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1

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CONTENTS

The principles and strategies for developing mixed ability teaching groups in mathematics

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1. Introduction

I have chosen to depart from the more traditional, theoretical structure of a paper because I wish to describe through anecdotes some of the key events that have been instrumental in causing me to develop my principles and strategies for teaching mixed ability groups in mathematics.
The structure of this paper will be to describe this background and then to explore the issues that were raised at the BSRLM day conference (21.11.92)

2. Background

The process of moving towards mixed ability teaching groups throughout the 11 to 16 age range took several years of planning, discussing issues and seeking effective teaching strategies. Outlined are some of the significant events that influenced the decisions taken and how these were put into practice. An important feature was to seek effective teaching strategies that supported and enhanced better learning environments.
This process continues.

Prior to moving to my present school in Telford in January 1986 I worked at Wyndham School in Cumbria. Here the mathematics department developed problem solving learning methods and from the early seventies used "investigations" as a way of encouraging students to use process skills to develop their mathematical thinking.
I believe that the word "investigations" has unfortunately been bastardised of late and the worst effects are seen in the way that pseudo open ended tasks (many of which end up at 1, 3, 6, 10...) are used at specific times rather than being recognised as a holistic way of working with and learning mathematics.

I wanted to bring a problem solving, investigative approach to the whole of the 11-16 mathematics curriculum and these opportunities presented themselves upon my being appointed Head of mathematics at a school where the management were seeking change. Given that two members of senior management were also teachers of mathematics I was given plenty of scope and encouragement.

The department had just kitted itself up with the SMP 11-16 scheme and operated setting from the end of 1st year. As I had worked with 1st and 2nd year mixed ability groups in my previous school I had a sufficient resource bank that could be used and we moved to teaching mixed-ability 2nd year groups from September 1986.

In considering the possibility of mixed ability groups for 3rd year classes my headteacher insisted that the rationale for the decision had to be an educational one, that best served the students and not one that served my own political ideology. This proved to be a positive motivation and having built up a sufficient resource bank we moved to mixed ability groups in the 3rd year from September 1987.

For the next two years we continued to teach setted groups in 4th and 5th year, mainly because we didn't feel we had the necessary skills that the wider range of potential achievement of the older pupils demanded. However during this time the Head of science, kept asking me what was preventing us from moving to mixed ability groups in the 4th year, reasoning that the principles and the strategies were exactly the same as they were for lower school students.

Throughout this period of time the department, along with six other schools, piloted the ATM-SEG 100% GCSE coursework syllabus. The continuous support that we received when we met to discuss methodology was an important contributing factor in departmental debates about philosophy and pedagogy and a catalyst for curriculum development thoughout the period from 1986 to 1991. Unfortunately the GCSE has become a political football and the demise of the syllabus also brought to an end the curriculum suppport and development network that had blossomed during this period. Indeed the focus upon curriculum fragmentation and testing has been a trademark of the educational dogma that has severely subverted curriculum development in late eighties/early nineties.

The worry about teaching a wide ranging mathematical content was the key issue, so I set myself the task of seeing how it might be possible to teach a high level content skill to a mixed ability 4th year group. I chose transformation matrices and decided to work backwards from there to a starting point that only required some relatively straightforward content knowledge. (A more detailed account is written up in MT132 in an article called "Seeding"). Basically the idea was to start with simple shapes on a co-ordinate grid which were then reflected in the x axis. The co-ordinates of the initial shapes and their images were recorded and a connection looked for between these sets of co-ordinates. Reflections in other lines were followed by rotations, combinations of reflections and rotations, and enlargements. At each stage the students were asked to explore what happened and the role of the teacher shifted from whole class interference to individual or small group interventions. The issue of teacher intervention is often seen as one where the teacher shouldn't 'tell' students how to do something.
I believe that there are certain truths that need to be told and discussed, such as how to cross the road, how to wire up a plug and how the co-ordinate system works in terms of moving horizontally then vertically. My main fear is that certain excellent teaching tasks such as "Octagon Loops", have come to be seen as exclusively assessment vehicles where 'telling' is seen as 'cheating'.
At some point I would tell some students very didactically how to transform a shape using a 2 by 2 matrix, they would then be set the task of finding the other transformation matrices using the data that they had already collected from their work on reflections, rotations and enlargements. What happened was that students were provided with several 'content' problems which they then investigated using and applying 'process' skills.

The <u>content</u> of the mathematics curriculum became <u>process</u> driven.

The focus of students' learning at their own pace rather than teachers teaching at their own pace became a key strategy that could be applied to other areas of the content.

A workable model - a first iteration as it were - had been constructed, but what was required now were other simple starters that could head towards higher-level content; eg:
- home-made rotating arms leading to trigonometry
- areas on vectors leading to Pythagoras
- cuboids with a constant volume leading to surface areas of cylinders

Within each module a range of content skills were being developed and used which moved us away from just teaching isolated skills, exercise by exercise and chapter by chapter.
In September 1990 we moved to 4th year mixed ability groups.

An important part of the planning was to find questions to ask that would lead on from initial starters. Of course some of these questions emerged naturally whilst at other times studenst created their own questions. Encouraging students to have a sense of personal responsibility to develop their mathematics, rather than respond to whatever level they felt themselves to be at according to the set number on the front of their exercise book, was a further important ingredient. As we developed these 'coursework' tasks, that didn't depend upon text books or repetative responses to worksheets we recognised that students no longer asked the often debilitating question "Why are we doing this?" A further reason for the absence of this type of question was because the students become engaged with the problem solving process and built-in is the question "Why am <u>I</u> doing this?"

We began to use the differences that exists between students in positive ways, whereas previously we had attempted to 'keep the class together' which had clearly not served the mathematical needs of the students.

In department meetings throughout this year of operating with 4th year mixed ability groups we kept asking ourselves the question "What about 5th year mixed ability?" On each previous occasion we had agreed that we would have to create setted groups. However a significant event took place which was that I combined my current 'bottom' 5th year group with another teachers 'top' 5th year group for the final term. The 'top' set teacher had asked me to help him develop some ideas with his group that I had been working on with my own top set in the other 'half' of the year group and one way to do this was to combine groups and team teach them. This proved to be a very useful way of working and we realised that it hadn't been the students abilities which had previously caused us to set students in the 5th year, but our own confidence as teachers. Consequently in the last meeting before the end of the summer term the categorical and unanimous view was that we would keep them as mixed ability groups. We therefore designed further two to three week modules and in September 1991 worked with Y11 mixed ability groups.

3. <u>The BSRLM Session</u>
The following statements, questions and further discussions followed on from this initial introduction.

<u>Developing a curriculum map</u>
The model of taking high level content ideas and working them backwards to simple starting points was considered to be a powerful way of developing a teachers' curriculum map. At the moment the National Curriculum is in danger of providing teachers with a curriculum map which is already worked out and effectively subverts teachers from making their own decisions about how to work with and develop this curriculum

"Do the less able students feel overwhelmed?"

I believe that this is one of the crucial issues.
If students are made to feel under-valued because they recognise that other student in the same class are working at much more complex levels than themselves this will clearly have a demotivating effect. The teacher has an important role to play and the philosophy that we continually compete with our own previous best rather than with each other is a fundamental axiom for our way of working. It is important therefore that students are encouraged to help each other, and in doing so the following three way of working occur:

i) Students who are struggling to understand an idea can receive help from a source other than the teacher. Discussing ideas and helping each other is a natural way for people to work and is a pleasureable activity.

ii) In explaining something to someone else, the concept becomes clearer to the 'teacher'

iii) It is a useful strategy to have more than one 'teacher' in the classroom. When students can help each other the teacher can direct his/her time to students that they particularly wish to work with.

The strategy has similarities to a technique that Dorothy Heathcote (retired senior lecturer in Drama, Newcastle upon Tyne) defines as "Mantle of the Expert".

