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CONTENTS

APPARATUS ALGORITHMS AND CHILDREN'S NUMBER DEVELOPMENT

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The use of apparatus to assist learners in developing their concept of number is not new. Simple one–to–one correspondence seems to have been used over the years to link the ideas of counting to the number system. In most, if not all, classrooms where young children learn arithmetic, the provision of sets of objects to assist in counting seems to be a requirement. In fact the term "counters" is assigned to a set of plastic objects apparently specifically designed for this purpose. Once the numbers in a calculation get larger, the one–to–one representation becomes less helpful. The notion of place value in the representation of the number becomes a more powerful means of recording the number, and in turn it becomes a computational aid. Apparatus designed to model this, or material to help the ideas for this (for example Dienes' apparatus, tens boards and coins) are introduced.

Observing children using apparatus is interesting and revealing. Consider the following scene in a classroom. Daniel (a pseudonym) who is an eight year old Year Three child, is working with some subtraction questions.

$$\begin{array}{r} 47 \\ -25 \\ \hline \\ \hline \end{array}$$

He used Dienes, when working it out. He put out four tens and seven ones, then removed 2 tens first, then five ones. He recorded the answer 22.

$$\begin{array}{r} 51 \\ -27 \\ \hline \\ \hline \end{array}$$

He set out five tens and one unit. Then he exchanged one ten for ten ones. He combined the ten ones with the single one to get eleven ones. He removed two tens first, then seven ones. Finally he recorded the answer 24.

$$\begin{array}{r} 45 \\ -19 \\ \hline \\ \hline \end{array}$$

He set out four tens and five ones He removed one ten and counted 9 up a second ten. He covered the 9 with he hand and positioned the one remaining next to the other ones. He then counted out loud, "10, 20, 1, 2, 3, 4, 5, 6. 26." He wrote down 26.

$$\begin{array}{r} 100 \\ -21 \\ \hline \\ \hline \end{array}$$

He picked up the hundred flat, turned it so the holed side was uppermost. He then counted, "10,20", pointing to columns on the flat then covered the next one with the finger. He then counted, out loud, from the other edge of the flat, "10, 20, 30, 40, 50, 60, 70, 1, 2, 3, 4, 5, 6, 7, 8, 9. 79"

These sorts of activities go on in many classrooms. Children often do similar things to Daniel. On the other hand the text books or mathematics schemes designed for the teaching of subtraction offer a slightly different approach to using apparatus. In particular they often stress that the apparatus has to be used in a particular way, and it is often related to a specific form of recording.

There seem to be two modes of using apparatus. The first we might describe as a *developmental mode* and the second as *demonstration mode.* In some respects the originators of the apparatus encourage the use of a developmental mode by encouraging teachers and learners to use the first introduction to the use of apparatus as 'free play' or exploration. The learner is very much in charge of the activities and from this a degree of familiarity is acquired by their working with the equipment. The demonstration mode tends to be more didactic, and in the case of base ten materials often relates to a method of recording. An example of this is the use of the base ten material from Dienes Multibase Arithmetic Blocks (MAB). These are often used to link with the recording related to decomposition. A typical teaching sequence would require the learner to work out subtraction examples with the apparatus and then link the apparatus use to the recording associated with decomposition. Habits and methods developed in the developmental mode may not readily transfer to the demonstration mode. Children who have managed to use

apparatus to subtract a two digit number from another two digit number can do so by removing tens first, find the instruction to start with the units rather puzzling.

In some cases it seems that they are following a procedure based on the apparatus and are using it algorithmically, that is following a rule of their own or their teacher's making. These would include behaviours such as using finger counting to perform addition, and obtaining the answers by a routine. Reflecting on how children use apparatus might produce different sorts of *Apparatus Algorithms.* Different types of apparatus algorithms probably exist. A spectrum with two extremes is suggested. At one extreme there is a *Mechanical Apparatus Algorithm* where children are applying rules mechanically, where they do not relate the actions to an understanding of the concepts of the operations they are performing. At the other extreme there is an *Interactive Apparatus Algorithm* where the children are able to transfer freely between the apparatus and the concepts, the ease with which they operate being a sign of their confidence to work interactively between the representation of the concept and the abstraction it represents. Advocates of the use of apparatus stress the need for a development of the use of apparatus. The apparatus is seen as analogous to scaffolding, which is essential for the construction of a building, and its function is less required as the building reaches completion. If it has done its job properly, once the building is complete the scaffolding can be safely removed.

Between the two extremes suggested there may be intermediate apparatus algorithms. As the learner proceeds more towards the interactive apparatus algorithm, there is a progression where the apparatus takes less of the strain, and the learner becomes more self reliant. Two intermediate points are suggested: *Dependent Apparatus Algorithm* and *Supportive Apparatus Algorithm.*

Mechanical Apparatus Algorithm

The term mechanical apparatus algorithm is used to describe the phenomenon where the person manipulating the apparatus is following a set of instructions and not necessarily understanding what they are doing. This may be a result of an over-dependence on the teacher having used the apparatus in demonstration mode. Additionally the learner may have not perceived the link between the apparatus and the recording expected. This could happen with Dienes' base ten, if an undue emphasis is placed on manipulating the blocks according to a fixed procedure.

Equally the proponents of the apparatus may be tempted to make exaggerated claims for the apparatus itself! There is a temptation to emphasise the value of apparatus, to the extent that it could be seen as a panacea. If it works to help the learner to understand one aspect of number, could it not be used to help them with another. Publishers and educational suppliers may be tempted to stress that a particular apparatus "teaches" the concept, whereas the originators might not actually wish to make such a claim. In the written description of the use of apparatus, there may also be a tendency to encourage apparatus algorithms, this can be seen in the way in which the Cuisenaire apparatus is used in helping to develop the idea of equivalent fractions. It may be that there is better alternative apparatus to aid the comprehension of equivalence, or it may be that a numerical algorithm might be clearer to the learner.

Another example of apparatus algorithm is the idea that counters can be used to model all addition and subtraction. It would be sensible to think of addition of small quantities to demonstrate number bonds up to ten. Although the addition of 54 and 66 *could* be achieved by using 54 and 66 counters, the implied 'count all' algorithm would not assist the learner to apply a more efficient method to obtain the result. The use of Dienes' apparatus in base ten might aid the learner to make the transition between the apparatus and a mental picture which could be useful in obtaining the result itself and to generate a method of obtaining results of other additions of a similar sort.

