness of transferability theory. Is lack of transfer really the responsibility of teachers or a convenient means for employers to charge their training to the taxpayer? People are now beginning to question the effectiveness of training based on this model. 3: There is no "pure" maths in the sense of a value free pursuit of uncommitted knowledge. 4: There will always be problems with naming eg. is it maths (Pythagoras) or is it scaffolding (3,4,5)? 5: If it is unreasonable to expect transfer (see Jean Lave) then it cannot be used as a final test of anything (see page 9 of the Nottingham report). This is too pessimistic for a final position but is preferable to a vague assumption of transferability. Walkerdine recommends translating between discourses. Research is going on that bears on these issues (eg. Wolf, Carraher, the Logo Projects, Hughes, Harris) - not many answers yet but Lave and Walkerdine have posed good questions. Own feeling is that "skills" need to be taught much more explicitly in context and for school leavers there is more commonality around contexts related to learners as citizens than learners as workers.

A topic that was briefly discussed at the end was that we do not know enough yet about how different approaches suit different people.

The discussion thus ended where it had begun, within the responsible discourse of mathematics education, that field so totally ignored by the Powers that Be in determining the numeracy education of so many vocational courses. I remain very worried about the situation and will offer another anecdote that I did not quote at our meeting. I once offered an entire committee of "committed-to-education" industrialists a seminar or two on developments in mathematics education. They had not heard of Cockcroft. I did not get one taker. The "interest" is just a little one-sided is it not?

Finally I would like to thank all the people who came to both discussion groups for their observations. If you have any comments on any points that our discussion raised, please write to Mary Harris, ULIE, 28 Woburn Square, London WC1H OAA.

COMPUTATIONAL ESTIMATION BY YOUNG CHILDREN

(Paper presented by Ann Dowker, Department of Experimental Psychology, University of Oxford.

Conference of the British Society for Research into Learning Mathematics, Brighton Polytechnic, Falmer, Brighton, May 6th, 1989.)

This is a report of a study of 5- to 9-year-old children's computational estimation. There are both practical and theoretical reasons why estimation is an interesting topic to study.

(1) It is an important practical skill. The increasing use of calculators makes estimation abilty increasingly important in comparison to mechanical calculation skills.

(2) From a more theoretical point of view, observing people's computational estimation strategies can give us information about their more general understanding of mathematical concepts and relationships.

Over the last two years, I have studied children's estimates for addition, subtraction and multiplication problems, but will here deal just with children's estimates for addition sums.

In all, 86 children from an Oxford primary school took part in this study. Each child was seen individually for all parts of the task (often on several occasions.) I devised a list of addition sums, graduated in difficulty from "4 + 5", "7 + 1", etc. to "235 + 349"; and also four sets of estimation problems, , ranging from a relatively easy set to a relatively difficult set. I first gave children the set of addition sums, and then, on the basis of their performance on these sums, I assigned the children to one of four groups and, according to their group, gave them one of the four sets of estimation problems. Each child was given the set containing sums that were just a little too difficult for them to solve correctly. For example, children who correctly solved such sums as "4 + 5" and "7 + 1", but had difficulty with such sums as "4 + 9" and "8 + 6", were given Level 2, which will be described below, and which includes such sums as "6 + 6", "8 + 9",etc.

Level 1 contained sums to be estimated such as "3 + 6", "4 + 4", etc. Level 2 contained such sums as "8 + 9", "7 + 4", "29 + 3", etc. Level 3 contained such sums as "18 + 59", "71 + 18", etc. and Level 4 contained such sums as "217 + 285", "251 + 53 ", etc.

The sets are here referred to as Level 1, Level 2, and so on. The children assigned to these levels are also, for convenience, referred to as "Level 1 children," "Level 2 children", etc. This refers only to the groups to which they were assigned on the basis of this task and for the purposes of the estimation study, and does not imply any evaluation of their general mathematical ability. Age was not used as a criterion for assigning children to levels; although there was some correlation between level and age, there was a wide age range assigned to each level, and considerable overlap between the age ranges at the different levels.

In order to introduce the task, I first presented the children with a set of estimates supposedly made by imaginary characters ("Tom and Mary") and asked them to evaluate the guesses as "good" or "silly".(The sums involved were similar but not identical to those that the children themselves were subsequently asked to estimate.) This report deals only with the children's own estimates.

32 children were also given levels above those appropriate to their arithmetical level, so that, for example, a Level 2 child might be given Level 2 and also Level 3 or Levels 3 and 4.

The rationale for giving these "difficult" levels was:

(1) I wanted to look at <u>what</u> strategies would be used and indeed <u>whether</u> any strategies would be used, if children had to cope with estimating answers to problems that were very difficult for them.

(2) I wanted to see the extent to which any differences between responses to the different levels were due to differences between the type of sum rather than to differences in how advanced the children were in arithmetic.

On the whole children, as they became more advanced in arithmetic, seemed to use more appropriate strategies and fewer inappropriate ones and to make increasingly more reasonable and less random estimates.

Level 1 children were more likely than were those assigned to higher levels to make grossly unreasonable estimates. More than a quarter of responses to Level 1 sums were less than the larger addend .Estimates less than the larger addend were much less frequent at the more advanced levels. Proportions ranged from 5% of responses to Level 2 to 10% of responses to Level 4. Extremely large estimates _ more than twice the exact answer _ accounted for about a quarter of the responses to Level 1, but only one-tenth of responses to Level 2 and a very small proportion of responses to Levels 3 and 4.

Like the 4- and 5-year-old children studied by Baroody (1985),some of the younger and less advanced children in this study seemed to use simple and rather inappropriate strategies. Strategies observed by Baroody included stating an addend (e.g. 6 + 3 = 6); adding 1 to one of the addends (e.g. 6 + 3 = 7); stating one of the addends as a teen (e.g. 6 + 3 = 16); and response biases (favouring a particular number, such as 7, 8 or 'eleventeen'.) The children in my study almost never used the strategy of responding with an addend-teen, or in general of adding ten to an addend. They did, however, use all the other strategies observed by Baroody.

13% of responses to Level 1 involved stating an addend (as in Baroody's study, usually the larger addend). Hardly any of the responses to the more advanced sets involved stating an addend.