"Do the students 'set' themselves within a mixed ability group"

This does happen, but not all the time and the situation is changeable and fluid. Who sits with whom is mostly determined by the students themselves. Of course if I want to organise the class in certain ways, such as in a circle or in pairs then I will make a postive decision to do this for the course of that lesson or sequence of lessons. What is important is that the students have opportunities to work with whom they wish to. This in fact reflects the way that people naturally organise themselves socially. People who are interested in playing bridge, badminton or share any common interest will seek each other out and join together to participate in their preferred activities.

Student self-perception

There is an issue about how children perceive themselves in relation to their peers when setting is the norm. No matter how much we might tell those in the bottom set that they should not see themselves as less worthy beings and that they are in a small group in order to receive extra help, they do 'know' that they have been separated out from their peers because they are seen as being less intelligent. (Often the children themselves, and even sometimes the teachers will use labels such as 'thick', 'no-no's', 'dumbo's' etc.). The reality for these students can often be a poor learning environment where they practice the same skills on a repetative basis that they have failed to come to terms with year on year out. The tenor of such groups is one of a dependency upon the teacher to tell them everything, built-in failure, low self-esteem and a lack of motivation. Many top-set children meanwhile have a false sense of superiority. Being in the top set must mean that they are the 'best' and there exists a question over personal responsibility to deal with and confront problems which don't have obvious solutions and demand a deal of thought and hard work. I am concerned also about the children who are close to the 'bottom' of the top set, those who had been at some point deemed as one of the best, but for such children the reality proves to be different.

"Teaching mathematics TO children - v - teaching children mathematics"

We explored this idea. The first part is about teachers deciding not just what mathematics they wish to teach to the children, but also, with regard to the current National Curriculum document, the level at which they might access childrens thinking according to which set the children are in. So for instance on a simplistic three-set model the starting point for teaching Pythagoras will vary according to the set, and for the bottom set there can often be a presumption that understanding Pythagoras will be too difficult anyway so why bother to start.

The second part is about using a common starting point which is accessible by all the children in the class. Some children will go through an accelerated learning process and will soon become Pythagoras proficient and eventually work on 3-D type problems. Some children will get sight of an understanding of Pythagoras, and other children will not become Pythagoras proficient.

The most important difference between these two models is that as their teacher I do not make choices for them which automatically pre-judges students' potential achievements or future levels of understanding.

"Have you been able to measure the benefits of the changes that you have made in any way"

Measurement in the current political climate is that progress, ability and understanding can be measured by testing. Whilst I disagree with this there is another more important aspect which can't be measured on a simplistic scale and this is attitude. My measurement of attitude is in the way that students engage with the problems; in the way that they aren't in a desperate hurry to leave the classroom at the end of a lesson; and that theywill tell me that they intend to continue with some mathematics at home and not just wait for me to remind them about their homework.

The department doesn't for instance operate anything like a detention system. It is about befriending in a very positive sense through the medium of doing mathematics.

I didn't have the figures to hand at BSRLM, and whilst I don't hold any 'store' by examination results in isolation from many other factors it is interesting nevertheless that the departments GCSE results have increased dramatically for the last two cohorts of Y11 students - who were the first two cohorts to be taught entirely in mixed ability classes from Y7.

The results show % A to C grades taking the whole of the year group into account, they are:

1988 (33%)
1989 (30%)
1990 (23%) - final year of setting
1991 (46%) - first year of complete mixed ability
1992 (50%)

"Do you get anxious parents?"

It is important to involve the parents in the way that mathematics is taught and learnt and it is with this in mind that we hold 'Mathematics for Parents' evenings, where parents are invited to take part in a choice of typical lessons and are put in to problem solving situations. A noticeable feature of one evening which had been set up to discuss GCSE and National Curriculum was that the parent became so involved in the mathematics that in the plenary session all they wanted to discuss was the problems that they had been doing and not one question was asked about the other issues.

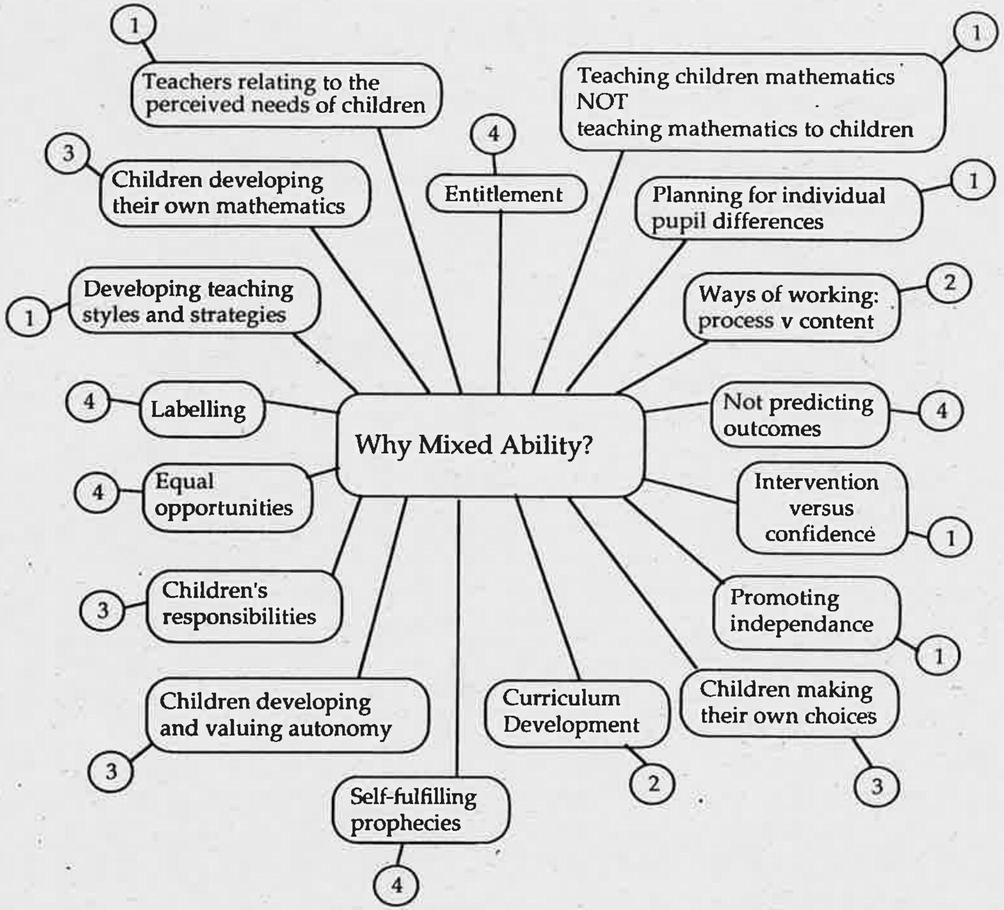
Clearly their own involvement in the mathematics had taken priority over anything else. I believe that the same is true in classrooms and if we can provide interesting and stimulating tasks that 'hook' the students, then we don't need to worry about which 'set' a student is in and whether or not they have just attained a certain level within NC.

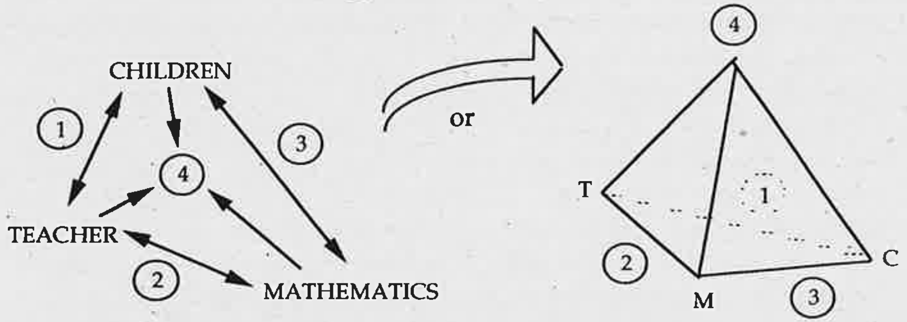
"Why do you think that a teacher who is in the position of creating a new structure might get 'cold feet'?"
This seems to be about setting the conditions in the first place of working with other people in the department, which might not necessarily be the whole department. Too often initiatives are stifled because there is a feeling that everyone must be in agreement. This can lead to a department working at the pace of the lowest common denominator rather than at the pace of the highest common factor. What is important is that colleagues work together, try out jointly planned ideas, reflect upon and discuss outcomes and look for ways of developing other ideas by building upon a small-scale success.

