Always, sometimes, or never true: The linguistic challenges of transitioning between examples and generalisations

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Both teachers and students encounter linguistic challenges in mathematics classrooms. Many of these challenges are also mathematical in nature and are often topic-specific. In this paper, we examine these challenges as they arise when teachers and students interact around a widely used task focusing on the truth-values of algebraic equations. The data arises from a collaborative project between teachers and researchers focused on developing materials to support language-responsive mathematics teaching. These materials focus on offering guidance on using widely used or known tasks in language-responsive ways. Videos from the lessons in which the teachers implemented these materials were analysed. In these lessons, the challenges encountered arose when the students and teachers were discussing the relationship between the examples generated and the conclusions drawn from these examples.

Keywords: language-responsive mathematics teaching; equality; classroom interaction; examples

Introduction

Today's mathematics classrooms are linguistically diverse and students and teachers experience a range of linguistic challenges in the teaching and learning of mathematics. Many of these linguistic challenges are topic-specific, and intrinsic to the mathematics being taught and learnt. How concepts are represented and talked about in lessons can influence the meaning students develop for these concepts. These linguistic challenges include the mathematical and academic registers (Wilkinson, 2019) that students are expected to develop, as well as forms of reasoning and explanation that are needed to solve problems, justify responses and make arguments, and ways of representing mathematics.

The majority of research on the relationship between language and mathematics in teaching and learning has focused on multilingual students who are learning the language of instruction alongside mathematics (e.g., Barwell, 2018; Planas, 2014). In England, in both policy and practice, there has been an emphasis on the vocabulary needed to succeed in mathematics rather than the broader linguistic challenges involved in the teaching and learning of mathematics. In recent years, research has shifted to focus on how students make sense of mathematics drawing on a range of resources, including more informal ways of talking about or representing mathematics, challenging the previously dominant deficit perspectives taken towards students who encounter difficulties with learning mathematics. Much of this recent research focuses on the mutually co-constructed nature of language in interaction during the processes of teaching and learning mathematics, which is the approach taken in this paper.

Language-responsive mathematics teaching supports all students to develop language proficiency that is relevant to the mathematics being learnt, without reducing the cognitive demand of the mathematical tasks and activities (Erath et al., 2021; Prediger, 2019). Central to language-responsive mathematics teaching is the focus on the epistemic function of language, that is language as a tool for thinking and learning, alongside the communicative function. Erath et al. (2021) identify several design principles and teacher moves involved in language-responsive mathematics that are supported by a range of research. It is these principles and moves that formed the basis of the design-based research project described below. In particular, engaging in rich discourse practices such as explaining and arguing has been shown to be particularly important for developing conceptual knowledge (Zahner et al., 2012). In this paper we examine the use of one task by several teachers which is designed to offer students opportunities to explain and develop arguments relating to algebraic equations.

Methods

The extracts offered in this paper arise from data collected as part of a collaborative design-based research project. This project investigates topic-specific language-responsive mathematics teaching for three key topics, solving linear equations, angles in parallel lines and probability. The collaboration between teachers and researchers focuses on developing and adapting mathematics tasks already being used by the teachers in their classrooms to make their use more language-responsive (see Prediger, 2019). The principles underpinning this work include that all students can benefit from language-responsive teaching, the use of words with multiple meanings or ambiguity can be productive for learning, language (and mathematics) needs to be amplified not simplified when facing challenges in learning, and that mathematics and language are intertwined and should not be worked on separately.

Lessons in which the teachers use these tasks are video recorded and analysed using both an observation framework and conversation analysis of extracts that focus on the tasks developed during the collaborative work. In this paper, the extracts are taken from videos from two different teachers from two different secondary schools in England using the same task and focus with their Year 9 (aged 13–14) students. One of these schools was an urban school with several multilingual students in the lessons. The other school was a coastal school with few multilingual students but with many socioeconomically disadvantaged students. The discussion following these extracts also draws upon the videos of all of the five lessons where this task was used with students.

The task

The task each of the teachers used with their class asks whether the equation $x+y = xy$ is always, sometimes or never true (DfES, 2005). This task was well known to the teachers and was one that the teachers and researchers had worked on and developed guidance to support the use of this task in language-responsive ways.

The task draws upon a relational understanding of the equal sign, where students need to consider both *whether* the expression on the left-hand side is equal to the expression on the right-hand side, and *when* they are equal. One of the mathematical challenges with this task is appreciating how the truth-value of the equation changes depending upon the values of both x and y. Other mathematical challenges can occur in seeing the letters x and y as variables, and appreciating that both x and y can take the same value. Each of these challenges is documented as difficult for students when learning algebra (Kieran, 2022). Some of the observable linguistic challenges include describing, using and understanding the notation involved, describing and talking about and with letters, and developing and refining explanations and justifications. The materials developed include focusing on using key terms in meaningful ways such as variable, values, and equal to, as well as anticipating different explanations students may offer and scaffolds to support the refinement of these explanations. These explanations focused on the challenges associated with the equal sign and the meaning of letters in expressions and equations.