If we look only at those items for which it could not yield the correct answer, about one-sixth of responses to Level 1 involved adding 1 to an addend: nearly always the larger addend. Just under a tenth of responses to Level 2 came into this category, but nearly half of these came from one child who used the strategy consistently. Almost none of the responses to Levels 3 and 4 came into this category.

As for response biases, only two children responded with the same number to all items. Seven of the 20 children given Level 1,four of the 23 children given Level 2 and four of the 30 children given Level 3 showed <u>some</u> response biases (responding with the same number to at least three items. None of the children given Level 4 showed such response biases.

I also observed a few other inappropriate strategies ; for example, a few responses to Level 1 and Level 2 were of the form "3 + 6 is 36"; "8 + 9 is 89".

We turn now to more appropriate strategies. In Levels 3 and 4, such strategies would include what Reys called the "front-end strategy" of adding the left-hand digits (for example, with 18 + 59, ignoring the units, adding the 10 and the 50, and getting around 60); and the still more accurate strategy of rounding the addends (for example, 18 and 59 might become "20 and 60 which is about 80.") Neither of these strategies is really relevant to questions involving the addition of single-digit numbers, so they are here discussed only in relation to Levels 3 and 4.

In Level 3 a front-end strategy appeared to be used for about one-third of responses to the items where it could have been used. Looking at those items in Level 3, for which one could distinguish a rounding strategy from a front-end strategy, 27% of responses involved the use of a front-end strategy and 15% involved the use of a rounding strategy.

In Level 4, a front-end strategy was used in about half of the responses to items for which it was a possible strategy, including those for which rounding was more appropriate. Rounding was used in about a quarter of responses to items for which it was an appropriate strategy and could be distinguished from other strategies. Thus, both a rounding and a front-end strategy seemed to be used more frequently by children given Level 4 than by those given Level 3.

The fact that at least some of the Level 1 children did use obvious, though relatively inappropriate, strategies lends support to Baroody's (1985) view that "the development of mental arithmetic ... cannot be understood entirely in terms of forming specific numerical associations" but that children "seem to manufacture estimates by employing strategies that are based on their knowledge of arithmetic."

When children were presented with levels that were much higher than their appropriate level, the result was generally a considerable reduction in accuracy and an increase in the number of unreasonable and apparently random estimates. One interesting finding is that children did show some signs of using strategies when dealing with estimation problems one level above their appropriate level. For example, Level 1 children used the same somewhat inappropriate strategies in dealing with Level 2 as they did in dealing with Level 1. Level 3 children used some of the same strategies (of varying degrees of appropriateness) in dealing with Level 4 problems as they did in dealing with Level 3 problems. But when children had to deal with problems that were very much too difficult for them (2 or more levels above their appropriate level), they began to respond wildly. An extreme example is the Level 1 child who responded to the Level 4 problem, "35 + 96?", with the guess, "14 and 200 pounds and 16."

Not surprisingly, unreasonable estimates, especially estimates lower than the larger addend, were much commoner when children were given levels above their appropriate level. 56% of responses by children from lower levels responding to Level 4 questions were less than the larger addend. Level 1 children were more likely than Level 2 and Level 3 children to make such responses to higher-level items. But even some Level 3 children made a number of such responses to Level 4.

As regards the use of strategies, Level 1 children used the strategies of stating an addend and of adding one to an addend (usually the larger addend) about as often when dealing with Level 2 as when dealing with Level 1. However, they did not seem to use these strategies _ or, as far as one could gather, any strategies _ when dealing with Levels 3 and 4.

Though the number of children involved is relatively small, Level 3 children seemed to use the front-end strategy and the rounding strategy almost as frequently when dealing with Level 4 as when dealing with Level 3. On the other hand, Level 2 children very rarely used these strategies when dealing with higher levels, and Level 1 children never did so.

The most striking finding of this part of the study is that children did often show signs of using strategies, of varying degrees of appropriateness, when presented with estimation problems not too much above their appropriate level. Once, however, they were presented with problems that were very much too difficult for them _ especially when Level 1 children were presented with Levels 3 and 4 _ they seemed not to use these strategies. In reponse to the questions posed above, it appears that differences in the actual sums (e.g. the size of the numbers involved) do affect the children's responses; but that there is also a more general cognitive effect: children are ready and willing to use strategies in dealing with problems that are not greatly above their appropriate level; but that, when faced with very difficult problems, they become disoriented and/or simply regard all estimates as equally likely and just guess wildly.

REFERENCES:

Arthur Baroody: Early addition estimates: retrieval or problem solving?; Paper presented at the biennial meeting of the Society for Research in Child Development, Toronto, April 1985.

REPORT ON THE DIAGNOSTIC ASSESSMENT IN MATHEMATICS (DAIM) PROJECT PRESENTATION AT THE BRITISH SOCIETY FOR RESEARCH INTO LEARNING MATHEMATICS; MAY 6th 1989

The first purpose of diagnostic assessment is to determine the nature and extent of pupils' misconceptions in the subject. The second purpose is to respond to those misconceptions in effective ways. It is with these purposes in mind that the DAIM project is working with around 20 teachers in six LEAs to assist in the development of their diagnostic skills.

Four main qualities that the project has identified as being crucial for teachers to work more diagnostically in their classrooms were discussed in the session. These qualities are that the teacher:

1. is willing to explore ideas about what factors assist or inhibit children's learning;

2. responds to pupils' misconceptions and errors in constructive ways, and enables pupils to do likewise;

3. uses an appropriate range of teaching-learning techniques in the classroom;

4. evaluates and develops the curriculum with regard to information acquired on pupils' behaviour in both the cognitive and affective domains.

Each of these qualities was highlighted with examples taken both from the project work itself and sources, such as the APU and CSMS, which the project has used to provide starting points for teachers in the process of diagnostic teaching.