"I think that the biggest fear would be having control of the whole class"
One of the most important principles underpinning mixed ability grouping is that students take an ever-increasing responsibility for their learning; that they value control of their learning and engender some kind of worth in what they do. Their involvement with the mathematics is about them as mathematicians, it is not concerned with what they believe someone else thinks of them as mathematicians according to whichever 'set' they have been placed in.

Simon Goodchild in his article "Active Learning, Reflection and Interpretation" in the Mathematics Education Review no. 1, May 1992 responds to the question *"how does the teacher with a mixed ability class of 30+ lively pupils ensure that they are working interpretively as they engage in the activities chosen for, or by them?"* by saying: The answer to this question is not merely a matter of adapting teaching and learning styles to include, for example, more discussion and open ended questions; although it is believed that these are essential in developing pupils' powers of working interpretively. Rather the answer lies in the development of pupils as independent learners who are critical in the broadest sense, of the activities in which they engage and who feel personally responsible for their own learning

This then begs the question that these aims would be the same for teachers who work with setted groups, and I feel sure that this is true, however there are some crucial differences between working with setted and mixed ability groups and it these that I am currently working on. In preparation for BSRLM (21. 11. 92) I wrote the enclosed sheet on "WHY MIXED ABILITY?" I wrote down many of the issues concerning teachers, children and mathematics and then classified them according to whether the issue was a link between, teachers and children, children and mathematics and teachers and mathematics. There were four classifications and it occurred to me that the fourth one was the inter-connection between all three and that the issues in this inter-connection were all idiosyncratic to mixed ability teaching.





Whilst I may not necessarily agree, it could be argued that 1's, 2's and 3's could be issues equally well met for teachers working with setted groups. However 4's are issues about mixed-ability.

Looking at Children's Writing of Mathematics

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Introduction

When children's written work is to be used as a basis for evaluating their mathematical attainment it is vital that their written communication of their mathematical activity should be *appropriate* , *clear* and *effective* . These are the terms used by the various examination boards as criteria for assessing children's communication in GCSE mathematics coursework. For example, assessment criteria include:

> 'Selects the **most appropriate** methods for communicating results' (LEAG, 1990)
>
> 'A **clearly-expressed** contribution with **effective use** of mathematical language, symbols, conventions, tables, diagrams, graphs, etc.' (MEG, 1989)

There are two problems with the use of such criteria. Firstly, they are undefined: I am sure that each mathematics educator reading this has a clear idea of what she/he individually understands by these terms and could exemplify them, but could we explain these ideas to pupils and would we even agree on the same examples? Secondly (and this is one of the problems with trying to define them), the terms are all relative to the context in which the communication is taking place. We have to ask: appropriate to what context; clear to whom; effective in achieving what? There appears to be some general agreement that appropriate forms to use include, for example, algebraic symbolism (Wolf,1990) and tables (Morgan,1991). But this agreement appears to be largely conventional with little analysis of why or how these or other forms of communication should be used.

When we read a text, we read not only about the apparent content of that text but also make inferences about many other aspects. For example, from the presence of a table we tend to infer that the child has worked systematically, and (with less justification) from the absence of a table it may be inferred that such systematic working has not taken place (Morgan,1991). From the presence of a very large number of examples we may infer that the child is conscientious and rule following but probably does not display high quality mathematical reasoning (and, moreover, is likely to be a girl). These examples of the influence of the form of the text on judgments about the mathematical activity that has contributed to it are in the context of explicitly mathematical forms of communication and, as such, are relatively easy for mathematics teachers and examiners to recognise and consciously take account of. I would argue, however, that there are many other aspects of any text that influence the reader's understanding and judgement of the content and of the author. In this paper, I intend to give some examples of such influences within the context of a piece of professional academic mathematical writing and suggest that looking at children's work in a similar way may raise our awareness of what features of children's writing influence our reading.

The functions of text

The following analysis is within the framework of Halliday's (1985) functional linguistics. The examples of academic writing which illustrate how this may be applied to mathematical text are taken from a single paper (Dye,1991) published in the Journal of the London Mathematical Society.

Halliday (1973) identifies three macro-functions of any text. The *ideational* function is to express what objects are significant in the text, what kinds of processes take place, which objects act and are acted upon. *Interpersonal* aspects express the role taken by the author and the role expected of the reader as well as the relationship between author and audience. The *textual* function determines what sort of text it is to be read as and the task that the author has undertaken.

Ideational aspects

One of the most striking features of Dye's paper is the absence of active human participation (this is a feature common also to much academic writing in other domains). For example, in the introduction to the paper, it is stated:

> 'The intial aim of this paper is to discover a picturesque object that accounts for this occurrence of A_5.'

It is the paper itself which has aims rather than the author and one mathematical object accounts for another rather than the author finding a relationship between them. Moreover, throughout the paper, mathematical objects themselves act on other mathematical objects:

> 'Then σ . . . permutes the vertices and sides of Δ cyclically.'
>
> and
>
> 'Any member of the group Σ_3 . . . fixes Δ and acts on X, Y, Z.'

Thus an impression is given of mathematics as an autonomous system in which human beings have a role as onlookers but not as actors.

Interpersonal aspects

In spite of this absence of active human participation, clear roles may be inferred for both author and audience. The author is a strong source of authority: he defines what may be taken for granted and what the reader's attitude towards the content should be. For example, he makes it clear which parts of the proofs he presents are significant and which are trivial:

> '. . . it is very easy to verify that . . .'
>
> and
>
> 'These are just the edges of the hexagon . . .'

The reader, however, is also assumed to be part of the mathematical community and to take an active role in reading and making sense of the arguments presented. This is largely expressed by the inclusive use of "we":

'Our next step is the following . . .'

and

'By Theorem 1 we may assume that H is H*.'

The relationship between author and audience is thus one of membership of the same community and this community is one where the author himself has authoritative standing.

Textual

One of the most significant textual features of the paper is the construction of the text as an 'analytical exposition' (Martin,1989) in that it presents a thesis supported by extended arguments as to why this thesis is true. This is manifested in part by the extensive use of expressions of causality and reasoning which form links between different parts of the text. A short extract from the proof of one of the theorems within the paper illustrates this aspect:

> '**But, by (4),**
>
> $$j^4 + j = (j+1)^2 + j = j^2 + j + 1 + 2j = 2(j^2 + j) = 2j(j+1) = 2j \times j^2 = 2j^3.$$
>
> **Hence, by (16)**, $B = C$. **Then, by (4), (15)**, $A = C(j^2 - j) = C$. **Hence** the only possible orthogonal polarity having $\Delta, \Delta_1, \Delta_2, \Delta_3, \Delta_4$ self-polar is that with respect to the quadratic form
>
> $$Q(x) = x^2 + y^2 + z^2.$$
>
> **Using (4)** it is easy to verify that the five triangles are self-polar with respect to $Q(x)$; **(III) follows**.'

By their prominent thematic positioning at the beginnings of many clauses, these expressions of reasoning produce a strong sense of the significance of the text as argument.

Analysing children's writing

In describing the application of this method of analysis by referring to a piece of academic mathematical writing I am not suggesting that the specific features described above are present or should be present in children's writing for GCSE mathematics coursework. The ideational, interpersonal and textual features that have been identified above are those considered appropriate, clear and effective within the genre of academic mathematics papers (evidenced by the fact that the paper has been published in a prestigious refereed journal). The question that needs to be addressed is what features of children's writing may be considered appropriate, clear and effective within the genre of GCSE coursework.

Those present at the BSRLM session studied a number of texts produced by children in response to coursework tasks set by LEAG. The following questions were considered:

- What are the ideational, interpersonal and textual aspects of the texts?
- What features in the text perform these functions?