The lessons

In each of the lessons, the students encountered challenges related to transitioning between examples and generalisations. The relationship between examples and generalisation becomes particularly visible when considering relationships such as the one in the equation in the task, i.e. statements that are sometimes true. In the discussion below we illustrate some of these challenges using extracts from the whole-class discussions that occurred after the students had generated a few examples both of when the statement was true and for when it was not true, but had not yet drawn conclusions during the whole class discussion as to whether the equation was always, sometimes or never true, though they may have reached these conclusions during pair work on this task that occurred before these extracts.

The first extract comes from a class with a history of low prior attainment in mathematics in a coastal school. The students have had some time discussing the task in pairs as part of a think-pair-share activity.

Extract 1: Classroom A

Student:	It’s true for like, one number, which is 2, but not for the rest.
Teacher:	So, when I rephrase it, [pointing to the equation x + y = xy] I’d say, this is sometimes true. It works for some numbers [pointing to the equation 2 + 2 = 2 × 2].
Student:	That’s one number.
Teacher:	Yeah. That still meets sometimes true, doesn’t it? That’s when I sometimes go to the shop. I might mean I go to the shop once and it might mean I go to the shop lots of times. Okay. Sometimes me… [inaudible] go to the shop, sometimes [inaudible] don’t. This sometimes works [continuing pointing to the equation 2 + 2 = 2 × 2] but it doesn’t most of the time [circling other equations e.g., 2 + 5 ≠ 2 × 5], doesn’t work. Okay.

In this first extract, the student focuses on the specific examples, drawing the conclusion that “it’s true” for 2, but not true for the other examples. It is not clear from what the student has said whether they appreciate that both x and y have the value 2 rather than just one of the variables. Notably, the student discusses the truth value of the statement for the specific value(s) without referencing the more general conclusion that the statement is sometimes true. It is the teacher who follows up with this general conclusion that the statement is sometimes true, explaining that it works for some numbers. This rephrasing by the teacher is treated by the student as repairing (correcting) their answer of one number, asserting that “that’s one number”. While 2 is “one number”, the truth-value of the equation relies on two values. The teacher continues but treats the difficulty as with the meaning of “sometimes”, expanding their explanation to include examples from everyday life to illustrate the meaning of sometimes. That is the teacher treats the difficulty as a linguistic one.

However, while both the student and the teacher accept the statements $2 + 2 = 2 \times 2$ and $2 + 5 \neq 2 \times 5$, the student is focused on when the equation is true, rather than the overall truth-value of the equation – that it is “sometimes true”. This distinction is a mathematical one focusing on the relationship between examples and generalisations from these examples.

The second extract is taken from an inner-city school. Before the second extract, two examples have been offered by students. First, a student had offered "1 and 2 is 3 … but 1 times 2 is 2". The teacher rephrased their answer into that when x and y "have the values of" 1 and 2 then the statement is not true. The teacher then asked for "a different version of this example" and "can we make a general statement" as well as "how can we be sure whether or not it's always, sometimes or never. Can we use these two examples alone?" Another student then offered "2 and 2" as an example. Extract 2 occurs shortly after.

Extract 2: Classroom B

Teacher: So what conclusion can we draw when we these two numbers [pointing to 1 and 2]?

Student: Sometimes.

Teacher: The statement was? [pointing to the space below '1 + 2 = 3 BUT 1 × 2 = 2']

Student: Sometimes.

Student: Never true.

Teacher: When we use these two numbers… [Students replied 'never true' 'never'] It was? … Not true, right?

In Extract 2, the challenge arises between the truth-value of $x + y = xy$ and the two specific examples "1 add 2 is 3 but 1 times 2 is 2" and "2 and 2". The teacher asks students to draw a conclusion about the specific example of 1 and 2, but the first student responds with a general statement about the truth-value of the statement. The teacher rephrases the question, but the next two students still focus on concluding whether the statement is sometimes or never true. In the final turn in the extract, the teacher makes it clearer that they were looking for "not true" and that they were focusing specifically on the example using 1 and 2.

Discussion

In each of the classrooms, the students were able to generate examples where the statement was true and examples where the statement was not true. However, communicating about the relationship between these examples and the general status of the statement as always, sometimes or never true was challenging. In particular, in both the extracts above, there is a difference in how the different students are talking about the task; for example in terms of specific examples or in terms of a generalised property of the statement. Moving between specific examples and generalisation is complex, particularly when there is ambiguity around what the examples are examples of. In Extract 2, while the teacher focuses on the specific example where 1 and 2 are used to substitute x and y in the statement $x + y = xy$, the students focus on the truth-value of the general statement.