The idea of 'misconception', as mentioned in quality 2, produced some lively discussion. Some participants taking a radical constructivist view felt that it was not a case of 'misconception' but one of 'partial conception'. Other participants disagreed, regarding the use of 'misconception' as appropriate. Some pupils' work on decimals and angles was then featured. In particular the fact that a large number of pupils treat decimals as though they were whole numbers or regard the largest number after the decimal point as having the smallest value. For the concept of angle, many pupils view this as length or area, rather than as amount of turn. Are these 'misconceptions' or 'partial conceptions'? Are some conceptions closer to the 'acceptable' conception than others? What constitutes making a mistake?

The session went on to give a flavour of some of the project work that has gone on since September 1988. One aspect the project is focussing on is the teaching technique of 'conflict-discussion' in which 'pupils are exposed to the consequences of their misconceptions'. Pupils are given critical questions which are known to result in a variety of answers being given. These answers are then examined by way of discussion between pupils and teacher and between pupils themselves.

Conflict about the mathematics arises within the class and this conflict may, or may not, be resolved by the discussion that ensues. Essentially, there are two strands to the 'conflict-discussion' technique. One looks at different answers to the same question, the other looks at the same answers to different questions.

Variations on this technique were shown including the idea of pupils being presented with common 'misconceptions' and having to use their reasoning skills to deal with them, and the idea of pupils marking the work of another pupil in a constructive manner.

One of the more important messages that has emerged from the DAIM project is that some teachers are inclined to set up situations in which pupils avoid misconceptions rather than are confronted by them. The teachers working on the project are aware of this and it is hoped that this message will be more widely received when the in-service materials are produced towards the end of the project in 1990.

Counting and Creativity

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The focus of discussion within this group was our understanding of creativity and the way in which it may be linked to counting methods used by below average children, between the age of 8 and 12 to solve, basic numerical problems based on the number of facts to 20.

The creative act was determined as that which is unique and flexible, consequently highlighting originality in the combination and relationship between mathematical ideas, techniques and approaches.

There is a considerable respectability attached to what may be determined as counting and creativity but any analysis of creativity within this framework is based on the underlying process and not on the product.

In the majority of cases identified processes were deemed to be the result of pedagogic intervention and so are hardly creative. However, many children spurned the use of pedagogic aids to help their counting process but used invented alternatives . It was the identification of these alternatives, coupled with counting approaches which might be identified as creative.

The use of any materials provided the children with a crutch which they could touch, see, draw or imagine; physical or quasi-physical, and their use was identified in those approaches used by the children

i) Sustained approach

The initial choice and use of physical or quasi-physical aids was identified as creative. However the implication seems to be that the chosen material, together with the counting approach always brings success. It is apparent that the approach is then used even if it is inappropriate and eventually leads to fixation - the anti-thesis of creativity.

ii) Fragmentary approach

The child faces each new problems afresh, has a short period of gestation and establishes an anchor which enables him/her to carry out the counting process. The approach is identified as fragmentary because a new anchor may be established for each successive calculation.

(iii Adaptive approach

sustained approach is used in a minimalist way and does not become fixed to the extent that it may be inappropriate. As the range of calculations implied the need for knowledge of the structure of the number system the creative approach was adapted.

The essence of the approaches of all of the below average children is that they failed to indicate understanding of the relationship between the numbers. This approach was in contrast to that used by brighter children and had consequences for dealing with calculations that involved exchange and/or decomposition.

OBSTACLES RELATED TO THE CONCEPT OF FUNCTION

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SUMMARY

Different factors affect the acquisition of the concept of function. Assimilation is complicated by the manner in which the concept is introduced, both within pre-university curricula and in the first year of university. Preliminary studies involving questionnaires on the concept of function which were tested with mathematics teachers at upper-middle and uppergrades in Mexico, showed that different stages of learning could be identified. On the basis of these findings, a further questionnaire was designed to discover the manipulation used by teachers at different cognitive levels. Another aim was to obtain more information about the obstacles found by mathematics teachers in relation to function. Finally, it was hoped to analyse the skills needed to prove mathematical statements; to this end, the questionnaire included questions designed to detect any difficulties experienced by teachers in carrying out these proofs.

INTRODUCTION.

With the development of the mathematics, the definition of the function has been changing, the concept has been going through a succession of stages. For exemple, in the eighteenth century, the function was considered as a change, or a variable depending on other variable, Leonar Euler in [1755] (Dhombres [1976]) writes: '*...Those quantities which depend on others such that, if the others change, they change too. They are usually called functions...*'

Later with the influence of set theory, a new definition of function appeared

-A function F is a set of ordered pairs of numbers
$F = \{ (x_1;y_1), (x_2;y_2), \ldots \}$ with the property that,
if $(x_k;y_k) \in F$, if $(x_p;y_p) \in F$, and if $x_k = x_p$, then $y_k = y_p$.

The acquisition of the concept of function was complicated with this approach, and new proposals came to be defined in textbooks.

Then, basically, we have several kinds of definitions of function

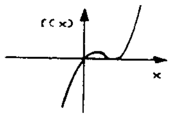
-Function in terms of variable
-Function in terms of set
-Function in terms of a rule of correspondance
-Function in terms of INPUT-OUTPUT

EULER AND THE CONCEPT OF FUNCTION

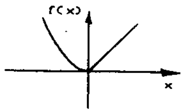
A major advance in the concept was made by Leonard Euler, who used a number of definitions in the course of his work. In one of his works, Euler writes: *'...the function is the analytical expression of any variable quantities and numbers or constant quantities...'*. That is, the function of x was considered by Euler as a single expression or formula containing x as a variable. Using this definition, there was no possibility of 'sticking two formulas together' to represent a function. Elsewhere, Euler used the definition of function as *'... the relation between y and x expressed on a plane by a curve drawn freehand...'*. In one of his writings (1734) he used brackets to denote a function evaluated at a single point: *'... Si f(x/a + c) denotent functionem quamcunque ipsius x/a + c ...'*. This work of Euler's was to be the first in which f was used to denote a function (Cajori [1929]).