Dependent Apparatus Algorithm

When children are taught to use apparatus, they often are directed towards procedures which give answers by following a sequence of steps. Sometimes this requires the apparatus to hold both the question and the answer. An example is the use of the number line for subtraction. The idea of counting back actually needs the presence of the number line. It does not generalise into a method of working out all subtractions. The process of counting back is a difficult one to execute without a written number line. If a subtraction 7–3 is considered, the point seven on a number line is the starting point, with a counting back of three. This process needs to be carried out on a number line.

Another example of a dependent apparatus algorithm is that of fractions as 'pieces of cake'. In this representation of fractions the cutting up of cake to give a picture of a fraction is helpful, only in so far as it gives *one* picture of the number. Any extension of the idea into a generalisation of fractions is difficult, *without the picture.*

Supportive Apparatus Algorithm

In this case the apparatus is taking some of the strain. That is it is holding some aspect of the thinking whilst the learner has time to think. If addition of quantities is under consideration, the first number in an addition sequence can be 'held' by apparatus. For example if the computation 6+3 is to be worked out, apparatus can be used to hold the number six, whilst 3 is added. Once the number six is counted, the counting does not have to be repeated. In this way the apparatus is taking the strain of holding the running total. The addition of three can continue, by the counting on from the six. An extension of this can be seen in the use of a number line. The process of 'adding on' 3 to six can be achieved by using the number line to hold the six, and then counting on three from there. This process can generalise into a method of modelling addition of other quantities.

Interactive Apparatus Algorithm

In this case the user is able to operate on numbers with confidence. They are able to explain what they are doing and can change from one representation to another. They can visualise the number both as a quantity and as a concept. In explaining their ideas they are able to enlist both aspects of apparatus and concept. The user does not really need the apparatus to work out what they are doing. If called upon, they can explain how they are working with reference to both the apparatus and the concept.

The influence of object size, number of dimensions and prior context on children's estimation abilities.

Michael Forrester and Beatrice Shire, University of Kent.

In primary mathematics education the ability to estimate is considered to be a core skill, and previous research has shown the importance of maturational and contextual factors on estimating abilities. This experiment examines the influence of object size, dimensionality and the previous estimation judgement on the estimates of primary aged children (aged 8 to 11 years).

In addition to the main task which involved estimating how many small cubes would fit into a larger box, the children were given a short assessment of their mathematical competence. This pre-test included simple addition, subtraction and multiplication using the two to ten times tables, and the children's responses were noted and also the manner of response: immediate, counting-up, thinking (without obvious counting) or no answer at all. We expected that the manner of response would reflect the children's confidence with and ability to manipulate numbers, and may be related to the accuracy of their estimates.

The main task used a small card cube covered in soft (as it happened, pink) material so that it was a 2cm cube with a feeling of 'give' to it enabling the children to envisage it being packed somewhat flexibly into a cuboidal space. This was discussed with the children as was the idea of giving a 'rough' answer to questions about how many cubes would fit into each of the boxes. The boxes, constructed out of stiff white card, came in three types: long, flat and brick-shaped (i.e. 1, 2 and three dimensions), all having one open side so the inside could be seen. We had previously established in a pilot study that children treated rows of cubes in an nx1x1 'long' box the same as rows of card squares on an nx1 card strip, and similarly for two-dimensional arrays of cubes and cards. The cube was a similar size to a unifix cube which the children were used to sticking in rows to count. Each of the boxes had dimensions that were 2, 8, 16 or 32 cm (1, 4, 8 or 16 cubes) long, and every possible combination of dimensions was available. Each child, however, was asked to estimate for only 10 boxes: 3 long, 3 flat and 3 bricks

plus the largest 32 cm cube. There were 8 different orders of presentation involving each possible arrangement of up/down size/dimension; two examples are:
Starting with the smallest 'long', next a small 'flat' then a small 'brick'; then a medium 'long', 'flat' and 'Brick'; then a large 'long', 'flat' and 'brick'; then the largest box.
Starting with a small 'long', then a medium 'long' then a large 'long'; a small 'flat', a medium 'flat' and a large 'flat'; a small 'brick' a medium 'brick' and a large 'brick'; then the largest box.

24 6 to 9 year olds and 43 10 to 11 year olds (roughly half boys, half girls) took part, these being the children of those ages attending a primary school in East Kent. They were tested individually. and as they made each estimate the box was left on the table visible to them for their next estimate. They were allowed to turn the boxes round to look at them, but were prevented from physically placing the small cube inside and counting. They were encouraged to talk about how they were thinking in order to make their estimates.

Results.
We looked at three aspects of the results.
a) The effects of age, size and number of dimensions.
b) The effects of age and prior estimate.
c) The relation to mathematical competence and manner of calculating.

a) In a 2 x 3 x 3 anova using estimate as a percentage of correct value as the dependent measure, there was a main effect of age ($p<0.02$), with the younger children tending to underestimate more than the older; a main effect of size ($p<0.001$), so that the larger the value the greater the underestimate; and a similar effect of dimensions ($p<0.01$). There was also an interaction between age and size ($p<0.01$), with younger children increasing their tendency to underestimate with the larger boxes more than the older children; lastly an interaction between size and dimensions ($p<0.01$), due to a greater increase in underestimation as size increased in three dimensions than in one.

b) The two age groups were investigated separately and showed different patterns. Younger children produced more pronounced underestimates if

the task involved a larger estimate than the previous one, and more so if there were 2 or more dimensions involved. Older children appeared to be taking the extra dimensions into account, and for two or more dimensions tended to slightly overestimate if the task required a larger estimate than the previous one, though not if it was smaller.
Some older children demonstrated their awareness of the importance of their earlier estimates: 'If one of them's wrong then all the rest is wrong'
c) There appeared to be no relation between mathematical competence and accuracy of estimates, but when children were categorised as *underestimators, average and overestimators* we found the *underestimators* were more prone to giving immediate responses in the pretest. The others, *average and overestimators*, gave more 'don't know' responses. (Chi-squared test, $p<0.001$).

