Exploring the design principles of examples to help primary school students' multiplication learning

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This study aims to find a set of principles for designing examples, which can serve as theoretical guidance or practical tools in mathematics education to assist educators in creating teaching materials and helping teachers organise their lessons. I use variation theory as the theoretical basis to explain learning, choose multiplication as the context topic in mathematics education, and apply design research to generate a set of design principles, then develop a set of examples based on them. I collected data through interventions with groups of three students and analysed their responses to each example I designed. So far, I have proposed a set of initial design principles (iteration 0), tested and improved those examples (iteration 1 and iteration 2). This study is currently working on iteration 3.

Keywords: examples design; variation theory; design research; primary multiplication learning

Introduction

From my experience as a teacher in China, I realised that it is important to present appropriate examples for learners to enhance their learning. Inappropriate examples may make it more difficult for students to learn new things. This was the impetus for my current doctoral study. I aim to find a set of principles for designing examples, which could be used as a tool in mathematics education to help educators develop teaching materials or teachers organise their lessons. This study is still in progress. In this paper, I will briefly introduce the theoretical frame that I am using, present a few examples I have designed and trialled, and some of the insights that I got from that set of examples towards my next work.

Theoretical framework

In this study, 'example' only indicates the examples related to the object of learning, such as an example of a concept or procedure (rather than the display of someone's behaviour or punishment). I use uppercase EXAMPLE to highlight its limited meaning in my study and also distinguish it from the common expression, example, in writing.

I use variation theory (Marton & Booth, 1997) to understand the role EXAMPLES play in learning. This is because variation theory is able to present a sufficient and detailed learning process, from how new information is experienced, to how the learner thinks, and to how the learner could respond. Variation theory can also help me analyse the object of learning as multiple critical aspects, allowing for exploration of how to present this object of learning to students. In order to develop principles for EXAMPLES design in mathematics education, I chose to focus on

multiplication. Below I will present key ideas related to variation theory and multiplication.

Variation theory

Variation theory focuses on a phenomenon (an object of learning) to be learned by a learner. It explains how a learner understands a phenomenon. In variation theory, there are critical aspects of a given phenomenon each of which learners must experience so that they are eventually aware of all critical aspects simultaneously to achieve understanding of the phenomenon (Orgill, 2012). For example, for children to understand the ripe apple from an unripe apple, the apple's size, colour, and taste may be the three critical aspects that they need to experience.

In terms of the specific learning process, there is a 'significant pattern of variation' used to present the changes in the learner's perception of the object of learning in variation theory. This pattern has been mentioned by multiple scholars (e.g., Ling & Marton, 2012; Marton, 2009; Orgill, 2012) in various orders. I introduced it in the order of: 'contrast' - learners come across a learning object and preliminary awareness of the learning object through comparing it with something else; 'separation and generalisation' - critical aspects are separated out from the whole learning object by teacher and experienced by learners to generalise the critical feature of each aspect; 'fusion' - the learner experiences the variation of multiple aspects simultaneously in order for learners to understand the whole object of learning. This pattern will be employed to guide my specifically EXAMPLES design.

Multiplication

I selected multiplication learning to provide a context for my study as multiplication is a key part of children's primary learning stage. The learning of multiplication occurs almost throughout the primary school years, from the most basic one-digit whole number times one-digit whole number to more complex multiplication of rational numbers. It is also closely related other mathematics topics that learners will learn (such as division, fraction, probability and so on). More importantly, throughout my teaching experience, the focus of learning and teaching for multiplication has still been on memorising multiplication facts. Rote memorisation of multiplication facts does not mean that learners have understood multiplication's meaning (Boaler & Williams, 2015).

To understand the concept of multiplication, many researchers, (e.g., Hart et al., 2004; Larsson et al., 2017; Post, 1988) describe a variety of conceptual models of multiplication. After reviewing these existing studies, I identified eight models for developing a 'complete' understanding of multiplication. Learners need to experience all of these (and recognise they are all related to multiplication): equal grouping, number line, branching, scaling, rate, array, area, and Cartesian product. In addition, by analysing multiplication through variation theory, I identified three critical aspects of understanding multiplication as a comprehensive concept: units, continuity and operating. Each model exhibits distinct features within these three aspects, which I refer to as the properties of the models. For example, in the 'equal grouping' model, there is one unit and one virtual unit (rather than two factor units and one product unit); it is discrete (rather than continuous); and it is a unary operation (rather than binary). The analysis for these models will significantly influence the sequencing of EXAMPLES in my future designs, as they will be based on different models.