It will be seen that with Euler's first definition (an analytical expression representing a relation between two variables) would produce functions like:

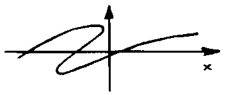
$f(x) = x^3 - 2x^2 + x$



but not: $f(x)=\begin{cases} x^2 & \text{if } x \le 0 \\ x & \text{if } x > 0 \end{cases}$



And with the second one, the *'free movement of the hand'* one would have something like:



DESIGN OF EXPERIMENT AND QUESTIONNAIRE (REAL-VALUED FUNCTION)

It was thought necessary to carry out exploratory studies to find out how mathematics teachers handle these concepts. The first pre-questionnaire was with 72 teachers, the second with 16 teachers and the definitive with 29 teachers.The definitive questionnaire consist of 25 questions, both open-ended (OE) and multiple choice (MC). In this paper we can not to mention all interesting points that have been discovered in the answers. We shall mention only the questions : 4, 9, 10, 13, 18, 23, 24 and 25 (At the end of the paper you can find them).

CURVES, FUNCTION GRAPHS AND THE DEFINITION OF A FUNCTION

Questions 9, 10 & 18: Graphs and the definition of a function.

Questions 9 and 10 were about representing curves in a plane, some of them being function graphs. Question 18 was about the definition of a real variable function.

Question	No answer	Incorrect answers	Correct answers
9	0	1	28
10	0	7	22
18	0	4	25

An analysis of the performance of the 29 teachers on these three questions shows us that they had most difficulty with question 10, but in general they were able to identify function graphs and correctly identify the definition of a real variable function. Their performance overall, however, shows that there was a gap between this knowledge and the applications of the principle that the teacher is able to make. This view is supported by the analysis of the remaining questions.

CONSTRUCTION OF FUNCTIONS

Question 4: Construction of functions.

As will be seen, of the 29 teachers thirteen gave correct answers to question 4a and only five to question 4b. The high level of errors may be due to the fact that 10 of the 13 who got the right answer constructed continuous functions (7 parabolas and 3 compositions of 2 straight lines). From this it may be inferred that, **all but three of them reached the limit of their ability to construct continuous functions.**

Question	No answer	Incorrect answer	Correct continuous function	Correct discontinuous function
4a	11	8	10	3
4b	14	9	3	2

Whereas for question 4a teachers were required to construct a 2nd grade polynomial with certain properties, a greater knowledge of algebra was required for question 4b. In this question attempts were, in fact, made to construct a 3rd grade polynomial with these properties. One teacher constructed a 4th grade polynomial and arrived at the correct answer.

The foregoing shows that not withstanding the 25 correct answers to the question on the definition of a function, when teachers came to construct functions they showed a marked tendency to think in terms of continuous functions and very little skill in actually constructing them.

Question 13: Using a graph to make a function explicit.

Question 13 is included in this analysis to provide a frame of reference in regard to the difficulty that was found in using correct terminology and quantifiers when solving problems.
The results are summarized as follows:

Question	No answer	Correct definition function in [0,4)	Incorrect attempted to define the function periodically	Correct definition of the function in $\mathbb{R}$.
13	15	2	11	1

Only one teacher was able to define the function correctly. The 11 who gave a wrong answer would have avoided this if they had defined the function in [0,4) and said that it was periodic about $\mathbb{R}$, the period being equal to 4; however, in 8 cases the answer was something along the lines of:

$$f(x)=\begin{cases} x \text{ if } x\in[i,\ i+2],\ i\in\mathbb{R} \\ 2 \text{ if } x\in[2(i+1),\ 4(i+1)] \end{cases}$$

All these answers show an intention to extend the function defined in [0,4) to $\mathbb{R}$, but were not able to do so.There was a notable absence of skill in handling the symbols required by the question.

Questions 23,24&25:Constructing functions with special properties.
The answers to these three problems show a marked tendency to go wrong after the first item. Of all the constructions made in answer to the three questions, only two corresponded to discontinuous functions; one of these was an incorrect answer to the second part of question 25 and the other was the correct answer to the same part of the same question.

Question		No answer	Incorrect answer to 1st item	Correct answer to 1st item	Incorrect answer to 2nd item	Correct answer to 2nd item
23	Explicit function	4	4	16	7	2
	Graph	4	6	17	2	2
24	Explicit function	14	1	9	3	4
	Graph	14	3	11	3	6
25	Explicit function	15	1	9	1	6
	Graph	15	1	8	2	4

The results show that there is a need for more emphasis on the construction of functions in pre-university curricula and in textbooks. Another finding of this part of our study is that function is closely linked to continuity. The linking of function-continuity would not appear to be dangerous at the pre-university level (to the pupil, that is, rather than the teacher). There does, however, seem to be scope for approaching the concept of function from the historical perspective, for teaching purposes, to isolate the concept and make clear the relationship between it and the continuity of functions. Fischbein [1982] states in his article 'Intuition and proof' that it is necessary to build up in pupils 'an intuitive acceptance' or, as Fischbein says in referring to Dr. Bell in the same context, it is necessary to build up 'a new basis of belief'. It seems to me that we need this 'new basis of belief' for our mathematics teachers.

CONCLUSIONS

The concept of function is associated with other mathematical concepts. This study shows that it is necessary to discover whether any of the associated concepts assists the assimilation of the concept of function. It can also be seen in the study that there are different levels of abstraction associated with the concept. The answers given to question 13 show that teachers have great difficulty with the use of the terminology of set theory. Consequently, their intuitive ideas were not able to find expression.

It should be noted that the definition of function used in the questionnaire is far removed from intuition, despite which the informants had no difficulty in identifying it. In fact, teachers showed they were able to find the image of a given pre-image and to identify the definition of a function. Teachers have a strong tendency to think in terms of continuous functions and, in general, lack the ability to construct them. This leads us to believe that there is a need analyse and suggest different approaches to the acquisition of the concept of function, in addition to those commonly presented in textbooks and in the classroom. The teacher engaged in teaching mathematics at the pre-calculus level certainly needs an operating definition of a function, but not the familiar one immersed in set theory. At this level, numerical functions are also necessary and the set theory-inspired definition could be reserved for pupils studying advanced topics in mathematics.

The problem that some authors see in using the definition of function in terms of variable 'A function is a variable so related to another variable that to each value of the latter there corresponds uniquely a value of the former', is that a definition of 'variable' is needed before the definition of real-valued function. In any case, if a teacher is forced to teach one or another definition of function because the textbook he uses, it would be better to make a reference to the development of the concept before the definition.