Though skill in multiplication did not appear to be an indicator of better estimation, many children reported strategies which involved multiplication as part of the process. What may be true though, considering b) and c) above, is that the more you know about the complexities of the task the less willing you are to hazard an answer unless you are sure it is correct.
Strategies other than multiplication used in estimating in more than one dimension generally involved counting along sides and then inwards in layers or a spiral, with various allowances fokr the fact that corners belong to two sides. Three dimensional shapes were usually visualised as layers of 'flats' but sometimes as two or more of another 'brick' - depending on the previous box used. Ingenious strategies seemed no worse than multiplication at giving estimates, and where the children were unhappy with the outcome (commenting that it seemed too large or too small) they occasionally adjusted it.
An interesting (and probably familiar) comment on accuracy and estimates came from one child as an explanation of how to estimate: 'I work it out, by multiplying, then knock a few off because Mrs. H. doesn't like us getting it exactly right'

MATHEMATICAL AND SOCIAL PROCESSES IN GROUPWORK WITH COMPUTERS

POZZI, S., HEALY, L. & HOYLES, C.

A great deal of interest has developed in recent years, not least within mathematics education, over the possibilities provided through learning from one's peers in group situations. Groupwork is already taking place out of necessity in contexts where resources are scarce - most notably where computers are involved. However, the integration of the computer brings a new dimension to groupwork, a dimension which many argue is qualitatively different. The use of programmable computer environments such as Logo and databases is said to support mathematical discussion – and hence learning – through providing a means with which to express pupils' mathematics. If groupwork with computers is to be exploited then understanding what the groupwork is for and how this aims can be achieved is critical. It was from these concerns that we began our study as a part of the ESRC project, Groupwork with Computers[1]. Our aim was to identify both background and process factors which influence the success of groupwork with computers - in terms of both learning and group outcome, though we shall focus only on learning gains here.

Our research was based on a multi-site case study design, working in six Primary/Middle schools. Before we undertook the study, all the teachers involved undertook a programme of in-service training to develop their own groupwork and computer use in the classroom. This was followed up by regular classroom visits to support teachers. For the study itself, we focused on groups consisting of six pupils (aged 9-12) - each consisting of three girls and three boys, a girl and boy from each of high, middle and low ability levels as assessed by their teacher. This design provided a reasonable number of groups of equivalent composition, though from different school contexts. Eight pupil groups were chosen, each drawn from a class of individuals who shared a common culture - both in terms of the school and the classroom. Each group undertook three research tasks at various points during the school year; two involving Logo programming and one a database. The tasks were named Letters, Spokes and Homes and each lasted about two and a half hours (see Fig 1a and 1b). This provided 24 group settings through which we could explore the influence of groups, tasks and software on both the group processes and individual learning. In each class, the teacher introduced Logo and databases to all their pupils and decided, in conjunction with the researchers, when the pupil group were sufficiently familiar with the software to work on the research tasks.

The tasks themselves were based around different mathematical ideas; namely modular programming, rotational symmetry and data classification. The content of the tasks were chosen and developed in consultation with the teachers to be both relevant to the curriculum and stimulating and enjoyable for the pupils. Importantly, the tasks were all carefully designed to include a set of activities - *local targets* - which could be shared out into subgroups and constructed with the computer and a network of mathematical components - *global targets* - to be considered by the whole group. This task design was used to facilitate peer interaction in two ways; on the local targets through products being constructed with the software at different levels of sophistication, and on the global targets through the exchange of ideas and comparison of alternative perspectives. We carefully described our tasks to each group after which we made no further interventions.

The pupils were completely responsible for all aspects of task management; how they organised themselves, the task and the resources. Allowing the group to control how they executed the task was important as it would allow us to assess how far the pupils could take responsibility for themselves and their own learning. Furthermore, it is common practice in classrooms for teachers tend not to intervene when pupils work with computers, though this is often for classroom management reasons as much as educational. One intended consequence of allowing group responsibility is that the composition of the groups allowed for a number of subgrouping possibilities which could follow very different gender and ability lines. This would allow us to look at what natural subgroupings emerged, and what consequences these might have on the group processes and potential learning.

Process data was collected by two researchers, through video recordings and field notes. We interviewed all the pupils together after each task, and probed their perceptions of the task and the

[1] Research project in conjunction with the University of Sussex funded by the InTER programme of the Economic and Social Research Council 1989 -1991, Grant Number 203252006.

Craigmiller Castle
I live in a home called Craigmiller Castle. It is a castle which is 600 years old. It is an large home with 107 rooms altogether. It is heated by wood fires. We have 31 bedrooms

Inglenook
I live in a home called Inglenook. It is a maisonette which is 25 years old. It is an average size home with 7 rooms altogether. It has gas central heating. We have 3 bedrooms and 2 floors. When I counted the doors (including the front

Sputnic 5
I live in a home called Sputnic 5. It is a space station which is 2 years old. It is a small home with 2 rooms altogether. It is heated by solar panels. We have 1 bedroom and 1 floor. When I counted the doors (including the front door) and windows, I counted 3 doors and 2 windows. We have some interesting features in my home it has a transporter, solar panels and rockets. My home is near space, planets and stars. In my home the oldest person is Ivan who is 45 years old. There are no children in the house. The youngest person is Raisa who is 42 years old. The total number of people living in my home is 2. We have 1 pet, a dog.

HOMES

Use the database to find out the answers to these questions.

Which home has the largest number of rooms?
How many homes have more than 5 rooms?
What is the most common number of windows?
How many homes have more than one floor?
How many homes have central heating and a garden?
How many homes have a garage or a cellar?

You have to put this data onto the database so that you can find the answers to our questions

Discuss together the fieldnames for your database, then write them down.

FIELDNAMES

Use the database to find answers to these questions.

