Investigating Science Education students' mathematical writing – the case of mental brackets

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This study reports on first-year Science Education students' mathematical writing when solving tasks involving functions, logarithms, derivatives, and integrals. The focus of this paper is on students' use of mental brackets, a concept which up to now has been mainly studied in primary school students' scripts. When using mental brackets, the students do not write the brackets however they evaluate and manipulate the expressions as if brackets are present. In this pilot study, forty first-year Science Education students were asked to complete tasks that required the use of brackets in the above-mentioned topics. Students' scripts were analysed focusing on instances where the students performed operations as if brackets were written. These occasions of mental brackets in students' writing were further categorised using thematic analysis. The findings show that mental brackets in students' scripts were used in instances related mainly to successive signs and grouping terms.

Keywords: undergraduate students' writing; mental brackets

Introduction

Brackets are frequently used in Mathematics as both a grouping tool to prioritise operations while calculating expressions and as a way to preserve the structure. For example, when rewriting a rational expression in horizontal form brackets are used to ensure that the new way of writing the expression is equivalent to the initial one, thus respecting the relation between the numerator and denominator terms. Moreover, in Algebra, brackets can be used to group terms not necessarily focusing on the priority of operations but rather aimed at forming a singular entity within an expression. For example, in the expression $y + (x - a) + y - (x - a)$ brackets serve to consider $(x - a)$ as a whole and facilitate identifying the general type of the expression, i.e., $y + z + y - z$ which is immediately simplified to 2y. In this study, the aim is to examine and discuss the non-use of necessary brackets by Science Education undergraduate students while solving tasks involving functions, logarithms, derivatives, and integrals. More specifically, the focus is on the use of mental brackets, a notion initially introduced very briefly by Linchevski and Livneh (1999) and defined in more detail by Papadopoulos and Gunnarsson (2020). This term refers to the case where the students do not physically write necessary brackets, but they evaluate the expression as if the brackets were physically there.

In this paper, we first review the related literature and discuss the theoretical background. Then, we provide information about the methodology of this pilot study and discuss the results by providing examples of students' writing for each of the categories presented. Finally, we discuss the results, consider the bigger study, and provide some insights on students' use of mental brackets.

Theoretical background and literature review

The theoretical background of this study builds upon the notions of structure sense and mental brackets. Linchevski and Livneh (1999) define structure sense as students' ability to identify an expression's equivalent forms and make the distinction between the forms that are relevant to the task and all the others. They suggest exposing students to the structure of algebraic expressions in a way that allows them to develop structure sense. Later, Hoch and Dreyfus (2004) refined the term "structure sense" by defining it as the students' ability to see an algebraic expression as an entity, recognise it as a previously met structure, divide an entity into sub-structures, recognize mutual connections between structures, recognize which manipulations can be performed and which manipulation is suitable for a given situation. Since the role of brackets is of critical importance for the students' use of structure sense (Hoch & Dreyfus, 2004), Marchini and Papadopoulos (2011) made an effort to assist very young students to develop it by using unnecessary brackets to emphasise structure (i.e., $(5+\square)=8$). They found that the use of these unnecessary brackets worked both as an external element of the structure shaping the form of the expression and as means to highlight the structure of the expression.

On the antipode of the unnecessary brackets is the use of mental brackets. Papadopoulos and Gunnarsson (2020) investigated the work of Grade 5 and 6 students on the evaluation of rational arithmetic expressions that should first be written horizontally. To do this, the use of brackets is necessary to preserve the structure of the initial expression, i.e., the numerator and denominator terms. Their analysis showed that several students did not use brackets. For example, fractions $\frac{12}{4}+2$ and $\frac{12}{4+2}$ were both written as $12 \div 4 + 2$. So, technically, from the mathematical point of view, the evaluation for both items should be $3 + 2$. Instead, the students wrote $3 + 2$ for the first but $12 \div 6$ for the second. They did the same to more complex fractions such as $\frac{8+12}{3+2}$. They wrote it as 8+12÷3+2, which should be calculated as 8+4+2. Instead, they continued with 20÷5. The students seemingly violated the order of operation rules, but this happened because the use of mental brackets served as "prostheses of the mind to accomplish actions as required by the contextual activities in which the individuals engage" (Radford, 2000, p.241). This use of mental brackets is an indication of the presence of structure sense.

