An analysis of students' reasoning about surface area and volume measurement: A focus on prisms

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The purpose of the study was to explore students' reasoning while solving tasks about surface area and volume measurement. For this purpose, one-to-one task-based interviews were conducted with three middle school students (11-14 years old). For the first task, the students were asked to build as many different prisms as they can with twelve unit cubes and to determine the volume and surface area of the prisms they built. For the second task, they were asked to build a large cube with twenty-seven unit cubes and to explain how the volume and surface area changed when the unit cubes from some parts of the large cube were removed. The findings indicated that the students' reasoning involved some misconceptions: Surface area changes depending on how the prism is positioned, and prisms with the same volume have the same surface area. The possible reasons behind these misconceptions were discussed.

Keywords: surface area; volume; student reasoning; misconceptions

Introduction

Volume measurement is a significant topic from elementary to high school grades since it provides context for enhancing students' knowledge of arithmetic, geometric reasoning, and spatial structuring (Battista, 2003; Lehrer, 2003). Surface area measurement expands students' understanding of volume measurement (Outhred, & Mitchelmore, 2000). Students' poor understanding of surface area and volume measurement were indicated in the previous research studies (Dorko & Speer 2013; Tan Sisman & Aksu, 2016). Rote memorization and application of formulas without conceptual understanding are likely to result in difficulty in interpreting two dimensional representations of three-dimensional objects, to prevent students from conceptualizing surface area and volume, and to cause confusion of these concepts (Battista & Clements, 1996; French, 2004).

Presenting purposefully designed tasks about surface area and volume measurement and analysing students' reasoning while they are solving these tasks are important to understand their (mis)conceptions, errors, difficulties and how they internalize concepts. This can provide teachers, researchers and curriculum developers with an insight to enhance students' understanding based on their reasoning. Previous research has focused on students' reasoning regarding volume or surface area measurement in separate tasks. However, it is important to shed light on students' reasoning about both volume and surface area measurement in the same task. Thus, the purpose of the study is to explore middle school students' reasoning while they are solving tasks about surface area and volume measurement, particularly the surface area and volume of prisms. We sought an answer to the following research question:

- How do students reason about surface area and volume measurement while solving tasks?

Method

In the present study, a multiple case study was used to understand the similarities and differences within and between cases (Yin, 2003). The participants of the study were three middle school students (5^{th}-8^{th} grade level/11-14 years old) named Yaman, Irmak and Ayaz (pseudonyms). While two of the participants were boys (Yaman and Ayaz), one of them was a girl (Irmak). Yaman had just finished 6^{th} grade, Irmak had just finished 7^{th} grade, and Ayaz had just finished 8^{th} grade at the time of the study. In the Turkish mathematics curriculum (MoNE, 2018), students encounter the surface area in 5^{th} grade level for the first time and learn to calculate the surface area of a rectangular prism. Volume is introduced in the 6^{th} grade level in which students are expected to understand that the number of unit cubes that exactly fill a rectangular prism is the volume of the rectangular prism, to build different rectangular prisms having the same volume with unit cubes, and to construct the volume formula for the rectangular prism. Hence, all participants had already learned surface area and volume of the prisms before the study.

In order to collect data, one-to-one task-based interviews were conducted with each student by the first author (hereafter, the researcher). The first task was adapted from Haylock (2010) and the second task was adapted from Burger et al. (2014) and modified by the researchers. For the first task, the students were asked to build as many different prisms as they can with twelve unit cubes and determine the volume and surface area of the prisms they built. For the second task (Figure 1), they were asked to build a large cube with twenty-seven unit cubes and explain how the volume and surface area changed when the unit cubes from some parts of the large cube were removed.

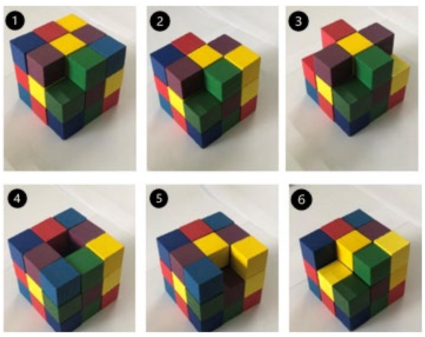

Figure 1: The second task

The interviews were video-recorded and transcribed for the data analysis. Interview transcripts were read several times and analysed by the researchers individually according to each student's explanations and actions during the tasks. Then, we compared the codes, discussed discrepancies and arrived at a consensus.

Findings

The case of Yaman

In the first task, Yaman was able to build all possible prisms (Figure 2). When he was asked the volume of the prisms, he correctly stated that the volume of all prisms is equal because all of them consist of twelve unit cubes.

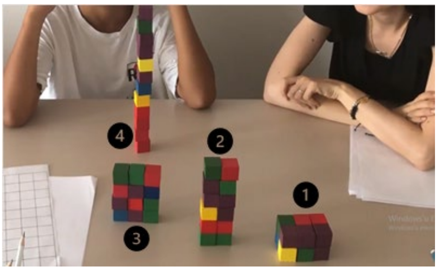

Figure 2: The prisms built by Yaman

Regarding the surface area of the prisms, he thought that the first prism has the largest surface area while the fourth prism has the smallest surface area.

> R: So, what can you say about the surface are of the prisms?
> Y: First one has the largest surface area because it covers more region which is 6 unit squares. The one with the smallest surface area is the fourth one because there is 1 unit square touching the floor.

Yaman considered the surface areas of the prisms as areas of the bases of the prisms. Then, the researcher rotated the first prism (Figure 3) and asked about the surface area of this prism once again. In this case, he asserted that surface area changes but still larger than the surface area of the other prisms.