An hypothesis that arises of the study is that with the use of the microcomputers, it might be more difficult to isolate the concept of real-valued function from the intuitive idea of 'function-continuity'. That is because the graphics packages designed up to date use the concept of function in terms of a relation between two variables which is expressed by a formula.

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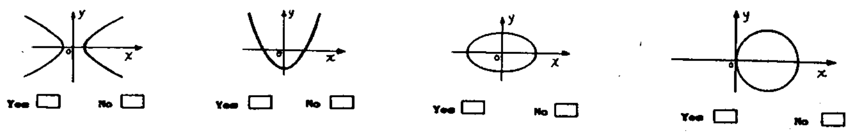
ANNEX

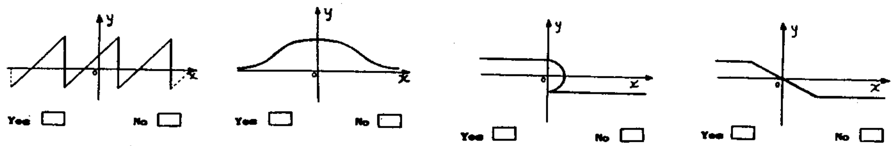
4.Construct two functions f_1 and f_2 with the domain ℝ and rank ℝ also, such that

$f(-5) = 2$; $f(0) = 1$; $f(5) = 6$

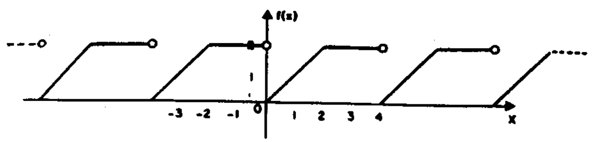
9.Examine the following graphs, and say whether or not each is a graph of a function, by placing a tick in the appropriate box.



10.Examine the following graphs, and say whether or not each is a graph of a function, by placing a tick in the appropriate box.



13.Given the following graph



make the function f explicit. Assume that the function is prolonged indefinitely

$f(x) =$

18.In each part of this question, indicate, by ticking the appropriate box, whether or not the definition of function given is correct.

a) A function is a set of real numbers, such that any two numbers in the set are different.

Yes ☐ No ☐

b) A function is a set of ordered pairs of numbers (x,y), such that no two different ordered pairs have the same second coordinate.

Yes ☐ No ☐

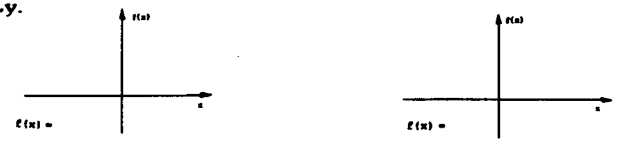
c) A function is a set of ordered pairs of numbers (x,y), such that no two different ordered pairs have the same first coordinate.

Yes ☐ No ☐

d) A function is a set of ordered pairs of numbers (x,y), such that there are always two different ordered pairs with the same second coordinate.

Yes ☐ No ☐

23.Given the property $(f \circ f)(x) = f(f(x)) = 1$ for any $x \in ℝ$
Construct two different examples, either by means of a graph or by making the function explicit, which have this property.



24.Given the property $f(x + 1) = f(-x+1)$ for any $x \in ℝ$
Construct two different examples, either by means of a graph or by making the function explicit, which have this property.

25.Given the property $f(x + 1) = f(x-1)$ for any $x \in ℝ$
Construct two different examples, either by means of a graph or by making the function explicit, which have this property.

DIFFICULTIES IN ANALYSIS

Accomodation or Assimilation

Bob Burn, School of Education, University of Exeter

In the Student's Preface to *Foundations of Analysis* (1929), Edmund Landau wrote "Please forget everything you have learned in school, for you haven't learnt it. Please keep in mind at all times the corresponding portions of your school curriculum, for you haven't actually forgotten them."

The almost universal discomfort and distress which is encountered by undergraduates as they begin their studies of analysis at unversity has stimulated a considerable and growing literature. A sample is listed. Generally, an author identifes one particular difficulty and either attempts an analysis of that one difficulty or attempts to construct a remedy for it. Exceptionally, authors may face more than one difficulty, as the cited literature reveals.
Publications which seek to analyse a difficulty are labelled below with a **D** (for Diagnosis). Publications which seek to alleviate a difficulty are labelled below with an **R** (for Remedy).

infinity

D Fischbein, E., Tirosh, D., Hess, P., 1979, The Intuition of Infinity, *Educ. Studies in Math.*, vol. 10, pp 3 - 40
R Hemmings, R. and Tahta, D., 1984, *Images of Infinity*, Leapfrogs - Tarquin
R Lieber, L. P., 1953, *Infinity*, Holt, Rinehart and Winston.
D Monaghan, J., 1988, Real Mathematics, *Math. Gaz.* vol. 72, pp 276 - 281
D Reynolds, J., 1967?, *Undergraduate Notions of Infinity*, Ph. D. thesis, Nottingham
D/R Tall, D. and Schwarzenberger, R., 1978, Conflicts in the learning of real numbers and limits, *Math. Teaching* vol. 82

limits

D/r Cornu, B., Ph. D. thesis, Grenoble 1983
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D Monaghan, 1988
R Swan, H. and Johnson, J., 1975, *Prof. E. McSquared's Original, Fantastic and Highly Edifying Calculus Primer*, Wm Kaufmann Inc.
D/R Tall and Schwarzenberger, 1978

numbers

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D Robinet, J., 1986, Les Réels: Quelles Modèles en ont Des Étudiants?, *Educ. Studies in Math.*, vol. 17, pp 359 - 386
R Wheeler, D., 1974, *R is for Real*, Open University

motivation

D Bibby, N., 1985, *Curricular Discontinuity*, M. A. thesis, University of Sussex Education Area Occasional Paper No. 12
R Edwards, C. H., 1979, *The Historical Development of the Calculus*, Springer
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geometry

R Gardiner, 1982, Ch. III
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functions

R Gardiner, 1982, Ch. IV
D Malik, M. A., 1980, Historical and pedagogical aspects of the definition of function, *Int. J. of Math. Ed. in Sci. and Tech.*, Vol. 11, No. 4, pp 489 - 492.

logic

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To understand the ε - δ definition of a limit,
to use the notion of infinity in a mathematically precise way,
to take on board irrationalty and uncountability,
to know when to appeal and when not to appeal to geometry and
to use the abstract definition of a function confidently;
each require the surrendering of part of the schema which a student has developed in doing sixth form mathematics.