Design research

Different from most other research studies, design research in education is the research to see what education could be or should be rather than what education was or currently is, which is widely used in developing new educational materials (e.g., computer tools, learning activities, or professional development programs) (Bakker, 2018). I am using a definition form McKenney and Reeves (2014, p131) to articulate what design research is: "a genre of research in which the iterative development of solutions to practical and complex educational problems provides the setting for scientific inquiry, and yields new knowledge that can inform the work of others."

There are four characteristics of design research, iterative, future-oriented, theory generative, practice-oriented (Campanella & Penuel, 2021). This study will design, test, and improve a series of design principles and EXAMPLES iteratively, in order to generate a relatively mature set of design principles. The design principles I offered will be helpful for the future learning and teaching materials design in practice, and also can also provide theoretical basis for researchers to critique or design EXAMPLES. These characteristics are all matched with my research aim, which is also why I chose to conduct design research.

Research gap

Some scholars have explained and classified EXAMPLES in detail (e.g., Bills et al., 2006). Some research made a comparison work between EXAMPLES from different textbooks (e.g., Zhang, 2019). Some research organises some points for helping choice and use of EXAMPLES through analysing a groups of novice teachers' lessons and interview them (Rowland, 2008). However, a more explicit framework is needed to assess the practical value of an EXAMPLE. The effectiveness of an EXAMPLE could be linked to its placement within a set of EXAMPLES, as well as the learner's attainment. The appropriateness of an EXAMPLE needs to be repeatedly tested in a specific context through design research. The points summarised from my study will provide a theoretical tool to other researchers for the evaluation or design of EXAMPLES.

There are also many researchers conducting studies on the basis of variation theory in different subjects education (e.g., Cheng, 2016; Marton & Pang, 2007). In terms of mathematics education, Watson and Mason (2006) who suggest using variation in some proper ways to highlight a certain point to learners. However, such previous studies did not offer systematic assistance for the design of EXAMPLES. In other words, they did not generate a set of rules, principles, or a model for EXAMPLES design. Due to differences in educators' teaching skills or knowledge frameworks, they might analyse critical aspects of a learning object and control the progression of variation in different ways. This poses a challenge in the current application of variation theory.

Therefore, based on my theoretical basis above and the research aim, I formulated the following research questions:

- What would be the characteristics of a set of EXAMPLES that need to be taken account into design principles?
- How do students respond to a set of EXAMPLES that is designed according to the principles?

Implementation

The initial stage of implementation for looking for design principles is still at the theoretical level, so I call it as my iteration 0. I analyse all EXAMPLES related to multiplication in two versions of the textbooks, one from Shanghai and one from Singapore. I examine the models they incorporate and the role of each EXAMPLE in the significant pattern of variation.

After analysing and critiquing EXAMPLES from current textbooks, I proposed initial set of six design principles. These six principles will not be fully test and improve within my doctoral study period, but I will test and improve a subset.

Iteration 1 involves testing the set of EXAMPLES designed based on first two principles (Figure 1). These EXAMPLES focus on making equal groups, serving as the initial set of multiplication EXAMPLES for learners to experience. Iteration 2 is a further test after improving the set of EXAMPLES and pedagogical skills through the first intervention. My participants are a group of three students, each nearly eight years old, in China in 2023. The intervention lasts the duration of one lesson.

1. An EXAMPLE should only highlight one new point/aspect to draw attention to learners.
2. Each EXAMPLE should have a clear aim. It should play a role of helping learners to experience **contrast** or **separation & generalisation** or **fusion** rather than be multirole.
   1) In order to let learners notice the object of learning and experience **contrast** stage, the designed EXAMPLE should exemplify a type of objects and a contrasting and distinct type or types of objects to learners.
   2) In **separation & generalisation** stage, the designer should separate the understanding of the object of learning into several critical aspects. Each EXAMPLE should highlight only one new critical aspect (it could include some other critical aspects in the case that they were highlighted one by one in the previous EXAMPLES), in order to help learners are able to generalise the critical aspect.
   3) In order to help learners go through **fusion** stage, the designed EXAMPLE should consist of all just introduced critical aspects and non-critical aspects of the object of learning.