- Which of these features do you react positively towards and which negatively?

In this section I will present some of the features that were highlighted during the subsequent discussion.

Most of the texts did not indicate that any exploratory or experimental work had gone on; a "tidy" version of what was done was included together with the results. In some cases "rough" work was attached at the back but the overall impression given was that mathematical activity is a straightforward path from problem to result. Given the assessment criteria which demand evidence of mathematical processes, such editing may not be appropriate. On the other hand, the benefits of drafting and redrafting writing to the development of both the writer's ideas and the form of expression are generally accepted and these processes are encouraged in other areas of the curriculum. The question of the role of drafting and redrafting writing in mathematics needs to be considered.

The author was strongly present in most of the children's texts. As in Dye's paper, the personal pronoun "we" was extensively used by some of the children. It was, however, used both exclusively and inclusively. The authors' own appearance as active participants in the doing of mathematics was expressed in the exclusive use, i.e. "we did . . ." describing the actions of a group of children working on the problem together. In other cases, where the child had worked alone or was claiming personal ownership of the activity "I did . . ." was used instead. It was felt by some of those present that this personal involvement is one of the features valued by teachers and examiners. On the other hand, the inclusive use of e.g. "we can show . . ." expressing a sense of community with the reader occurred less frequently. When it was used it gave an impression of confidence on the part of the author.

An interpersonal feature of one child's work was commented upon negatively. He wrote:

> 'When I had finished writing out this table I had seen another pattern. Can you see it?'

This way of addressing the reader was seen as inappropriate. It is, however, typical of the way that children are themselves addressed either by the writers of mathematics text books and work cards or (usually orally) by their teachers. This boy seems to have identified and copied one of the features of the mathematical texts provided for him without realising that it might be considered inappropriate in his own writing. One of the problems that both children and teachers encounter in attempting to develop appropriate forms of writing for coursework is that there are few (if any) examples of adult writing in the same genre that may be taken as models.

For many of the children, the textual nature of their writing was at least partially determined by the fact that the task provided for them asked them to answer specific questions. In the less structured parts of the task, however, the favoured form of text appeared to be narrative. This contrasts with the argument form found in academic mathematics writing. One point that arose in the discussion was the case of a child who sees the answer to the given task immediately and writes only a symbolic generalisation without any accompanying narrative. Such a child is likely to be given few marks for such a piece of work in spite of clearly demonstrating mathematical "ability" in some sense. How can mathematics teachers help such a pupil to write a more acceptable, "appropriate" response? One of the pieces of advice given by some teachers is to "show everything you have done"; in other words, to construct a narrative leading from the posing of the problem to the achievement of the result (Morgan,1992). Unfortunately such advice is not actually helpful to the child who has "done" nothing but has intuitively seen what the answer must be.

In the short time available it was inevitable that only a few features of a handful of children's texts could be examined. Given this limitation, however, a number of important issues were raised about the way in which we read children's work and about the support that teachers might provide for children to help them to produce appropriate, clear and effective writing. As Kress (1990) argues, unless teachers are themselves aware of the functions of features of writing they are unlikely to be able to help their pupils to learn to write effectively in the apropriate genre.

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Personal technology in the mathematics classroom: - graphic calculators and equal opportunities

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> "I've had my graphic calculator for six months, but I haven't even taken it out of the box. I saw no point in learning to use the graphic calculator because I couldn't see how I would use it in my classroom. But my male colleagues immediately wanted to 'master' the graphic calculator by working through the manual".

For three years I have been working together with a group of teachers. We are exploring ways to integrate the graphic calculator into the mathematics classroom. We have found that the graphic calculator is a powerful tool applicable over almost the full range of the curriculum.
The graphic calculator is a scientific calculator with a larger than normal screen. It draws the graphs of functions entered in their algebraic form. As well as being a "hand-held" graph plotter, the calculator can be programmed and used to process and display statistical data. It has many of the functions of a classroom computer but it is hand held, relatively cheap and can become personal to the student to be picked up whenever required. However, as the above comment made by a teacher at a graphic calculator workshop illustrates, the introduction of the graphic calculators into the mathematics classroom has some implications for gender equality. We wish to work in ways that allows **all** children access to benefits of this exciting technology. Working in mixed ability classrooms in the 11-16 age range we have found, contrary to our own experience (several of us took well over 6 months to open our own calculators), that young people find the graphic calculator easy and enjoyable to use. Tasks on the calculator lead to collaboration and discussion. We see that introducing the graphic calculator as a tool for collaborative and investigative mathematics is the way forward to empowering all pupils.

Background. Research has shown that boys tend to dominate the use of computers at school and home (EOC 1983, Culley 1986, 1988, Hoyles 1988). However, equal opportunities spreads beyond issues of gender. DeVillar and Faltis in their book Computers and Cultural Diversity point out the inequable distribution and uses of computers along racial, gender and ability lines. Furthermore they state that "without equity in the use of technology for instructional purposes, the existing disparities in academic achievement between groups can and will only widen" (DeVillar & Faltis 1991) We propose that the graphic calculator as a form of personal technology can provide young people with a useful bridge to the use of computers

Learning the lessons from computers. We are aware of the research that shows that work on computers can provoke discussion and collaboration (Healey,1988-89, Hoyles 1989, Hoyles et al 1991) and in particular that girls benefit from this collaborative work (Johnson et al 1985). DeVillar adds to the debate. He quotes from an extensive piece of research that shows that students who had been working in a cooperative learning situation (as opposed to competitive or individualistic) have in particular "more positive race/ethnic relations within the classroom'" as well as "higher acceptance by 'normal progress' students of their 'low achieving' mainstreamed peers" (DeVillar, 1990)

Can the graphic calculator satisfy the demand for collaborative work:
Research shows us the gains for pupils working with computers in a collaborative way. What is the role of the graphic calculator? Is there a contradiction here? We see the graphic calculator as personal to the pupil to be picked up by her whenever required. The privacy is important. Girls feel happier making a mistake in private without the fear of ridicule from the boy "computer experts". So we need to encourage students to work collaboratively on the graphic calculator while still enabling experimentation and mistake making in private. How do we do this? The graphic calculator is small and unless held carefully it is difficult to see by one person let alone shared by two. However, children are wonderfully flexible. I have seen many successful lessons with two pupils sharing. At times they press alternate keys, or each inputs a graph or a point. I should emphasise that the important point is not just the physical sharing of the calculator but the type of activity. The pupils can benefit greatly by having their own calculator but the activity must be one that encourages pupils to query, and to look at or use each other's screen output. It cannot be like two adults I saw "sharing" a calculator but siting with a large empty chair between them, and not speaking to each other.

Some activities: I have attached three graphic calculator activities that I have found to be particularly successful in encouraging pupils to investigate, share, talk and argue. For the task ***Not so Square***, students have found it impossible to sit on their own wondering whether they really have a parallelogram. Pupils compare with a neighbour or argue with a group on the other side of the room. Arguments that have to be debated by the whole class that start off with the exclamation "Here I've found a parallelogram .... silly,that's not a parallelogram, that's a square!" Sharing a mathematical activity on a small personal computer, while sometimes leading to squabbles, also leads to enjoyment and laughter. Maire Rodgers (1990) points to "the use of humour and laughter are very useful in dissipating the tensions created by learning difficulties." (Rodgers 1990)

The challenge for the future.
I would like to see mathematics lessons where pupils, in groups or at times individually, pick up graphic calculators to solve a wide range of problems. There is talk, debate and I hope fun. I have seen classes like this (in the 11-16 classroom) and in these girls and particularly bilingual girls have been in the forefront. Why? And how can we keep them there? It is ironic that one explanation could be that the calculators do not have the "street cred" of the computer that will play the latest game. Bente Elkjaer points us towards an explanation. She sees that computers and the related discipline of computer science are endowed with "male symbolism" (a quick glance at adverts selling computers and software, bear this out) and hence the gender identity of boys depends on their dominance in this area. She goes on to say that girls gender identity is not challenged at all by the subject. They are able to develop, if they wish, without anxiety. This explains why girls feel able to walk around and help one another: "girls enjoy talking together about the programing, and they are very aware of taking turns keying. They laugh a lot and are very relaxed when something unexpected happens, for example when the programme was lost" (Elkjaer 1992). I wonder if this interpretation helps to explain why, as yet, the graphic calculator scene is not dominated by boys. Girls are doing well (Ruthven 1990). My conjecture is the lack of high status of the calculator have ensured that boys see no need to dominate its use in the maths classroom. As I

have reported before (Smart 1992) some boys wish to classify the "power" of the calculator in terms of its cost, what it will do, lack of colour and moving parts etc. They seem less interested in exploring one facility fully but in knowing "what is the most difficult thing it will do". When they see it as less ***powerful*** than their own computer they leave the way open for girls and less demanding boys to take over. We are of course faced with a dilemma here. We want to award status to the graphic calculator as a very powerful and exciting tool in the mathematics classroom but not to take on the "masculine" identity of computers. Maybe if we can keep hold of the *personal* in personal technology and the *sharing* in the way of working, then girls will stay in control.