If we ask whether *history* has any lessons to teach us about the difficulties of today's students, I suggest initially just three.

1. The century preceding the publication of Cauchy's *Cours d'Analyse* (1821) (the first textbook to contain a substantial number of the theorems now found in a modern analysis course) saw much work on bounds to approximations calculated with inequalities. The use of inequalities in the construction of proofs in analysis is inescapable but students may well have never come across a proof depending on inequalities before they come to university. The novelty of style in such a proof is rarely acknowledged by a lecturer. Moreover, a student's fluency with the relevant techniques takes time to develop. The modulus sign for absolute value was not used by Cauchy or his predecessors, and the compression of meaning which it achieves may create a further barrier.

2. The historical development of the concept of function is well documented and pupils at school may have been offered a modern abstract definition. However at school the virtue of the abstract definition may not have been recognised by the pupils, and many sixth formers still only know functions as formulae. This means that pupils going to university may have to take the step from Euler to Weierstrass, and a single formal definition does nothing to acknowledge the size of the step which took the mathematical community a hundred years.

3. Eighty-five years passed between the publication of Bolzano's fallacious proof that Cauchy sequences converge (1817) and the publication of a book in English describing the completeness of the real numbers with modern conventions of rigour (Whittaker and Watson's *Modern Analysis*). Those eighty-five years saw the most distinguished mathematicians failing to recognise the unproved assumptions they were making about the real numbers. The reorganisation of ideas concerning the number line which was brought about by the work of Dedekind, Cantor, Liouville and Weierstrass, even when achieved, proved difficult to assimilate. Completeness is the most modern of the recognised high hurdles in analysis. In the earlier years of this century, universities faced this by offering a formal development of Dedekind cuts; for many students this treatment failed to shed light where it was needed, and the existence of dense but incomplete subsets of the real line remained unimaginable.

Exploring young children's understanding of estimation

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University of Kent

This exploratory study set out to identify the processes involved when young children are engaged in carrying out estimation tasks. In order to separate out various aspects of the 'estimating context' a framework was outlined distinguishing between factors brought to bear and prior to the act of estimating (background skills; specific problem-solving skills and language comprehension abilities) and those processes and procedures specific to estimating actions (strategies; judgements skills; approximation abilities, etc.).

The three questions we sought to address were

(A) What skills do children need in order to estimate?
(B) How do childrens' estimation abilities develop?
(C) In what way might language understanding and other non-mathematical contextual issues influence estimating activities?

Two experiments were carried out; the first involved an exploratory study of estimation skills in young children (employing a range of tasks), the second an experimental study of the influence of task language and stimulus value on children's abilities in an estimating task.

Experiment 1

In order to obtain an indication of the degree to which task context might influence children's estimation skills, a range of tasks were constructed involving both area/volume and distance estimation. For the area/volume tasks one involved estimating 3 dimensionally (how many cubes would fit into this box) and two involved 2 dimensional estimations (how many rectangles/triangles would fit on this paper?) The children were always provided with the 'measuring unit' (different sizes of each of the objects) and asked to estimate how many would fit in/on different sizes of the box/paper). The children were not actually permitted to 'fit them on', only asked to estimate. The three area/volume tasks then were (a) cubes (b) rectangles and (c) triangles, and within each task the children were asked to estimate three different quantities (small, medium and large).

For the distance estimation tasks the children were asked to estimate how many (for each measure) it would be to one of three distances - small, medium or large - from where they were standing. Thus a child might be asked "how many steps from here to the other side of the class?" The three distance tasks were (a) steps (the child's shoe) (b) jumps (the child was asked to jump, this was measured and then the child would be asked how many jumps, etc.) and (c) lying down (the child would lie down, be measured and then asked "how many of you?" to whatever distance they were being asked to estimate. Steps then were defined by the size of the child's shoe; jumps by how far they could jump; and lying down by the lenght (height) of the child lying on the floor. Great care was taken to ensure that the measurements used in the subsequent analysis were calculated appropriately (i.e. working out of the correct estimates for each child, etc.).

90 children over three age groups took part in the study - the youngest aged between 5.7-6.4; a middle group aged 6.9-7.4 and the oldest children 7.5-8.4 years. In addition to carrying out the tasks, each child was asked how they worked out their answers and asked what they would do if the size (of the measuring unit) was changed (What would happen if your feet were smaller? Would you need fewer or more steps?).

Results (Exp.1)

To compare estimating performance across tasks, each response was scored as a percentage of the correct estimate, indicating the degree of over or underestimation for each question. Appropriate ANOVAs were performed on the resultant data with respect to the Age, Task and Task-size variables. The children's verbal responses were coded using a content analysis category system and analysed with reference to age and task.

The most interesting observation was the fact that, while there was no overall age effect, each estimation task revealed very distinct response profiles, either with respect to the task-size interactions or task-age effects. These can be summarised as:

(1) Excluding the triangle and rectangle estimation tasks, the longer any child was at school, the better they were at estimating (as measured by 'correctness').
(2) The 'cubes' task always led to underestimates (not surprisingly given the complexities involved in calculating on three dimensions), with evidence of an age effect (the older children were slightly better).
(3) Estimations involving actions (lying down, jumping, etc.) improved at an earlier age (than other tasks) for the children.
(4) The oldest age group (aged 8) performed signficantly worse than the younger two groups on the rectangle and triangle tasks (they always overestimated).

Furthermore, when asked how they went about carrying out the task, the children's replies fell into four main categories (don't know; guessed; thinking and imagined),
e.g. don't know: *Really don't know*, guessed: *Guessed how many*
thinking: *Thought of numbers; think numbers*
imagined: *measured it with my eyes, imagined the space and counted them*

The verbal responses highlighted that:

(5) Over 55% of the replies involved answers based on imagery, and this was particularly prominent with the 8 year olds (78% of the responses).
(6) The "just guessed" answers came most prominently from the 6-7 year olds.
(7) A significant number (43%) of the youngest children said they didn't know how they did it (note there was no overall age effect in terms of performance).
(9) While there were no startling differences across the area/volume and distance answers (the same propensity to imagine, etc.) within the imaging category, there were clear differences (e.g. no repeated addition with distance estimation, but a lot with area/volume).