Which home do you think is the darkest? [ ]
Why? [ ]
Which do you think is the warmest? [ ]
Why? [ ]
Which do you think is the noisiest? [ ]
Why? [ ]

Homes Task

Fig 1b: The Group Tasks - local targets (above) and global targets (below)

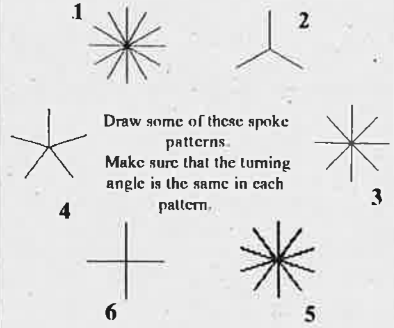
LETTERS

Z P Z A I

All the letters should be the same size

Can you write a procedure for each letter in the word PIZZA

SPOKES



Draw some of these spoke patterns.
Make sure that the turning angle is the same in each pattern.

PIZZA
PIZZA
PIZZA

The gap between the letters should be the same

Put your letters together in one superprocedure which will draw the word PIZZA

Letters Task

Try to find a connection between the number of spokes and the turning angle

[ ]

Discuss within your group why this connection is always true

[ ]

Finish the procedure to draw a seven spoke pattern

```
TO SEVENSPOKE
REPEAT ____ [ SPOKE RT ____ ]
END
```

Use the connection between the number of spokes and the turning angle to write a variable procedures that can draw any spoke pattern

```
TO ANYSPOKE :N

END
```

Spokes Task

Fig 1a: The Group Tasks - local targets (above) and global targets (below)

group and what they believed they had learnt. We also talked at length with each teacher to find out as much as possible about the group members, both individually and collectively. We also asked about the class as a whole, and the culture of the school with respect to computer use and groupwork in particular. Finally, all the pupils in the research classes completed two sociometric questionnaires to assess perceived status in terms of 'cleverness' and popularity. Individual progress in relation to each task was measured through pre-, post- and delayed post-tests administered to all pupils in the research classes; a week before, immediately after and four weeks after each task.

The main thrust of the analysis was qualitative, in which these data were synthesised in the development of case studies of the 24 group settings. The case studies were analysed in order to draw out the most salient aspects of the process factors involved and develop appropriate descriptors. Associations between background and process factors were then explored, together with how these affected individual learning. Quantitative techniques would then be used to explore these associations more formally, using two measures of learning. One used the improvement in raw scores on all the items in the test, while the other involved designating each pupil as a *knower*, *learner* or *non-learner* of a key concept for each task, and this was based on one of the items in the tests.

Our findings suggest that pupils involved in the groupwork improved significantly more than the rest of the class across all three tasks - both in the post tests and delayed post tests. Confidence intervals (at 95% or $p<0.05$) for the benefit due to being involved in each task are; Letters (8.2% $\pm$ 7.1%), Spokes (12% $\pm$ 7%), Homes (9.3% $\pm$ 7.2%). Although the rest of the class were not engaged in any comparative instruction - individualised or otherwise - two points need to be made. First, the improvement was sustained and in some tasks increased at delayed post test. Second, our tests were not simply individualised versions of the group tasks but placed the mathematics in very different contexts, both in terms of being paper-and-pencil based and in the form of mathematics. Thus, the gains would indicate substantive mathematics learning.

The next phase of the analysis involved examining the progress of the case study pupils themselves, to see whether background and process variables had any effect on learning:

Background factors: The two main background variables of interest were gender and ability, as these have often been reported to be significant. However, we found that girls progressed as much as boys after being involved in the groupwork for each task. Similarly, pupils designated as high, middle and low ability by their teachers all progressed to the same extent. Other background variables such as age and friendship also proved to be unimportant. The software and groupwork culture of each class; i.e. the extent to which groupwork and software use were an integrated part of the on-going classwork, also seemed unrelated to the extent of progress. This should be qualified however, as the first year of development work in the project meant that all our classes had a more established software and groupwork culture than most.

Process Factors: We identified two main process variables from our analysis, both of which were very influential in how far pupils learned:

Three different *styles of organisation* were identified which categorised the ways in which the groups organised themselves, the tasks and the resources. We termed these *collaborative, competitive and co-active* . In collaborative settings local targets are shared out and global targets considered by the group as a whole, working either across or away from the computers, resembling the way the task had been planned. This style only emerged if one or two pupils take on co-ordinating role whereby they make the majority of task and group management decisions. Where the group splits into subgroups which attempt the whole of the task separately, the style of organisation is termed competitive or co-active. In competitive settings, rival single-sex sub-groups do not communicate and instead concentrate on constructing computer products. This style invariably emerged where there was some pre-established antagonism across gender, which resulted in the single-sex subgroups. In contrast, subgroups in co-active settings, though working towards separate goals, maintain channels of communication through which help is given and task demands discussed. In contrast to collaborative settings, co-ordination of any kind was rare within competitive and co-active styles of organisation.

We further identified four *patterns of interaction*, based on whether interaction between peers over local and global targets of the task were dominated by particular pupils or more evenly distributed

amongst the group. Two patterns – *directed* and *mediated* – occurred within all three organisational styles. Directed interactions are characterised by asymmetric patterns of influence with one or two pupils – *directors* – dominating both local and global targets. In contrast, in mediated interactions pupils have a more equal influence over all targets with no apparent interaction roles. These two patterns represent opposite ends of a continuum in that, in the former, all targets are dominated and in the latter all shared. In collaborative settings only, two further patterns of interaction were identified – *navigated* and *driven* . In navigated interactions, one or two pupils — *navigators* — take control of the global mathematical issues while influence on the local targets remains evenly distributed. In contrast, in driven interactions, global target discussion is symmetric in terms of individual pupil input, but the construction of the local targets, at one computer at least, is dominated by one pupil – a *driver*.

The pupils who took on co-ordinating roles to establish a collaborative style as well as the pupils who dominated in asymmetric patterns of interaction all had a number of similar characteristics. Though over half such pupils had a high pretest score (23 out of 44), they were more likely to be designated by the teacher as high ability (32 out of 44) while the most common attribute of all was being perceived as 'clever' by their peers in the class (40 out of 44). Gender was unimportant, with similar number of boys and girls taking on these roles.