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Physical Metaphors in the Construction of Mathematical Knowledge

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1 The Beneficial Role of Physical Metaphors

Resnick & Ford (1981) refer to physical metaphors as physical representations of mathematical content. Their role is described as allowing *"...overt manipulation of materials in ways that help link performance algorithms to their underlying mathematical principles"*(p.248).

Physical metaphors can be defined as teaching materials which contain external representations of a mathematical content. Manipulative devices like Dienes blocks, Cuisenaire rods, phenomenological analogies (i.e imagine a line as a string), physical experiences with apparatus, pictures and objects extracted from people's everyday and cultural life, are suggested for use in the mathematics classroom. Since they are related to a certain mathematical content, they can be useful tools for both the teacher and the learner, in the process of learning mathematics, as theory and research findings have shown. The use of physical metaphors holds a valuable and important position in the process of meaningful mathematics learning, at least from a phenomenological point of view (Freudenthal 1973, 1983). It has been argued that they assist to establish relatedness between the mathematical knowledge and contexts of sensory motor and physical activity, a view that goes back to Piaget, and has been expanded in the structuralism of Bruner and Dienes (Bruner 1986, Dienes 1960). This view is also compatible with a constructivist epistemology, which argues for the close relation of mathematical knowledge to contexts of sensory motor and physical activities (von Glasersfeld 1987).

In the following, the physical metaphors will be more closely examined in terms of how they assist development of mathematical knowledge, taking into account the engendered difficulties, as seen from a constructivist point of view.

2 The Challenge of Physical Metaphors

Although the use of physical metaphors is argued as beneficial, a deeper insight into the kind of mathematical understanding which may be promoted, reveals that the situation is not quite so. Incidents have been pointed out in mathematics classrooms where the development of abstraction and generalisation of the mathematical concept embedded in the physical metaphor has not been assisted. Gates (1987) criticizing the experience with physical metaphors as a mathematical experience itself, argues that *"..the use of apparatus may make it difficult if not impossible to generalise"*, and that *"..reliance on physical experience may not provide the critical or sufficient experiences for developing mathematical thinking* (p.18).

Since there is evidence for their beneficial role, it is certainly reasonable to ask why physical metaphors may not eventually prove efficient. What might be the factors that prevent learners from meaningful mathematical understanding? When and under what circumstances do they prove problematic?

Gates (1987) analyses the students' experience in a physical activity of constructing right angled triangles in an attempt to enable students in conceptualising Pythagoras' Theorem. However, most students could not construct and measure accurately the sides of the triangles, and they either failed or had difficulties in "seeing" the evolution of Pythagoras' Theorem. Gates (1987) argues that mathematical generalization is "beyond experience" and that reflection is required on students' own mental activities. He observes that the successful use of physical experience is *"temporally and spatially dependent"*. It is temporally dependent, as it depents on students' ability and willingness to make connections between multiple physical metaphors. Sometimes students find it difficult to control the regularities and fail to recognise any invarient. It is also spatially dependent, because students may not have the appropriate spatial skills and accuracy to complete the task. Due to these reasons the mathematization process may be delayed.

Holt (1982) and Bereiter (1985) point out the paradox, known as the "learning paradox", of expecting the students to form mentally the mathematical meanings represented by the materials before they have constructed such meanings. The "learning paradox" can be seen in the use of manipulative materials, such as Dienes blocks. Dienes blocks are physical materials which contain a transparent representation of place value notation. They are assumed to be the primary basis from which students construct mental representations of preformed numerical relations, and thus give meaning to the steps of the written algorithms. Holt and Bereiter argue that this leads to incompatibility: students cannot grasp the mathematical meaning perceived by the expert teacher. The mathematical concepts embedded in a preformed environment are more complex than those learners have available, and thus they cannot grasp them. (Bereiter 1985).

In the above examples, the physical metaphors used have failed to assist pupils in developing awareness of the embedded mathematical content. It may be said, that the choice of the physical metaphors was not adequate enough in relation to the students' own abilities and mathematical knowledge. Students either did not have the spatial skill to complete the task (spatially dependent), or the collection of physical metaphors was not adequate enough, in order to enable the learners to make the appropriate relations by connecting various incidents and thereby to generalise (temporally dependent). Furthermore, the learner's prior mathematical experience does not permit to grasp the mathematical content embedded on the physical metaphors. As a result, students do not always interpret the mathematical meaning of these physical metaphors as they are expected to do. The above factors seem to prevent students from developing the physical experience into a mathematical experience.

However, besides the inappropriate choice of physical metaphors, the epistemological perspective from which physical metaphors are approached seems to be another factor which influence the use of physical metaphors (Cobb & al. 1992). In other words, their succesful use may also be influenced by the way the teacher approaches the phenomenon of learning. The teacher's expectations concerning the learner's development of mathematical understanding in relation to the physical metaphors may prevent the evaluation of their use as successful or not.

In the following, the epistemological perspectives of the "representational view of mind" and constructivism, will be closely examined.

3 The Role of Physical Metaphors from a 'Representational View of Mind' Perspective

Cobb et al (1992) analyse the pedagogical consequences that the epistemological perspective, of the "representational view of mind", may have upon the use of physical metaphors in the mathematics classroom. They argue that the 'learning paradox' is the result of accepting the illusion that students' internal mathematical constructions could ever "match" with the mathematical content of an external representation. The mathematics learning through physical metaphors, and in particular the (psychological) relation between the external representation of the mathematical content and its internal interpretations in the learner's mind, may be conceptualised differently from various learning theories. One such a conceptualisation is the so-called "representational view of mind". As Cobb & al (1992), explain: *"The representational view of mind in mathematics education is evidenced by theories that characterize learning as a process in which students modify their internal mental representations to construct mathematical relationships or structures that mirror those embodied in external instructional representations"* (p.2). Students, in the teaching process, are expected to construct accurately in their minds, the mathematical content represented by the physical metaphors.

As Cobb et al (1992) have argued, theoretical problematic issues result from the use of physical metaphors as external representations to be accurately "matched" with students' mental constructs. The view of mathematical learning as an accurate "match" or recognition of mathematical relationships and conceptions, implies that mathematical meaning is possible to be transferred from external representations to students' minds. This view is widely unacceptable in a constructivist learning framework.

Firstly, it is incompatible with the constructivist epistemology which sees knowledge as a subjective construction of the individual. Cobb & al (1992) argue, that a consequence of the above view is the problem of dualism between internal mathematical representations in students' heads, and external mathematical representations in the physical metaphors which result in the "learning paradox". Also the learner's active process of constructing mathematics knowledge is limited, because it promotes the theory that *"meaning is analysed in terms of fixed mappings between arbitrary symbols and objects or events in the world"* (Cobb et al 1992, p.6).