Experiment 2

The second study was aimed at teasing apart the influence language comprehension and/or stimulus value might have on children's responses in a Piagetian conservation task. Young primary school aged children (aged 4, 6 and 8) took part in an estimation experiment which involved comparing and estimating whether there were more apples in one or the other of two boxes.

Comprehension of "*more*" in such a context might be adversely affected by the ambiguity of the term (note "*more*" can be used sequentially (that is more than before, adding another object to a collection) or simultaneously (comparing two collections)).

The estimating game involved the children preparing boxes of fruit for old people in the community and formed part of an ongoing school project. Children in 3 age groups (10 in each group) were all taught a novel word *Piu* to convey the sequential meaning of 'more'. This word was used to describe the action of adding another item to a collection.

All the children then prepared boxes of fruit in 3 conditions:

(1) in which they were presented with 2 boxes of apples (6 in each, with red apples in one box and similar sized green apples in the other) [the standard condition],
(2) in which they "counted" the apples into the boxes saying *Piu* as each apple was added [the *Piu* condition] and
(3) as in (1) with the red and green apples replaced by similar sized potatoes and chocolate oranges [the modified condition].

After the child had agreed that both boxes contained the same amount of fruit, the experimenter suddenly 'discovered' that one of the boxes was faulty and they would have to transfer the fruit to an already constructed and somewhat altered alternative box (it had a raised bottom). The standard conservation question then followed: Now, are there more red or green apples (or more potatoes or chocolate oranges) or the same of each?

Each child completed 6 trials in each condition and the order of presentation of

conditions was counterbalanced. Further, in order to examine possible effects of rehearsal (as opposed to use) of the word *Piu*, an additional 10 children of each age from the same school were asked to carry out the task in the standard condition (i.e. not taught the new word). The number of correct comparisons (out of 6) for each child in each condition were used as the response scores.

Results

An age (3) * condition (3) Anova was performed on the responses for the first group of children and the results showed:

(1) There was no main effect for age (children at each age level were equally good at the task).
(2) Children in the 'Piu' condition were significantly better in their conservation judgements than in the standard condition.
(3) In the modified condition (and note great care was taken to ensure that the potatoes and chocolate oranges were the same size), the children's performance was significantly impaired.

Responses in the standard condition were compared between children who had or had not been taught *Piu* and:

(4) rehearsal of the word 'Piu' considerably enhanced performance (that is being introduced to distinct and alternative meanings of 'more' facilitated performance in the conservation judgement task),
(5) being given a separate label for one of the meanings of 'more' was particularly beneficial for the youngest children (age 4-5).

Discussion

With respect to our original questions, we can suggest that (A) children appear to employ imagery as a primary component of estimating (we are currently analysing in more detail the kinds of imagery they used) and (C) language understanding and non-mathematical contextual issues do bear upon children's abilities to estimate correctly. In particular, the effect for stimulus material was very striking and is of obvious importance with regard to testing.

As for (B), we did not isolate any clear developmental picture. Estimating performance appeared to relate to the kinds of tasks being carried out and, while there are indications of age trends, it seems more likely that this reflects exposure to number (it has been suggested to us that one reason for the poor performance of older children in the rectangle and triangle tasks (in experiment 1) arises from the fact that children appear to overestimate in the range of numbers they are beginning to feel at ease with).

The results reported here suggest to us that we need to separate out certain critical aspects of the estimating context if we are to build an adequate model of children's estimating skills. In particular, we wish to investigate further what is involved in the child's recognition of what is relevant for the act of estimating. Previous research has tended to concentrate on the strategies employed in estimating (approximation, etc.) and it can be argued that, without an adequate model of what is involved in the first place (perception before action), we are 'jumping the gun'.

A NATIONAL CURRICULUM

The Education Reform Act for England and Wales was passed in the summer of 1988, and Statutory Orders for Science and Mathematics in the National Curriculum were laid before parliament early in 1989 ready for implementation in September 1989. Thus May 1989 is a suitable time for taking stock of the implications of a National Curriculum, in particular a national curriculum in mathematics. I attempt to focus this stock-taking by considering four questions.

1. What is a curriculum?

Perhaps the most well-known early curriculum was that defined by Plato to enable the identification and further education of the philosopher rulers for a city state. It is of interest to note that in this curriculum students passed through the filter of a general curriculum to reach mathematics. Now mathematics is often itself a filter to reach further education and training.

Traditional curricula grew according to local conditions. Closed communities trained their young in traditional skills, while more abstract education provided social mobility for some, through church or state. Rarely was a need felt to document the offered curriculum. Only when we reached the complexity of 20th century life and the attendant explosion in communication that curriculum design became the subject of wide analysis and questioning.

The writings of John Dewey drew attention to the notion of curriculum development as a continuous process:

> Those responsible for planning and teaching the course of study should have grounds for thinking that the studies and topics included furnish both direct increments to the enriching of lives of pupils and also materials which they can put to use in other concerns of direct interest. Since the curriculum is always getting loaded down with purely inherited traditional matter and with subjects that represent mainly the energy of some influential person or group of persons on behalf of something dear to them, it requires constant inspection, criticism, and revision to make sure it is accomplishing its purpose. (Democracy and Education)

He also introduced the notion of collateral learning, or what has now come to be known as the **'hidden curriculum'**.