Interaction Pattern	Organisation Style		
	Collaborative	*Co-active*	*Competitive*
Directed	6	6	24
Mediated	53	6	12
Navigated	24	n/a	n/a
Driven	13	n/a	n/a

Table 1: Distribution of pupils across interaction patterns and organisational styles

Looking more closely at the association between these two process variables, collaborative settings are associated with two symmetric patterns of interactions - namely mediated and navigated. Competitive settings, on the other hand, are clearly associated with directed interactions (see Table 1)

Organisational Style	Pattern of Interaction	Knowers	Non-learners	Learners
Collaborative (n = 96)	Directing	1	5	0
	Driving	5	7	1
	Navigating	5	15	4
	Mediated	11	23	19
	Total	22	50	24
Competitive (n = 35)	Directing	10	13	1
	Mediated	6	5	1
	Total	16	18	1

Table 2: Distribution of progress within Organisational Style and Pattern of Interaction

The effects on progress of organisational style are such that pupils within groups adopting a collaborative style of organisation progressed more than those who were within groups adopting competitive styles (see table 2). Co-active settings are left out as there were too few to be a reliable indicator. Within this pattern, we found that pupils in collaborative settings who were involved in navigated or mediated interactions progressed further than dominated pupils within driven or directed interactions (see Table 2), with pupils in mediated interactions progressing furthest. Moving to competitive settings, this pattern seems to disappear. The most effective groupwork is therefore based on two features of the group processes; the emergence of a collaborative style of organisation, and maintaining symmetric patterns of interaction - especially at the computer. But why should these group processes prove to be so important to mathematics learning?

The effect of the different organisational styles on learning may be based on understanding the effect of pupils' goals in groupwork. Competitive settings brought about goal structures that were less to do with understanding the solution process and more to do with completing the computer aspects of the tasks as quickly as possible. The visibility of the computer screens accentuated this by allowing subgroups to compare each others progress. One could therefore offer a social psychological account of the demotivating effect of seeing other subgroups 'doing better', and hence less on-task behaviour. Some competitive subgroups in our study clearly did become demotivated, and at times off-task and disruptive. This does not however explain the fact that even the subgroups who were quite productive and seemingly highly motivated did not improve as much as pupils within collaborative settings.

We offer two further explanations. First, competing subgroups ignored or paid cursory attention to the global aspects of the task. This meant that they concentrated only on the possibly incorrect strategies developed at their own computers and were less likely to be confronted with differing mathematical strategies constructed by other subgroups. Collaborative settings, on the other hand, are more likely to allow this conflict of strategy within the context of whole group interaction. Second, the importance of symmetric patterns of interaction suggests the benefits of mutual discussion between peers. However, given pupils can progress even if whole group global target discussion is dominated by others, it was found that discussion was most fruitful when in the context of constructing with the computer.

Our view is that the software plays a crucial role here. It allows pupils to construct the mathematics for themselves, at their own level of sophistication, but in a way which requires them to formalise their mathematics at an appropriate level. The nature of the formal language of the software may provide the pupils with the means to do this - providing a common language with which to talk about the mathematics. Then, in the light of this, they are more able to make sense of the possibly more sophisticated ideas and perspectives of others in the whole group activity.

Maintaining both a collaborative style of organisation and symmetric patterns of interaction - especially at the computer - are clearly important in creating an effective environment for mathematics learning. However, as evidenced by our data, social processes can significantly affect the emergence of these two process factors:

Firstly, competitive styles of working invariably emerged from groups with cross-gender antagonism, though on closer inspection the antagonisms were not necessarily based on gender. Rather, they were in the context of isolate pupils whom others would not work with, inseparable single-sex pairs who did not want to work with others, etc . The fact that all these problems resulted in cross-gender splits in the group may partly be a function of the age, and we would speculate that a wide variety of antagonisms in different age groups could easily result in a splitting of a group along different lines but with similar detrimental results. The social system of the group, in terms of the peer perceptions between the group members, would therefore seem to have a very influential effect on how the group organise themselves, and hence the extent of their learning.

Secondly, pupils' perceptions ability also proved to be an important social effects, with dominant pupils tending to have high knowledge status, more than other characteristics such as high pretest knowledge, etc. However, this status stemmed not from the peers in the same group but from the class as a whole. Thus, it is an aspect of the class culture from which the group came rather than just the inter-personal perceptions of particular group members which results in certain pupils emerging to dominate interaction during the task. Also, the pattern between dominance and learning is not a straight-forward one. Without the emergence of dominance over management issues, a collaborative organisational style did not emerge. Dominant co-ordinators is thus a necessary feature of effective groupwork. If, however, dominance occurs over the aspects of the task, especially in terms of the monopolisation of the computer, this leads to ineffective groupwork.

In conclusion, our research suggests that social processes can have important influences on groupwork with computers, but so long as particular pupils co-ordinate group activity and there is no antagonism between group members, a collaborative organisational style can emerge. This is beneficial to mathematics learning when pupils can benefit from first engaging in mutual discussion with peers in the context of constructing with the computer, then coming across the perspective of other pupils in whole group discussion, without necessarily contributing to the discussion themselves. Without the former, pupils may not have developed any kind of strategy or understanding of the problem, so cannot make sense of any possibly conflicting strategies from their peers. Without the latter, pupils may remain centrated on their own way of understanding the problem, so are less likely to learn.

Natural, Conflicting and Alien: A Forum for Discussion.

by

Janet Duffin* and Adrian Simpson.**

Incidents at Hull University's Mathematics Workshops [1] were the catalyst for the thinking which, eventually, culminated in this presentation. The workshops, run by the staff and students in the School of Mathematics, are Saturday morning events for 12-14 year-olds and incidents which occur during the sessions are the focus of conversation as everyone clears up at the end.

Two incidents stood out for us and sparked off our discussions. Pupils' perceptions seemed to differ widely from those of helpers and yet, the way in which the pupils talked about their perceptions made them seem 'natural' ways of thinking for the individual pupils. In discussing them, we found that the word 'natural' along with phrases like 'it has to be' and 'it couldn't be anything else' cropped up.

The first of these related to the idea some pupils had that two solids made from the same surface pieces must have the same volume and, in the second, we found that there was a marked difference between the perceptions of the meaning and value of 2^0.

We began to think hard about, and discuss in depth, the meaning we attached to the word 'natural' which one of the student helpers had coined to explain his account of one of the incidents.