This task highlights these challenges. In mathematics we are generally discouraged from generalising from examples (in the sense of inductive reasoning), in which case no number of examples will be sufficient for a generalisation, instead algebraic reasoning involves students focusing on the underlying structures (Kieran, 2022) and seeing the general in the particular through these structures (Mason & Pimm, 1984). In contrast, a single example is needed as a counter-example. In this task, two examples are needed to demonstrate that the equation is sometimes true, one where it is true and one where it is false. In contrast to show that an equation is always or never true, an explanation is needed where examples may play an illustrative role, and potentially a convincing role, but do not constitute a justification without attention being explicitly drawn to the structures that result in the equation being always or never true. This complicates the relationship between examples and generalisations in mathematics, particularly as students' experiences of these relationships will be varied.

In Extract 1 there is the added ambiguity of whether the number 2 is treated as the number which both *x* and *y* need to be for the statement to be true, or the value of *x* and *y* which makes the statement true. In other lessons, this challenge of whether *x* and *y* can take the same value was made more explicit and was also identified in the instructions accompanying the original task. However, the juxtaposition of this challenge visible in the student's turns, and the challenge of the relationship between the examples and the generalisation as visible in the teacher's turns, in itself poses an additional linguistic challenge, that is communicating mathematics in ways that ensure a shared understanding what is in focus in an interaction.

Erath et al. (2021) argue that these challenges should be made explicit in teaching in ways that offer students opportunities to communicate mathematical ideas for themselves. However, as the teachers work on tasks such as the one considered in this paper, there are a range of mathematical and linguistic challenges, and different students may be facing different challenges both simultaneously but also concurrently as they work on the task. This places significant demands on mathematics teachers in recognising and responding to these challenges, but also in making decisions about how to proceed given than handling some challenges may distract from the learning focus.

As seen in Extract 1, it requires teachers to recognise that the student was experiencing some challenge when replying "That's one number" and to identify if the challenge was merely linguistic or mathematical. While the teacher in Extract 1 treats the challenge as a mere linguistic one, concerning the meaning of "sometimes", the challenge can be a mathematical one where "one number" may refer to the number of values required to substitute the variables of the equation. Making the challenge explicit can help identify the type of the challenge, enabling communications that help resolve the challenge.

Similarly, it requires teachers to recognise the challenges when transitioning between examples and generalisations. In Extract 1, the student focuses on the values of the variables when the equation is true as well as when it is not, rather than the truth-value of the equation. On the other hand, in Extract 2, the challenge arises when the teacher focuses on whether the equation is true or not true for specific values, whereas the students focus on the overall truth-value of the equation.

This shows that facilitating a language-responsive mathematics classroom not only requires teachers to have strong knowledge about the mathematical structure involved in the task implemented, but also requires them to be sensitive to the language involved, enabling the recognition of challenges, and to offer students opportunities to discuss and resolve these challenges. More research is needed to support teachers' professional development in the development of language-responsive mathematics classrooms.

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References

DfES (2005) A4 Evaluation algebraic expressions. in DfES (2005) *Standards Unit. Improving learning in mathematics: challenges and strategies.* Department for

Education and Skills. Available from: https://www.stem.org.uk/resources/elibrary/resource/26956/evaluating-algebraic-expressions-a4

Barwell, R. (2018). Situational and distal sources of meaning in a multilingal mathematics classroom. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.), *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 123–130). PME.

Erath, K., Ingram, J., Moschkovich, J., & Prediger, S. (2021). Designing and enacting instruction that enhances language for mathematics learning: A review of the state of development and research. *ZDM - Mathematics Education*, *53*(2), 245–262. https://doi.org/10.1007/s11858-020-01213-2

Kieran, C. (2022). The multi-dimensionality of early algebraic thinking: Background, overarching dimensions, and new directions. *ZDM – Mathematics Education*, *54*(6), 1131–1150. https://doi.org/10.1007/s11858-022-01435-6

Mason, J., & Pimm, D. (1984). Generic examples: Seeing the general in the particular. *Educational Studies in Mathematics*, *15*, 277–289.

Planas, N. (2014). One speaker, two languages: Learning opportunities in the mathematics classroom. *Educational Studies in Mathematics*, *87*(1), 51–66. https://doi.org/10.1007/s10649-014-9553-3

Prediger, S. (2019). Investigating and promoting teachers' expertise for language-responsive mathematics teaching. *Mathematics Education Research Journal*, *31*(4), 367–392. https://doi.org/10.1007/s13394-019-00258-1

Wilkinson, L. C. (2019). Learning language and mathematics: A perspective from Linguistics and Education. *Linguistics and Education*, *49*, 86–95. https://doi.org/10.1016/j.linged.2018.03.005

Zahner, W., Velazquez, G., Moschkovich, J. N., Vahey, P., & Lara-Meloy, T. (2012). Mathematics teaching practices with technology that support conceptual understanding for Latino/a students. *The Journal of Mathematical Behavior*, *31*(4), 431–446. https://doi.org/10.1016/j.jmathb.2012.06.002