> Perhaps the greatest of all pedagogical fallacies is the notion that a person learns only the particular thing he is studying at the time. Collateral learning in the way of formation of enduring attitudes, of likes and dislikes, may be and often is much more important than the spelling lesson or lesson in geography or history that is learned. For these attitudes are fudamentally what count in the future. The most important attitude that can be formed is that of desire to go on learning. (Experience and Education)

Indeed not only is a pupil inevitably involved in collateral learning, but that which he learns from the lesson itself will rarely if ever be precisely what the curriculum designer intended. Recent writers in

Mathematics Education have attempted to analyse this mismatch, for example

> First it is essential to draw attention to the three levels on which the content of the school mathematics curriculum can be viewed:
> (a) the **intended curriculum:** what is prescribed in national and examination syllabuses;
> (b) the **implemented curriculum:** what teachers teach;
> (c) the **attained curriculum:** what students learn.
> (Howson and Wilson, 1986)

I would like to use the notions of an intended curriculum and an implemented curriculum and make a further refineent of the attained curriculum. Here I distinguish between **the received curriculum** - the pupils interpretation of the implemented curriculum through the perspectives of their previous experience, and **curriculum outcomes** - those changes in mathematical behaviour that can be assessed and attributed to the implemented curriculum.

2. In what senses can a curriculum be national?

A national curriculum is often something that is described, or perhaps prescribed, in a document issued by some central authority and which applies to all schools in a nation.The extent of the prescription varies but in mathematics can include:

1. A listing of topics to be taught.
2. An indication of how they are to be taught - this can include direction to use approved materials especially textbooks.
3. When a topic is to be taught and how much time is to be devoted to it.

The first confines itself to prescribing an intended curriculum, but the second two are overt attempts to control the implemented curriculum. However this control can never be complete, for the implemented curriculum necessarily depends on teachers' perceptions of mathematics and mathematics learning. It is still less possible to control the received curriculum, for pupils' perceptions and attitudes to mathematics are affected by outside influences such as parental opinion and peer pressure as well as prescribed educational experiences.

If popular perceptions of mathematics differ widely, one must question whether a national curriculum can exist. Conversely, if a strong common image of mathematics and the means of learning it does exist, then it seems likely that a national curriculum will evolve, even if it remains undeclared. Whilst acknowledging that societies and cultures are likely to appear more uniform from the outside than the inside, I suggest that cultural and societal differences can help us explain how apparently similar intended curricula can become very different received curricula in different communities. Furthermore a curriculum is likely to become more genuinely national through shared exploration of the image of mathematics than through legislation.

3. (a) Is the National Curriculum a curriculum?

We now have a variety of documents defining and describing the National Curriculum. One useful summary is to be found in the document "From Policy to Practice" (DES, 1989). This reads:

The National Curriculum comprises:

- o **foundation subjects** - including **three core subjects** and **seven foundation subjects** which must be included in the curricula of all pupils;
- o **attainment targets**, to be specified at up to 10 levels of attainment, covering the ages 5-16, setting objectives for learning;
- o **programmes of study** specifying essential teaching within each subject area;
- o **assessment arrangements** related to the 10 levels of attainment.

Thus legislation has given us an intended curriculum, but does not attempt directly to control the implemented curriculum. Legislation will prescribe a means of measuring particular curriculum outcomes. It is likely that these outcomes will become the criteria by which implemented curricula are judged to be acceptable.

(b) Is the National Curriculum national?

As a prescription, the National Curriculum applies to England and Wales, and not to Scotland or Northern Ireland. It applies to maintained schools but not city technology colleges or independent schools. It has already been pointed out that here legislation may be concealing an inevitable contradiction.

> Consider the objective of a national curriculum. The idea of a broad but differentiated curriculum being the entitlement of all children has general political and professional support. Its attraction lies in the implied promise of continuing the attempt to improve equality of opportunity. However the market will offer different kinds of school. Markets are about differences ... A national curriculum and a market in education cannot be compatible in any logic we understand.
> (Tomlinson in "Take Care Mr Baker", Havilland 1988)

However within contexts in which it <u>does</u> apply, the targets are thw same for all students regardless of educational needs, gender, ethnicity or culture. It is therefore pertinent to ask how common is the image of what is meant by National Curriculum orders for mathematics among those to whom it applies. Is there a common image among teachers, pupils, parents and employers? Is there even a common image among teachers? If not, what might make it common?

The National Curriculum provides, at least, a common vocabulary for discussion. However definitions alone do not produce images. Examples may help, though we must beware of mistaking examples for exemplars. The examples in the Statutory orders for mathematics may be useful in illustrating the statements of attainment, but they should not suggest that every programme of study for that level should include that particular example, nor even that schemes of work should consist of activities directed at single statements of attainment. Many of the richest starting points for teaching in mathematics span several attainment targets as well as a variety of levels.

4. What research questions, pertaining to the learning of mathematics, does the National Curriculum Pose?

I suggest that there are at least three ways in which such questions will arise or be posed. These are:

(a) **Explicit assertions that are testable.**

These would include the kinds of statement which are made in the DES Circular No 6/89

> Assessment at or near the end of each key stage <u>will demonstrate</u> which of these attainment levels, each of which is defined by a statement or a number of statements of attainment, an individual pupil has reached. The level reached by a pupil at any given age <u>will reflect</u> the number of years of schooling he or she has had, what they have been taught and how well, and individual ability and maturity.
> (My emphases).

(b) **Implicit assumptions that can be revealed.**

It is assumed that all subjects, including mathematics, can be analysed according to a framework of levels and groupings of subject matter within attainment targets and profile components. It is also assumed that there are implementations of the intended curriculum which will allow (almost all) children to achieve attainment targets in mathematics at levels deemed to be appropriate for their age.

(c) **Questions of implementation.**

These are questions about the impact on teachers and children of implementing the National Curriculum. These include questions about

- o the load on primary teachers who have the task of integrating demands of several documents simultaneously;
- o the potential anomalies in Secondary schools where Mathematics and Science teachers could make default assumptions about the national curriculum, which will define school policy not only on their own behalf but on behalf of their colleagues in langugaes, arts and humanities; and
- o the impact on the role of teacher and her relationship with pupils, parents, and colleagues.

Concluding remarks

We are currently embarking on a vast educational experiment, unprecedented in England and Wales in its scope, but with a minimal timescale. In many ways it is an exciting time for those of us engaged in educational research, but it is a salutary reminder to us that the subjects of our research are human beings. They are teachers who need, like all human beings, time to assimilate and adapt to changing conditions and requirements, and children who have a one-way ticket through the educational system.

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May 1989.

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