It also comes in contrast with recent anthropological findings, which claim that mathematics

knowledge is socially and culturally situated and is the result of justification, argumentation and negotiation (Bishop 1988, Saxe 1991).

Finally, the ethos which such a view promotes in the mathematics classroom, seems to value teacher's authoritarian role over students. As Cobb & al (1992) claim, emphasis is placed almost exclusively on the teachers expert mathematical interpretation of the materials which are projected into the student's environment of learning. As a result, the learners' active construction is underestimated.

Accepting the values of using physical metaphors, a vital question is then, how can they be used in mathematics classroom without imposing a "representational view of mind", so that to avoid the resulted problematic issues. In other words, how can physical metaphors be used in order for the learner's active construction of mathematical knowledge to be valued and promoted.

As seen above, sometimes the expert teacher can easily accept that physical metaphors can be used as external representations of mathematical relationships. Arguing that mathematical meaning in physical metaphors is self-evident, then the possibility that students may interpret them in a variety of alternative ways is not considered. However, a constructivist perspective cannot accept that the correspondences considered as natural by acculturated expert members are self-evident for the novice student. As Cobb & al (1992) explain, *"the experienced naturalness of certain mathematical interpretation is relative to the taken-as-shared conceptual schemes that we have actively constructed in the course of our mathematical acculturation"* (p.10).

4 The Role of Physical Metaphors in the Constructivist Framework of Learning

In a constructivist framework mathematical understanding is the result of reflection on the individual's actions through a physical and social environment.

According to von Glasersfeld (1989), the basic principles of radical constructivism's epistemology are:

- *A: Knowledge is not passively received either through the senses or by way of communication, and it is actively built up by the cognizing subject.*
- *B: The function of cognition is adaptive, in the biological sense of the term, tending towards fit or viability, and it serves the subject's organization of the experiential world, not the discovery of an objective ontological reality* (p.4-5).

The constructivist epistemology centres attention at the learner as active constructor of his/her mathematical knowledge. A constructivist perspective: (1) does not accept the "match" between students' internal constructions and the external representations of the mathematical content on the physical metaphors, (2) emphasizes the value of negotiating ideas through discussion and argumentation in the process of mathematical knowledge's construction (von Glasersfeld (1987, 1989), Confrey (1990).

The problem of learning through physical metaphors in a constructivist framework, can be approached as a problem of constructing mental representations by acting on them. As Vergnaud

(1987) claims, this is an important problem, because of two strong epistemological reasons: *"(1) Mathematics plays an essential part in conceptualising the real world; (2)mathematics makes a wide use of homomorphisms in which the reduction of structures to one another is essential"*(p.227).

Vergnaud (1987), building on the work of Janvier, Goldin, Lesh, Kaput and Mason [in the "Problems of Representation in the Teaching and Learning of Mathematics" (Janvier 1987)], conceptualizes the problem of representation as an interactive "translational" play between three levels of entities; the referent, the signified, and the signifier. Avoiding the oversimplified duality of the representing/represented and the referent/reference, he welcomes the distinction between signified and signifier on a psychological level as mentioned by Mason, which enables for an in depth analysis of the problem of representation.

The three interactive levels of representations are described by Vergnaud as follows:

the referent

- the real world experienced by the subject
- the learner acts on it in ways that "please him, or that are in accordance with his/her conscious expectations and representations"

the signified

- the cognitive level consisted of internal representations
- the learner recognizes invariants, draws inferences, generates actions, and makes predictions.

the signifier

- the different symbolic systems, like natural and mathematical language
- the learner uses these symbols for communication purposes.

As Vergnaud (1987) explains, the learner's problem of representation, may be found in the interaction between these levels. He argues that the problem of learning through physical metaphors is not merely a distinction between novice and expert, or between able and less able. He argues that the states of knowledge in the process of representation in the interactive three levels of entities are more than just the two of the novice and expert, suggesting that the process of representation is a developmental one.

In particular, he discusses the reciprocal problem between the levels of signified and signifier. From the signifier towards the signified, *"the problem of the unicity of meaning"* is identified. In other words, more than one interpretations of the mathematical meaning can be found at the signified level. And when the learner moves from the signified level towards the signifier, the problem of an appropriate symbol's availability takes place. The learner may not be able to find the adequate symbol (word, mathematical language) to express his/her mental representations (thinking). At the same time, the referent/signified interface may prove problematic, due to

inadequacy between the signified level of mental representations and the referent level of physical metaphors. It is argued that this is due to the subjective nature of the learner's mental representations (von Glasersfeld 1987).

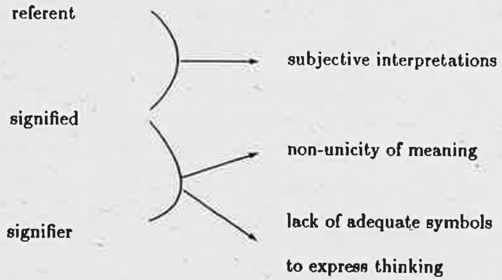

Figure 1: The Problems of Interpreting Representations

The above problems as identified by Vergnaud are problems of interpreting and communicating mental representations, an issue discussed and elaborated by the constructivist epistemology. Vergnaud's analysis assists to understand the complications that the process of learning through physical metaphors engenders. It illuminates the difficulties of misenterpretation, as well as the lack of communication due to that.

From the above, it may be conjectured that, physical metaphors, from a constructivist point of view, are not materials for teaching mathematics by themselves, but they are integral components of the mathematization process. When physical metaphors are used, attention should be given in the process of communication between teachers and students. A *"...communicative perspective conceptualizes teaching of mathematics, not as an attempt to focus students' attention on things we see in their environment in explicit ways, but instead to guide students' efforts in an interactive environment which involves negotiation and sharing of mathematical meanings on a communicable basis"* (Cobb et al 1992, p.2).

Concluding, a constructivist perspective may suggest that:

- The learner's internal representation of the mathematical concept, does not match with the external representation of it as provided by the physical metaphor.
- Physical metaphors are components of the mathematization process. They are used along with social discourse (discussion, argumentation), in order to facilitate the negotiation of mathematical ideas.

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TOWARDS A RESEARCH PROGRAMME FOR MENTAL IMAGERY

I have been working on and with imagery explicitly for a long time, and I feel I have a sense of roles imagery can play and of ways of working, but formulating research questions seems to be unexpectedly difficult. I can make conjectures, but I am not clear in what sense they are testable.

I proposed this session for two reasons: first, because several people have come to me for research advice, started to work on mental imagery, and then slid sideways into some other topic that arose during their initial investigations; second, because I am interested in methodological issues, such as how research proposals are formed and formulated, and this seemed an ideal opportunity to develop both at the same time.

By way of throat clearing, I began by indicating the basis of my being perplexed about research in mental imagery, and referred to the intimate connection between methodology and question framing. Just as in the midst of a difficult problem you reach first for a familiar tool, so in the midst of asking questions, you tend to reach for a familiar and confidence inspiring paradigm, and that in turn fashions the sorts of questions likely to be asked.

I also rehearsed the idea that mental imagery is important because it lies at the heart of human meaning making, and is the means of preparing in the now, actions to take in the future. Furthermore, we are now at the beginning of an image revolution with the rise of CD-ROM technology. People can select and juxtapose a bewildering array of static and dynamic images on a screen (most taken from a catalogue provided), and then offer these to students either as visual or hyper-exposition, or as a resource for their own scrapbook construction. Images can now be generated, assembled, and modified easily, with the real possibility of students being saturated by images. There are certainly questions to explore concerning the effective use of such image sources in education, and the transformations of teaching and learning which may result.

I am particularly concerned that lessons are learned from past experience. A constructivist perspective on text, namely that having told someone something, it is unwise to assume that they now know what you have told them, applies also to images: having shown someone an image, it is unwise to assume that they now have access to that image, or see the same things in it.