In doing so, however, we found ourselves considering how 'natural' misconceptions are eventually overcome and we came up with the notion of 'conflict' - an experience leading the learner to re-evaluate his/her way of thinking. This new notion also caused a conflict for us. Initially we were in some doubt about whether teaching should include conflict, or whether teaching could be designed to allow only the natural building of structures.

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Even expressing our doubts in such terms, however, led us to see that conflict was an essential ingredient in learning.

Our perceptions of these two notions: natural and conflicting, began to crystallise into an early definition of them. We saw that

Something is **natural** if it:
- fits an already formed mental structure
- is comfortable and acceptable to the learner
- is the basis for actions and decisions in mathematical situations.

Something is **conflicting** if it:
- comes from an experience which does not match with existing structures
- forms the basis for a reorganisation of the mental structure
- can result in a new mental structure that accommodates both the old and new experiences.

With these in mind, we found ourselves re-evaluating other incidents outside the workshops to see how they matched our evolving definitions. We examined three incidents we had encountered with a six year-old boy, a twelve year-old girl and an undergraduate woman. In all three cases we were able to analyse many of the learners' actions and explanations in our new terms as we attempted to account for what they did and said by reference to their previous experiences.

In each case, however, there appeared to be something about the learner's behaviour which was not adequately explained by our two concepts. In the case of the twelve year-old, the learner returned to the safety of her calculator in trying to evade what we initially saw as a conflict between her actions in attempting to perform a division and her abilities with other arithmetic processes. The undergraduate also ignored what we, at first, saw as a conflict, when she produced an adequate proof of a conjecture she had made, but then returned to checking individual examples to demonstrate the truth of it. In terms of their previous experiences (the twelve year-old girl in the CAN project [2] and the undergraduate's early experiences of proof) we could see their reactions as 'natural' but somehow there was what could be termed an 'unawakened conflict' which stood out for us in both incidents.

In the case of the six year-old, this 'unawakened conflict' was quite startling. Following a strategy of taking the smaller digit from the larger in a subtraction (a method we saw as 'natural' given the early experience of 'taking away' in a concrete setting) he wrote beside his working 'but the real answer is 277.' What we saw as a major conflict between *his* answer and the 'real' answer was of no concern to him at the time.

In all three cases we had perceived the opportunity for conflict but found that the learners had evaded or ignored them without reformulating their mental structures. We began to term these experiences 'alien': situations where learners were 'going through the motions' and producing contradictions which, because of a lack of connection with their stronger, more 'natural' ways of thinking, had not resulted in the re-alignment of internal structures.

After more discussion, we polished our three definitions in a setting we feel to be basic: that all human beings try to make sense of their experiences and the environment in which those experiences occur. This we take to be innate.

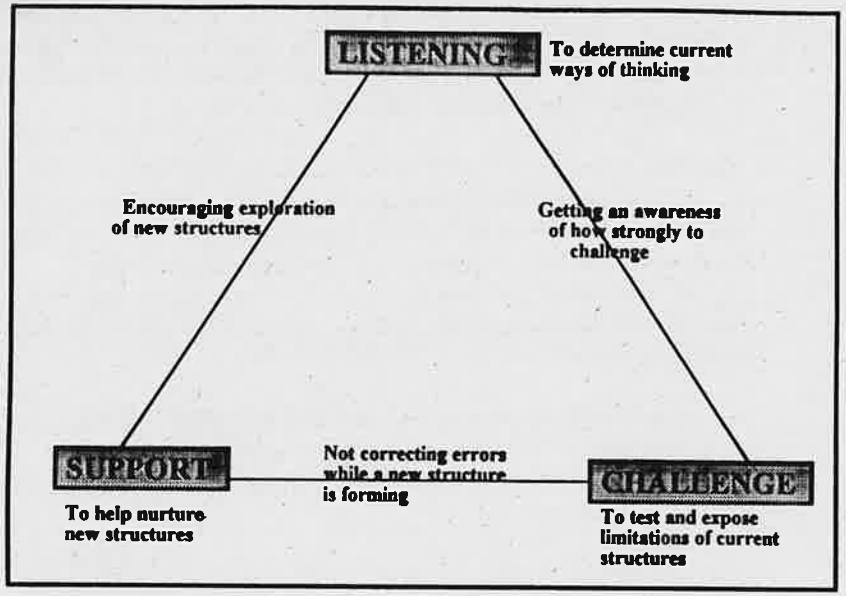
Something is **natural** if it
- fits an individual's internal framework
- results in actions consistent with this framework
- evolves as the learner attempts to make sense of experience.

An experience is **conflicting** if it
- highlights the limitations of an internal framework
- results in internal inconsistency which needs to be resolved
- can effect a merging or modification of internal structures.

Something is **alien** if it
- has no connection with an individual's internal framework
- does not result in *internal* inconsistency
- may result in various avoiding actions
- may be the foundation for a new, but separate structure.

Armed with these definitions we found that they began to colour the way we observed both our own learning and that of others. We began to think about ways in which they might be used to support our approaches to teaching and learning.

We became convinced that if learners are to be helped in the learning process, they need both challenge and support to enable this to evolve successfully; that the ability to listen must take on a greater significance as part of the teacher's role. A powerful aid in our thinking about this has been this diagram:



Most of the discussion at the presentation centred around this as an issue and suggested that our new ways of thinking formed with these definitions also had implications for the learner (and indeed, that there may well be some overlap between the role of the teacher as listener, challenger and supporter and the roles of the learner and his/her peers). We also discussed the resonances that our presentation made for others' views of alternative theories of learning.

Many of the incidents and discussions related to here are expanded upon in [3] and we hope to take the ideas further as we develop them.

Notes and References.

[1] Kopp, E., Duffin, J. and Simpson, A., "The Hull Mathematics Workshops: Content and Outcomes", *Mathematics Education Review*, (to appear).

[2] CAN - the Calculator Aware Number project in England and Wales, 1986-1992.

[3] Duffin, J., and Simpson, A., "Natural, Conflicting and Alien", *Journal of Mathematical Behaviour*, (to appear.)