Methodology

The participants of this study were forty first-year Science Education undergraduate students from a state University in Turkey. During their secondary school studies, the students were taught the concepts of functions, trigonometry, logarithms, derivatives, and integrals. Therefore, the tasks of this study were familiar to them. The participants were asked to complete a short collection of tasks (see Fig. 1). This collection comprised of tasks on functions, trigonometry, logarithms, derivatives, and integrals. The students were asked to solve these tasks during one session of approximately 40 minutes. The tasks were designed with the consideration that brackets were needed to be used to achieve the solution. Students' written solutions to these tasks constitute the data of this study. The data were analysed using an inductive thematic analysis approach (Mayring, 2014). The analysis gave emphasis on instances illustrating the use of mental brackets and took place in two stages. The first stage focused on identifying instances where mental brackets were present, and the second stage focused on further exploring this use. The whole set of data was examined separately by each one of the three researchers, the results were then compared and discussed until an agreement was reached regarding the categorisation.

1) Let $f(x) = x^2 - x - 1$ *and* $g(x) = -x + 2$ Find the following: i) $f(g(x))$ ii) $f(g\left(-\frac{1}{4}\right))$ iii) If $h(x) = -\frac{1}{4}g(x)$ find $foh(x)$	2) Simplify the following: $\frac{cosx}{1+sinx} + \frac{cosx}{1-sinx}$ 3) Simplify the following logarithm and rewrite it in terms of $logx$, $logy$ and $logz$: $log\sqrt{\frac{2(xy)^{-1}}{5z}}$
4) Find the derivative of the function: $f(x) = \frac{(x-2)^2}{1-\frac{2}{x}}$	
5) Find the following integral $\int \frac{x-\sqrt{x}}{x^{\frac{1}{3}}} dx$	

Figure 1: The tasks used for data collection

Results

The first stage of analysis illustrated that mental brackets were used in four of the five tasks. These were the tasks on functions, logarithms, derivatives, and integrals. Our analysis showed that mental brackets were present in 26 of the 40 students' scripts. The second stage of analysis illustrated two different uses of mental brackets: *successive signs* and *grouping terms*. In what follows, we discuss each of these categories and present representative examples from students' written responses.

The category *successive signs* appeared in students' solutions to the function task (task 1 – Figure 1). More specifically, as shown in Figure 2, the student substituted the expression of the function *g(x)* and the corresponding numerical value $-\frac{1}{4}$ in function $f(x)$. However, when substituting the value of $x = -\frac{1}{4}$ he did not add brackets around the negative number and continued working as if brackets were there.

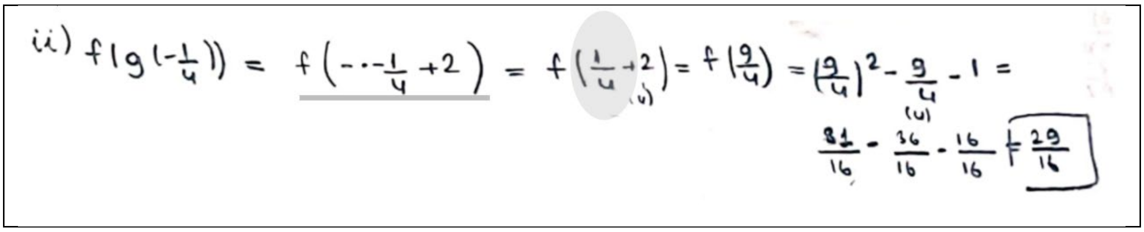

Figure 2: Student's 7001 solution to task 1ii)

The category *grouping terms* appeared in students' solutions to the logarithms, derivatives, and integrals tasks (tasks 3, 4, 5 – Figure 1). It should be noted that there were instances related to *grouping terms due to an application of a rule* related to logarithms, derivatives, and integrals (Figures 3, 4, 6) and also instances related to the use of algebra (Figure 5).

One instance related to *grouping terms due to an application of a logarithmic rule* can be seen in Figure 3, where the student is using mental brackets to apply the law of logarithms ($loga^n = nloga$). Initially, the student is changing the logarithmic expression from a root to an expression with a fractional power. However, in this process, he has not used brackets to encircle the whole expression which was earlier placed underneath the root. The power $\frac{1}{2}$ as its written is only applied on the 2 in the numerator. His next step though, the application of the logarithmic law, illustrates that he is using mental brackets as if the power $\frac{1}{2}$ is on the whole $\frac{2}{5xy^2}$.