Figure 1. My initial design principle 1 and principle 2

An example of a designed EXAMPLE

Below, I will demonstrate the work I have done through an EXAMPLE I designed and used. Figure 2 shows the third EXAMPLE used in both Iteration 1 and Iteration 2, but with different purposes. The preceding two EXAMPLES were intended to help learners experience the contrast between equal and non-equal groups, followed by the experience that arrangement is not a critical aspect for identifying equal groups.

Figure 2. EXAMPLE 3 in my EXAMPLES set in iteration 1 and iteration 2

In iteration 1, I designed this EXAMPLE was intended to help students consolidate what they had just learned—that arrangement is not the critical aspect in identifying equal groups. This EXAMPLE falls into the 'separation & generalisation'

stage. Here, the variation aspect is the non-critical aspect of the items used, shifting from plates and biscuits to trays and cakes. My original intention was to place three same cakes as a line on the left-side tray and put three same cakes as a triangle on the right-side tray. However, two children did not think they are equal groups, as the cake in the middle of the right-side tray appeared larger than the others. Through my subsequent analysis and reflection, I realised that, on one hand, the number of the items in each group is not the only critical aspect for identifying equal groups, the nature of the items should also be considered critical, such as the size or type.

So, as Iteration 2 commenced, I decided to keep this EXAMPLE. I aimed to see if the new group of participants would respond similarly to the first group. In addition, I introduced a new EXAMPLE after EXAMPLE 3, aimed at highlighting the nature (size and type) of items involved, which I identified as two new critical aspects in this set of EXAMPLES (as shown in Figure 3). During the actual intervention of Iteration 2, a similar scenario unfolded with the new group of learners. I firstly explained the original intent of EXAMPLE 3 and asked for their thoughts. At that time, they all pointed out that "they are equal groups, because, despite different arrangements, each tray has the same number of cakes, three." Such response means they have been able to discern that arrangement is a non-critical aspect for identifying equal group, even in new situations where the items involved were different from EXAMPLE 2. Next, I presented them with EXAMPLE 4(1) and asked, "if the situation as you thought (the cake in the middle is indeed larger than others), are they equal groups?" This discussion led the learners to recognise that the size of items involved in a group is a critical aspect for identifying equal groups. Furthermore, I showed them EXAMPLE 4(2) to highlight that the type of items involved in a group is also critical for this identification.

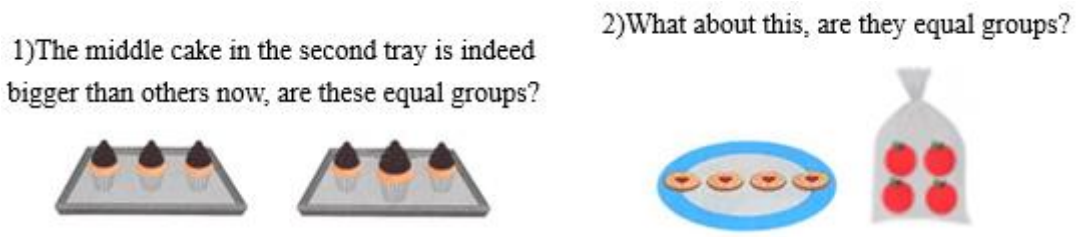

Figure 3. EXAMPLE 4 in my EXAMPLES set in Iteration 2

Conclusion

From the example I demonstrated, based on initial design principle 1 and principle 2, the EXAMPLE I designed and used successfully separated and varied only one new aspect (the items involved). The learners' responses show they can discern the new aspect (exclude varied items) while also discerning the last aspect they just experienced (exclude items arrangement). Although I did not initially realise that the size and type of items involved should be considered critical aspects when I first designed the EXAMPLES, I made further improvements to the set of EXAMPLES based on participants' responses during the first intervention. As a result, a new EXAMPLE was added after EXAMPLE 3 to highlight the size and type of the items involved so that learners can gradually generalise them. Regarding the design of additional sets of EXAMPLES or even more extensive sets, questions arise: will my initial design principles still be effective? How should all EXAMPLES be sequenced properly throughout an entire chapter? These considerations, involving my design

principles 3 and 4, as well as content beyond the initial principles, will be the focus of my subsequent steps in research.

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