The first thing to do of course is to review the literature: a huge task because people from all sorts of disciplines have written about imagery, including many volumes challenging its very existence as a phenomenon. But not a lot strikes me as informing my practice. Indeed, most of the main questions and concerns which have been raised in the literature emerged in our session as well. It was not my aim to carry out such a review in this session, but interested people might consult the three review papers of Alan Bishop, and the plenary papers in PME XV and XVI by Tommy Dreyfus, Wille Dorfler, and Gerry Goldin.

You can also consult your own experience. And that is how I introduced the way I use terms like imagery and visualisation.

> Imagine ... a classroom in which you are going to teach on Monday, or a room in which you will work on Monday.

What ever you do in response to what I said, is imagery.

> Imagine a triangle. Imagine also a straight (infinite) line in the same plane. Let the line move about so that you get a sense of the freedom available to it, and to the ways in which it can interact with the triangle.

Whatever you do in response, you are using your powers of mental imagery. Language can evoke activity in the mind other than straight processing of words, and that is mental imagery. Mathematicians act rather like poets in this respect:

> Think, when we talk of horses, that you see them,
> Printing their proud hoofs i'the receiving earth.
> For 'tis your thoughts that now must deck our kings,
> Carry them here and there, jumping o'er times;
> Turning accomplishment of many years
> Into an hourglass.
>
> (Shakespeare: Prologue to Henry V)

I showed some pictures as well, illustrating a difference between looking at a photograph and a diagram of the same object: different purposes, different actions, different effects. The photo tends to generate links to other contexts, while the sketch-like aspects of a diagram tends to evoke a gestalt closure action so that the viewer fills in details not shown. A diagram provides stressing and consequent ignoring, and so can be used for communication of analysis in ways which are much more difficult with a photograph. I am particularly interested in analogues with mathematics: offering enough that others will use their natural powers to fill in detail, rather than providing photographic completeness. Discussion of samenesses and differences brought the role of affect and metonymy to the fore: images often spring to mind through unconscious metonymic channels.

There are a variety of words used in this area, of course:

> Seeing: figuratively as well as literally;
> Visualising: making the unseen visible
> Imagining: imagery as the power to imagine the possible and the impossible.

I then invited people to indicate questions that they had about imagery. One repeating theme was the difference between personal, idiosyncratic images, and conventional images as cultural tools, and the role of the teacher in offering and supporting some images and not others.

- Natural images and natural questions: exploring MaxBox, under what sorts of circumstances would the idea of investigating the features of a graph near a local extremum arise as a natural question in most groups? Some people seek economical experiences in which such mathematically significant questions arise naturally.

- When do images stop being useful? How do you recognise that an image is no longer sustaining or valid? How do you achieve flexibility?

- Student working on decimal names, came up with "3.1 and a half" as lying between 3.1 and 3.2, and after discussion reached 3.15 and seemed confident about it. When asked later for a number between 6 and 7, said "There isn't a number between 6 and 7 ", but here there is a new element in the discourse: use of the word *number* switches student attention out of decimals and back to whole numbers. Words can trigger unintended but dominating frames of mind. What role does conscious use of images play in this process?

- Robustness: what makes an image stick with you, what makes it ephemeral and transitory? What makes it robust and resistant to alteration or displacement? Some images develop, others stick.

  Classic example is .9 recurring, which feels different to 1 when you look from a perspective of uncompleted unfolding of a decimal naming process; later it may be overlaid with the fact that it is equal to 1, but the old image is often just under the surface, and readily comes to the surface under probing. It is often not until you try to convince others as a teacher that you may begin to be fully convinced yourself.

- How do you communicate your image? Frustration as expressed with vivid inner experiences which do not easily translate into words. Is your image quintessentially yours, or are there conventions, grounds of commonality, among people? Evidence from experience, and a constructivist position, both question the possibility of giving someone an image, though you may be able to provoke someone into forming an image for themselves. Salomon speaks of supplanted imagery, arising particularly from film and television, and illustrated by the sentiment "I don't want to see the film of that book, because I prefer my own images". There is something very powerful about visual images, which advertisers try to exploit, and which we as teachers could also exploit.

- Is it acceptable as a teacher to be explicit about my own images, or is that an intrusion into a private domain? Should I be communicating my image? Might be appropriate for half the class. Allowing children to have their own images ... do I have the right to deny them their own images? Exposure to the desirability of flexibility might be useful, and to how some images may not always be helpful.

  "Something can be clear inside my head... I have an image... but not the words to communicate a sense of what I experience." This highlights for me the value in working on *saying what you see*, in a variety of contexts as part of strengthening powers of mathematical thinking generally, as well as exploiting images and senses-of that we may have fleeting glimpses of. Exemplified by being asked "what do you mean by *sense of*?", to which a reply is... "I have a sense of it but it is hard to express in words". I showed a diagram which associates the various senses with specific forms of imagery, such lists having been drawn up by a variety of authors.

  Types (Presmeg quoted in Bishop 1989):

Pictorial (pictures in the mind)
Pattern (pure relationships depicted visually-spatially)
Memory images of a formula
Kinaesthetic
Dynamic

Types (Sense-based)

Seen (visual)
Felt (physically: kinesthetic, tactile)
Felt (emotionally)
Heard (auditory)
Tasted (gustatory, olfactory)
organic (eg toothache, thirst, drowsiness)
Sensed

Types (psychologically based)

Cognitive (relational)
Affective (emotive)
Enactive (kinesthetic)

Types (Purpose, Lakoff 1989)

for recognition
for classification

Recently, particular attention has been drawn to the possibility of stressing features of a diagram inappropriately, and so being misdirected. There are figural aspects which arise from the particular presentation, and conceptual aspects which belong to the thing being represented. Examples abound in geometrical figures, such as the base of a triangle always being horizontal. (Laborde [1988], Mariotti [1991] et al)

Imagery often forms a key for a rich network of connections and associations, and so has a crystalising effect. Often procedural knowledge is or can be tied to images (shades of Luria and mnemonic devices). For example, Descartes used the examples

imagine a chiliagon (a regular 1000 sided polygon);
imagine a regular 999-gon

to distinguish between imagining and conceiving. You can conceive of a difference, but you cannot imagine them (in the sense of Descartes) differently, because as figures, they are indistinguishable form each other and from a circle. But associated with the odd and the even number of sides are very different patterns of chord intersections. Once you use an image as access to such procedural knowledge, it becomes almost metonymic, a trigger to further awareness and thought rather than an end in itself.

An example was proposed of a child with no sense that an odd number plus an odd number is even. It suggests no access to a conventional image of two rows of squares, with one row one square longer than the other, and two of these fitting together to make a two-rowed rectangle.

Notice the difference in activity in working with words and with pictures. the pictures are particular but generality is intended to be read through the particular; words are general, and need to be particularised and then re-generalised.

Some images are more useful than others, so presumably we want to expose students to useful ones, ones which extend rather than break down. I am thinking here of multiplication by -1 as rotation of the number line through 180 degrees around 0, since this extends later to multiplication by $\sqrt{-1}$ by rotating through 90 degrees instead.

"I think in houses... even on one side and odd on the other... linked to experience as a child during the war when certain houses were missing and I used to worry about whether the missing houses really had had the missing numbers, or maybe something else on them. "

Might children get image-fatigue, being overwhelmed with a plethora of images?

How do we find out what children's images are? Is it necessary to find out?

Image as access to potential, waiting for what particulars, what details are likely to be helpful when a problem arises.

We had several images connected with 'minus minus makes plus'. Asked to find 2 - (-3) everyone said '5', and asked how, came out with something like 'two minuses make a plus'. Once automated, we don't use the supporting images, and the inner incantations we employ (if they are at the surface) do not actually describe what we are experiencing or perhaps even really what we are doing. There is a temptation to utter words that describe my immediate reaction in the hope that if children learn that incantation, they too will be able to do similar questions. But attempts to shortcut meaning are usually unsuccessful in the long term.

As might be expected, we allowed the time to be used in exploring some questions, and succumbed to the temptation to discuss, even address some emerging questions. We did not then get to looking at how such questions might be approached, and how the approach reflects a methodology. Perhaps another time.