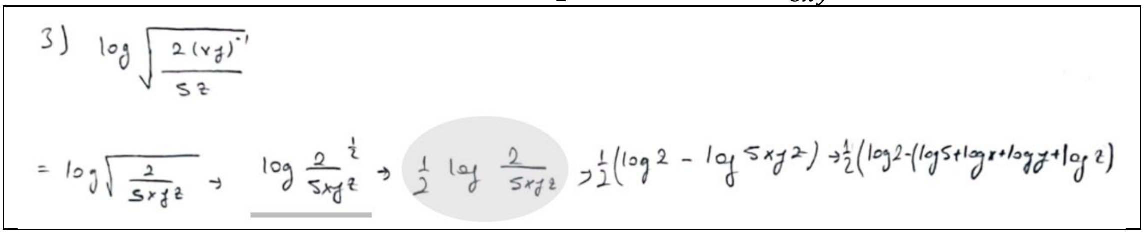

Figure 3: Student's 7038 solution to task 3

Similarly, in Figure 4, the student uses mental brackets twice in his writing. The student applies the *quotient rule for differentiation* (if $f(x) = \frac{u(x)}{v(x)}$ then $f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$). In this application of the rule though the student uses mental brackets twice: once to write the $v = 1 - \frac{2}{x}$ and the other time to write the $u = x^2 - 4x + 4$. These are considered instances of using mental brackets as the student continues his work considering these expressions as being enclosed by brackets in the next step and performs the appropriate calculations.

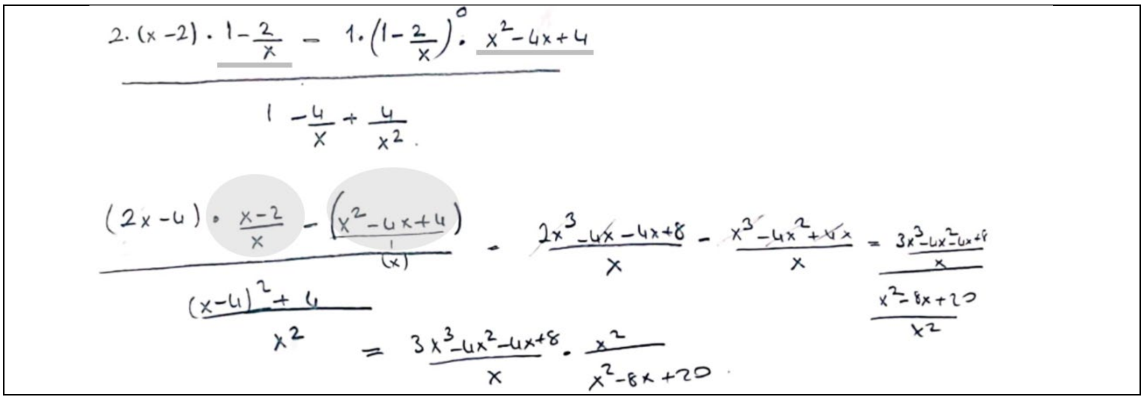

Figure 4: Student's 7028 solution to task 4

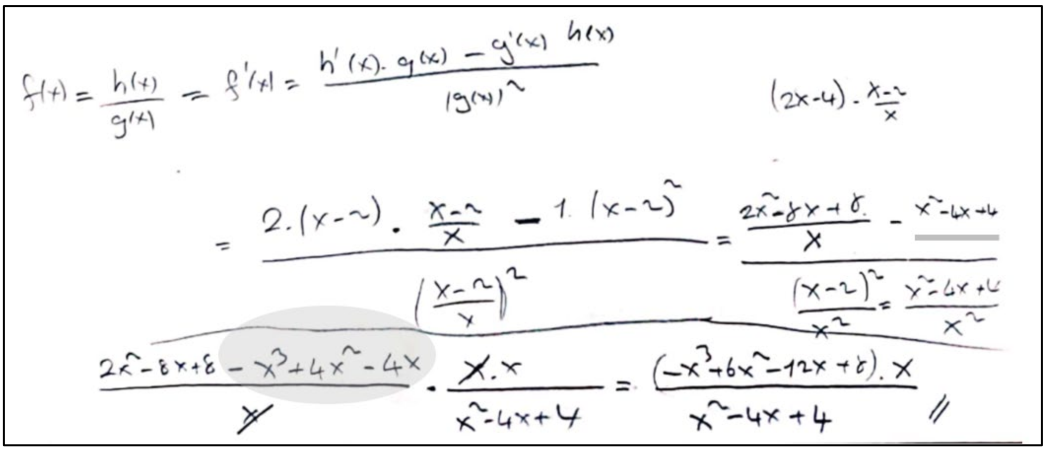

Figure 5: Student's 7020 solution to task 4

Similarly, the student's solution in Figure 5 uses mental brackets. However, this time they are observed only once and not in the first application of the quotient rule. The mental brackets in this script appear when the student expanded the expression $(x-2)^2$. This application of mental brackets relates to the use of algebra rather than differentiation rules.