METHODOLOGY, WHO NEEDS IT? CURRENT DEBATES ABOUT EDUCATIONAL RESEARCH

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The French mathematician Henri Poincaré is reported to have commented that, unlike real scientists, sociologists are preoccupied with method. I should perhaps begin, then, by admitting that not only am I a sociologist but that I have spent an increasing amount of time in the last few years thinking and writing about educational research *methodology*, rather than actually doing educational research. The question in my title is not, therefore, purely rhetorical. I often wonder whether my time might not have been more productively spent on research rather than methodology.

Worse than this, though, it is not just that methodology, like other forms of theory, may be useless for practice, but rather that it can sometimes actually be detrimental to practice. Recent developments in methodological thinking about educational research increase rather than quell my doubts. The developments I want to talk about are most common amongst qualitative researchers and are prominently displayed in recent publications produced on the other side of the Atlantic.[1].

These developments are certainly of direct relevance to how one goes about research. They relate to such fundamental matters as what the purpose of research is: is it simply to produce knowledge, or is it, for instance, to improve education, to promote educational and social equality, or to emancipate the oppressed? And this connects with the question of how we should judge the products of research. What do we mean, for example, by the term 'validity'? and so on.

What concerns me most is that some of these developments imply radical reconstructions of research

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[1] For example, Smith 1989, Guba 1990, Eisner and Peshkin 1990, Lather 1992, Eisner 1992, Guba 1992 and Barone 1992

that effectively turn it into something else. It seems that some educational researchers and methodologists would rather be political activists or novelists than be researchers. That is understandable, but it does not justify the transformation of research into politics or literature.

What I want to do in this talk is to examine two major themes in recent literature on educational research methodology, and to suggest briefly why I think that many of the arguments surrounding them are misconceived. I will call the first constructivism and the second activism; though this use of terms is not standard, these terms are sometimes used to refer to other positions, and other labels are occasionally used for the ideas I am going to discuss.

Constructivism

We can find the idea of constructivism in many different fields, for example in symbolic interactionist sociology and in cognitive psychology, as well as in writing about educational research methodology. It starts from the idea that human beings construct their actions rather than simply acting out internal drives or responding to external stimuli. People's actions are seen as based on interpretations of their environment, and those interpretations are themselves constructions in the sense that they rely on presuppositions. Furthermore, those interpretations are to be treated as more or less rational products; and, in that sense at least, as valid in their own terms. In particular, they are *not* to be faulted by the researcher as invalid, unscientific, ideological, etc. They should be appreciated rather than evaluated.[2]

Furthermore, one is likely to find in any social context multiple, perhaps conflicting, interpretations of that context. From a constructivist point of view, this amounts to saying that people live in different social worlds, or more graphically that there are multiple realities. And, it is suggested, the constructivist must simply accept the existence of these different realities, rather than seeking to

[2] On appreciation, see Matza 1969.

legislate amongst them. Here we have the sort of cultural relativism that has long been central to social anthropology and is very influential in other fields as well.

What is particularly significant, from my point of view here, is that this implies a questioning of any claim to epistemological privilege on the part of Western science. Of course, today, many educational researchers, and especially qualitative researchers, reject the model of science anyway---they do not see themselves as engaged in scientific research. However, constructivism leads to problems even for them. This is because it seems logical that constructivism should be applied to research itself. And once we do this research must also be seen as involving the construction of a world, one whose relationship to the worlds it claims to describe is problematic, to say the least. On this basis, we might say that any research report tells us more about the researcher than it does about the phenomena it claims to describe.

Some methodologists and researchers have embraced this conclusion. They argue that the phenomena that research describes can be no more than textual constructions, that they are merely products of the rhetorical devices used by researchers.[3] And it is suggested that, as such, they should be judged in aesthetic terms, in terms of their political correctness, and/or in terms of their practical usefulness. They certainly cannot be judged in terms of their validity, in the sense of how accurately they represent events in the world, because constructivism denies the possibility of this.

Constructivism has some dramatic implications, then. Above all, from this point of view it becomes unclear how research differs from fiction or ideology, or if it does why we should prefer it to these. Of course, if the argument were sound, we would simply have to live with these implications. Fortunately, it is not.

I am sure it is correct in many respects to see actions and accounts as constructions, and as involving presuppositions; though we must not forget that all construction of action

[3] See for example Tyler 1986.

and accounts takes place under various kinds of constraints. But to see interpretations of the world as constructions, and to recognise the value of investigating the construction process, does not require us to assume that that process was rational, even though I think we would be wise to ***begin*** from that assumption. More importantly, to treat the construction process as rational does not mean accepting its products as valid. It is true that we may need to suspend our assessment of the validity of people's accounts and knowledge if we are to understand how these were produced. If we begin by evaluating them, we may well fail to understand them properly. But there is no reason why that suspension of evaluation should be more than temporary. The claim that it has to be, that we must simply accept others' views as rational and valid in their own terms, is usually founded on the argument that there is no way in which the validity of any claim about the world can be established with certainty. This is true enough, but it is false to conclude that all claims must therefore be treated as equally uncertain in validity. This does not follow at all, and it flies in the face of our everyday experience, where much of the time we have no difficulty in judging the relative likelihood of various beliefs being true, and with some success.[4] Applying constructivism to the research process itself, and treating all accounts of the world as mere constructions whose validity cannot be judged with justifiable confidence (apart from relative to a framework that is itself necessarily unjustifiable) undercuts the whole research enterprise.

So, I suppose you could say that I am not rejecting constructivism as such but only the sceptical and relativist conclusions that often seem to be drawn from it. But it is those conclusions that are influential in the methodological thinking of some educational researchers today, and their influence seems to be quite widespread, and growing. And

[4] While judgments of success are always open to challenge, in everyday life we do not bow to the influence of radical scepticism and there is no reason why we should as researchers. For elaborations of this argument, see Hammersley 1991 ch3 and Hammersley 1992 chs 3 and 4.

that influence poses a serious threat to the pursuit of research, in all fields.

Activism

The second currently fashionable line of methodological thinking that I want to question is what I shall call activism. This is the idea that research should not be directed solely towards the production of knowledge but rather towards the solution of some practical problem or the furthering of some political cause.