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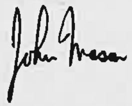

The Role of Spreadsheets in the Development of an Algebraic Approach to Mathematical Problem Solving

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Within this presentation I discussed the results of a study which investigated the role of spreadsheets on a group of ten 14-15 year old pupils' developing use and understanding of algebraic ideas. All these pupils had a history of being unsuccessful with school mathematics and had very little previous experience of algebra. The pupils worked on two blocks of spreadsheet activities over a period of 6 months working in pairs for approximately 12 hours of "hands on" computer time. The activities consisted of:

> *Block 1-Introductory Problems* Within this sequence pupils were introduced to the following spreadsheet and mathematical ideas; entering a rule; replicating a rule; function and inverse function; symbolising a general rule; decimal and negative numbers; equivalent algebraic expressions (for example 5n and 2n +3n). The problems were analysed so that mathematical and spreadsheet ideas were introduced simultaneously.
>
> *Block 2-Algebra Story Problems*. Within this sequence pupils were introduced to solving algebra story problems in a spreadsheet by: representing the unknown with a spreadsheet cell; expressing the relationships within the problem in terms of this unknown; varying the unknown to find a solution.

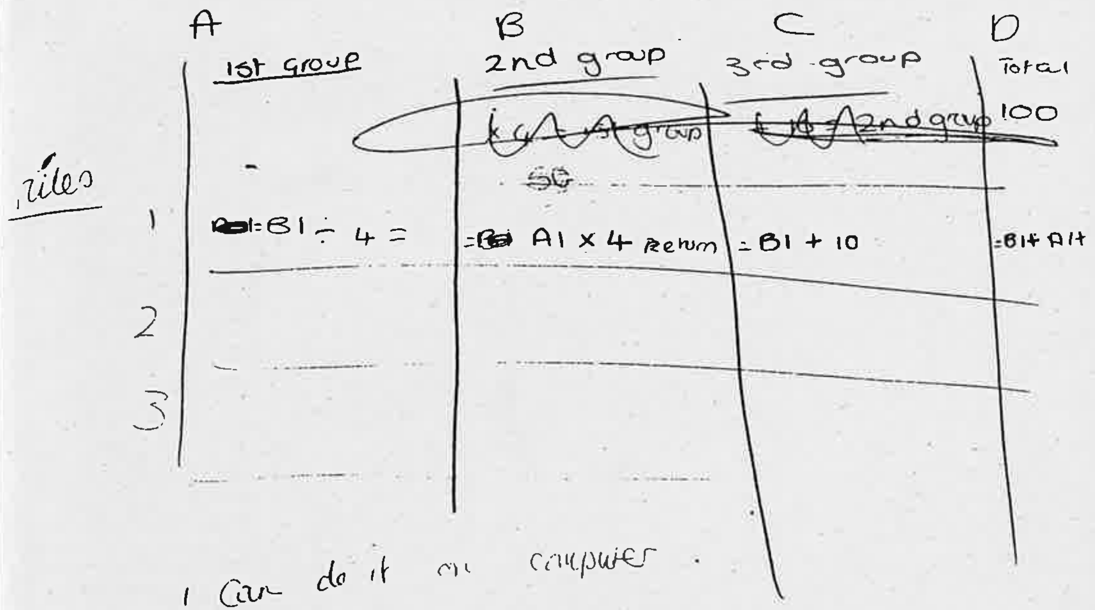
All of the pupils were interviewed at the beginning and end of the study in order to trace their developing use of algebraic ideas. The majority of the 14-15 year old pupils could not answer any of the pre-interview questions which focused on the algebraic ideas of: expressing generality; symbolising a general relationship; interpreting symbolic expressions; expressing and manipulating the unknown; function and inverse function. All of the pupils had great difficult expressing very simple general rules in natural language (for example "add 3") and none of them were able to answer questions on inverse functions. Responses to questions were very brief and it was noticeable how difficult it was for these pupils to articulate their ideas in natural language. All apart from one pupil (who had transferred from a previous school) were unfamiliar with literal symbols exhibiting the classic "misconceptions" reported in the CSMS research. For example Jo thought that the higher the position in the alphabet the larger the number represented. This clearly related to experiences from primary school *"A starts off as one or something....when we were little we used to do a code like that...A would equal 1..B equals 2..C equals 3"*.

When the class first started to work with the computer their lack of confidence was very noticeable. They minimised their engagement in the mathematical problems, continuously asking the teacher for help, were resistant to trying out their ideas at the computer and never produced more than had been asked for. It seemed as if they did not engage in the problems in order to avoid failure. As teachers we responded to this situation by providing more structure to the spreadsheet activities, whilst continuing to emphasise that the pupils had to solve the activities themselves. Their behaviour had noticeably changed by the fifth computer session. They learned to try out their ideas in the spreadsheet, modifying them until they produced satisfactory results and began to experience success in solving the mathematical problems. They were particularly successful with the algebra story problems which we had predicted they would find difficult.

It is important to stress that pupils were initially taught to enter a spreadsheet rule by pointing with the mouse to the cell which was being referenced. They were never explicitly taught to type in the spreadsheet-algebraic code (for example A5) although they had been explicitly shown how to display the "formulae" produced by the spreadsheet. Analysis of transcripts of the conversation between pairs of pupils indicated that they used this code in their talk (*"so what will it be....B2 take 4"*) and further questioning of the pupils in the final interviews revealed that they all knew the code for the spreadsheet formulae which they had entered with the mouse. They also knew how this code changed when being copied using relative referencing (for example from A3 + 1 to A4 + 1). The fact that they noticed and knew this code is, we suggest, related to the nature of the Excel spreadsheet environment in which the spreadsheet code is transparently displayed in the formula bar. The pupils learned that this was the language to communicate with the computer and began to use it as a language to communicate with their peers.

Analysis of the results from the final interview revealed that the spreadsheet-algebraic code played a mediating role in pupils' developing ability to solve the algebra problems which were the focus of this study. In the post-test the majority of pupils could express a general rule for a function and its inverse and often expressed these rules in spreadsheet-algebraic code. This contrasts with their performance on the pre-test. When asked how she could answer so many questions successfully in the post-test, when she had not been able to answer any in the pre-test Jo said *"because you have to think before you type it into the computer anyway...so it's just like thinking with your brain"*. The pupils said that they thought of a spreadsheet cell as representing any number and many of them were able to answer traditional algebra questions in the post test. For example when Jo had been asked the following question in the pre-test "We know that the length of this rectangle is double its width. The width is called X. Can you write a formula to calculate its length. Can you

write a formula to calculate its perimeter?" he had suggested that the length would be y and could not give an answer for the perimeter. In the post test he said that the length would be X x 2 and that the perimeter would be X x 6. The following algebra story problem was given to the pupils in the post-test and is similar to the Block 2 algebra story problems. "100 chocolates were distributed between three groups of children. The second group received 4 times the chocolates given to the first group. The third group received 10 chocolates more than the second group. How many chocolates did the first, the second and the third group receive?" Jo's solution (with no computer present) illustrates the way in which the spreadsheet code played a mediating role in her solution process.



In the post-interview Jo was asked "If we call this cell x what could you write down for the number of chocolates in the other groups" and she wrote down:

= X   = X x 4   = X x 4 + 10

It must be emphasised that Jo, who had always been unsuccessful with mathematics, had successfully carried out what is considered to be the most difficult part of solving an algebra story problem, that is representing the problem in algebraic code.

It was remarkable how effectively the pupils learned about the spreadsheet-algebraic code through using it in a problem solving situation which involved communicating with the computer. Pointing with the mouse seems to be an important part of the process of acquiring the code. Pupils do not appear to feel any anxiety towards this computer-based algebraic code, which is in contrast to the prevalent attitude to traditional algebraic code. We suspect that in traditional settings algebra code is associated with the authority of the teacher. This no longer needs to be the case in computer environments, because pupils can try out their ideas without turning to the teacher for the correct answer. Pupils develop a positive attitude to the spreadsheet algebraic code which can then take on a mediating role in their problem solving activities.