Finally, in the integrals task (Figure 6) we note that the student is not consistently using the integration notation in the first line of his writing. However, this is not the focus of this paper. The mental brackets appear at the end of the first line of the student's writing which comes after the student has manipulated the integrand. The written expression does not explicitly clarify what is the integrand in this occasion. Although there is an integral sign there is the absence of brackets and the dx. However, the next line of writing indicates that the student performed the integration process as if the integrand contained both $x^{\frac{2}{3}}$ and $x^{\frac{1}{6}}$.

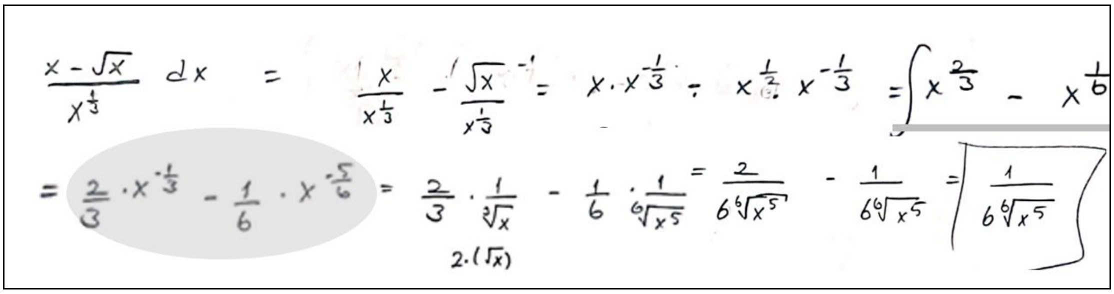

Figure 6: Student's 7036 solution to task 5

Discussion and conclusion

This paper reports the results of a pilot study where the aim was to investigate the presence of mental brackets in university students' solutions and further exemplify students' use of mental brackets. Our results extend further the notion of mental brackets, introduced by Linchevski and Livneh (1999) and defined in the context of numeracy in Papadopoulos and Gunnarsson (2020). The current literature on mental brackets mainly focuses on numeracy and primary schools students' writing (Papadopoulos and Gunnarsson, 2020), our results illustrate that the appearance of mental brackets is not specific to the mathematical topic and the students' age. Rather it seems to be appearing in multiple mathematical topics and students' writing at different educational levels.

We have to note that the task itself also plays a big role in whether mental brackets might appear in students' scripts. For example, the trigonometry task (Figure 1 – task 2) was not conducive to the appearance of mental brackets. This could be due to the fact that the students did not have to apply a particular rule that would necessitate the substitution of one term. Our results indicate that most of the times mental brackets appeared when the task demanded the application of a rule (e.g., the quotient rule or the product rule for differentiation) or when the students had to substitute a value or an expression (e.g., substituting a value for a variable or an expression in a composite function).

Also, it is important to note that we observed students' use of mental brackets related to *successive signs* and *grouping terms*. The latter category can be further explored by highlighting even further whether the *rule* use was related to the mathematical topic (e.g., logarithms, derivatives, integrals) in question (Figures 3, 4, and 6) or algebra (Figure 5).

Our analysis showed that students are using mental brackets in different tasks and multiple times in the same task (as seen in Figure 4). The students' writing signals that they are aware of the rules for the order of operations. However, their use of mental brackets illustrates that they consider this a reliable way of writing. It is important to further examine the students' views through interviews and explore whether they are using mental brackets due to speed and/or their focus being on the final goal of the task rather than checking the logical cohesion of their step-by-step arguments. Furthermore, studies exploring students' engagement with technological environments (Drijvers et al., 2010) discuss the importance of the use of brackets when entering formulas and expressions in various technological environments. Some ICT-applications provide one-line formulae entry, which means that students should use brackets very carefully to ensure that the entered one-line formula corresponds to their given formula. The same holds in case students are working on mathematical tasks in programming environments (e.g., working in a Logo-based environment).

In the next steps of our study, we will be investigating further university students' use of mental brackets in differentiation and integration tasks. We also want to interview students who have used mental brackets in their solutions to provide further insights as to why they are using mental brackets and also get insights into their lecturers' perspectives on students' use of mental brackets.

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