Now, I recognise that inquiry may sometimes legitimately be directly subordinated to another activity. Putting it the other way round: the pursuit of many activities sometimes requires inquiry. This is the sense in which all teachers, indeed all social actors, are researchers. In the course of an activity we may run into a problem for the solution of which we require some information, and if we *can* we suspend the activity in order to get that information. Similarly, in the course of political campaigning, in order to mobilise support, we often need to gather information to construct our case. Such forms of inquiry are of course quite legitimate, though whether inquiry is the best route to solving a practical problem in any particular case is a matter of judgment.[5]

However, these forms of practical inquiry, subordinated to other activities of one kind or another, are different from research institutionalised as a form of activity in its own right. Here we do not suspend our current activity because of a problem and engage in research until that problem is solved. Nor do we collect information simply for the purpose of producing a convincing case for a well-defined audience. We may be motivated to engage in research because of some practical problem that haunts us. And we almost certainly will want our research findings to be

[5] I think there is a tendency among those of us who are educational researchers to overestimate the value of inquiry for solving problems compared to other means. This is what we might call the intellectualist fallacy. See Lindblom and Cohen 1979.

relevant to practice in some way. But the goal of research as an activity is to discover the truth about something. And we engage in research as members of a community of fellow researchers who are similarly motivated and who, in my view, ought to constitute the initial audience for our findings.

Seen in this way, research cannot be integrated with other activities, such as teaching, without dilemmas arising that may be impossible to resolve without damage being done to research or to the activity it is combined with. We can of course ask whether research of the kind I am defending is of any practical value. And my answer to this is that while I think it may be of considerable value in helping practitioners to clarify their thinking and widen their perspectives, it is of less value than is often claimed. It does not, and cannot, tell people what to do on Monday morning, or on any other day of the week. And in my view we should rejoice in that fact.[6]

Conclusion

I have outlined and argued against two influential trends in current thinking about educational research methodology: what I called constructivism and activism. Let me stress that I am not rejecting all the elements of those positions. I think that there is much in constructivism that is correct about human behaviour. It is some of the conclusions drawn from its starting assumptions that I reject. Similarly, in relation to activism I do not wish to deny the value of practical inquiry. I engage in it frequently myself. But I do want to deny that all educational research should be, or inevitably is, practical inquiry, in the sense of being properly subordinated to practical or political activity.

So, finally, to return to my question: who needs methodology? Methodology is itself a form of theoretical inquiry. And to ignore methodology is to be a methodologist without being aware of it, and thereby potentially to overlook presuppositions that need questioning. The

6 For an expansion of these arguments, see Hammersley 1992, chs 7 and 8.

sociologist Howard Becker has argued that all researchers must be their own methodologists, and I think there is some truth in that (Becker 1970 ch2). On the other hand, I believe that the methodological ideas I have discussed in this paper are of questionable value. They seem to me to abandon the basic presuppositions that make research what it is.

More generally, I think it is easy to exaggerate the significance of methodology for the practice of educational research. It is often forgotten how thoroughly *practical* an activity research is. For that reason our methodological thinking as researchers should be closely tailored to the particular purposes of the research we are doing, and the context in which we are doing it. Above all, we must not get carried away with asking methodological questions. Otherwise, like me, we shall all end up spending our time doing methodology instead of research.

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'Understanding' as a social process: the culture of the mathematics classroom

notes from session at Oxford meeting, 4th April 1992

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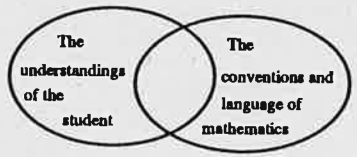
The aim of the session was to examine where we, in the mathematics education community, are at (more truthfully where I am at) in our theories about understanding. I started with a rough sketch, a caricature of the 'psychological' process of understanding, emphasising its private nature. In particular I suggested that constructivism, which has become, amongst mathematics educators, the accepted theory of the way people learn assumes a very private process of coming to know through reflective abstraction. At the same time the emerging ideas of social constructivism, and constructionism, suggest that many researchers are unhappy with a theory that cannot fully explain intersubjectivity and they have resorted to 'adding on' to constructivism a social element. This is a problematic approach in my view and I have outlined my concerns in my PME paper[1].

The focus on a unitary self who constructs concepts, with social interaction being just one of the factors that may bring about adaptation of theories, does not take into account recent research, some of which draws on post-structuralist ideas, that suggests that cognition is situated; that is, it is embedded in particular social and linguistic practices. I am referring here to the work of, for example, Teresina Nunes and others on the mathematics of Brazilian street market children, Jeff Evans on the way in which different social contexts call up different subjectivities for people, Valerie Walkerdine on the home practices within which children come to use and understand words like 'more', and David Pimm on the shifting teacher/student relationships in the classroom brought about by the use of linguistic 'gambits' by the teacher. Not only are concepts and knowledge socially constructed, as social constructivists would claim, so too are we ourselves, in the sense that our multiple subjectivities are constituted in and through particular practices. The centrality of social interactions is also the focus of Vygotsky's work. He maintained, "In our conception, the true direction of the development of thinking is not from the individual to the socialised, but from the social to the individual"[2].

I do not want to suggest that individuals do not construct their own knowledge (the individual is not reducible to the social), but that the construction of knowledge takes place in social interactions; indeed I would suggest that it is difficult to explicate otherwise. 'Misunderstandings' or 'misconceptions' are not aberrations from some normal process of correct learning but typical and characteristic of all learning; through activity and in particular through language, we conjecture and experiment with words and ideas and their appropriateness or otherwise is found in use and through talk. Thus learning mathematics is perhaps most fruitfully thought of as initiation into the language and practices of mathematics, *as well as* other social practices of the individual. A visual representation (due to David Blundell) might be the following - with the teaching aim of increasing the intersection!:



An illustration: we need to recognise that it's OK for students to talk of "My half is bigger than your half" so long as they realise that it is appropriate in some practices, whereas in the mathematics setting, if they are of different sizes they aren't halves.

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1 "The function of language in radical constructivism: A Vygotskian Perspective" *Proceeedings of Sixteenth International Conference of Group for Psychology of Mathematics Education*, New Hampshire 1992

2 *Thought and Language* L. Vygotsky1986 (Revised Edition, A. Kozulin (Ed.)), MIT Press, Cambridge, p. 32

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