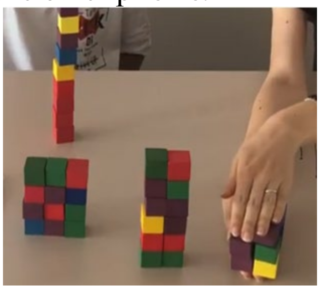

Figure 3: Rotation of the first prism by the researcher

Yaman thinks that surface area changes depending on how the prism is positioned. In the second task, like in the first task, he only considered the top face (base) of the large cube while making judgements. Therefore, he stated that surface area does not change in each case since new faces of the cubes appear in place of the removed faces. When he was asked whether there are other cases in which surface area increases or decreases, he declared that surface area does not change in any case. In terms of volume, he was aware that the volume decreases as the unit cubes were removed.

> R: How does the volume of the large cube change as we removed the unit cubes?
> Y: At the beginning, the volume was twenty-seven unit cubes. As we take out cubes, the volume decreases. For instance, if I remove 1 cube, the volume is twenty-six unit cubes. If I remove four cubes, the volume is twenty-three unit cubes. The more cubes we take, the less the volume.

The case of Irmak

Irmak built four prisms with twelve-unit cubes (Figure 4). However, two of the prisms were actually the same. Then, the researcher asked if the lastly built prism (4) was different from the previously built prisms.

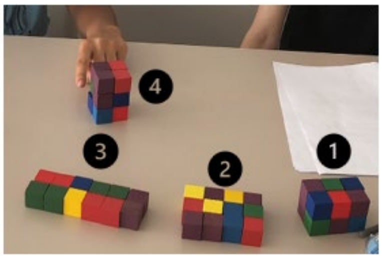

Figure 4: The prisms built by Irmak

> R: Do you think that this prism (pointing to the fourth prism in the figure) is different from the other three prisms you built?
> I: Yes, they are different. It looks like this (pointing to the first prism), but this one (pointing to the fourth prism) has three cubes in height and the other one had two cubes.

Irmak thought that two prisms were different prisms because their heights were different. When she was asked about the volume of the prisms, she gave a similar response as Yaman's and correctly stated that the volume of the prisms is twelve unit cubes. However, her reasoning about the surface area of the prisms involved a misconception.

> R: What about the surface area of the prisms? What can you say when you compare the surface areas of the prisms?
> I: Since the volume of the prisms is the same, the surface areas are also the same. Only the appearances of the prisms are different.

This excerpt shows Irmak's misconception that prisms with the same volume must also have the same surface area. In the second task, she thought that as the unit cubes are removed, the surface area always decreases and removing more cubes results in less surface area. Similarly, she stated that volume decreased since the number of unit cubes decreased. Thus, her reasoning in the second task was in alignment with the one in the first task. She believes that there is a constant relationship between surface area and volume.

The case of Ayaz

Ayaz initially built two prisms with twelve unit cubes (2x2x3 and 3x2x2). Then, the researcher asked him whether the prisms are different from each other. After thinking for a while, he realized that by rotating one of the prisms, the other prism can be obtained and hence, they are the same prisms. After that, he was asked if he could build different prisms. He was only able to build the second prism in addition to the first prism (Figure 5).

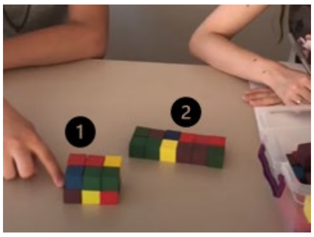

Figure 5: The prisms built by Ayaz

While finding the volume of the prisms, he did not explicitly refer to the volume formula but identified the linear dimensions of each prism and multiplied the three numbers. In this way, he realized that the volume of two prisms was twelve unit cubes. Unlike Yaman and Irmak, Ayaz was aware that the sum of the areas of bases and lateral faces gives the surface area of the prism. He counted aloud and correctly found the surface areas of the two prisms to be thirty-two and forty unit squares, respectively. By asking the underlying reason for obtaining different surface areas, we challenged him and tried to uncover his reasoning about surface area.

> R: What could be the reason why the surface areas are different although the volume of the prisms is the same?
> A: Arranging cubes in different ways changes the surface area. If the cubes are closer and the prism is more compact, the surface area is smaller.

In the second task, Ayaz could notice that the surface area would increase or not change depending on where the cubes were removed. Then, he was asked whether the surface area can be decreased by removing the cubes.

> R: You said that the surface area increases or does not change. Do you think there are cases where the surface area is reduced by removing the cubes?
> A: To reduce, I need to remove the cubes with the largest number of visible faces.
> R: Which cubes have the maximum number of visible faces?
> A: Cubes in the corners have three visible faces (after thinking for a while, he removed 3 cubes from the right side (Figure 6)) I can reduce the surface area by removing these. Initially, 8 faces were visible. When I removed them, the number of visible faces went down to 6.

Figure 6: Reducing surface area by Ayaz removing the cubes

He was able to reduce the surface area by removing the cubes. On the other hand, he was confused when he was asked how the volume changed because he could not apply the volume formula.

> R: How does volume change as we removed cubes?
> A: (attempted to use the volume formula and looked puzzled) I could not calculate.
> R: Why?
> A: Here (length) is 3 and here (width) is 3. But I do not know if this is 2 or 3 (pointing to height).

Ayaz was not able to find the volume because his formula-based approach did not work for finding the volume after removing the cubes.

Discussion and conclusion

All students' reasoning in the first task was consistent with their reasoning in the second task. In the curriculum there are some objectives about surface area measurement in the 5th and 8th grade levels, but not in the 6th and 7th grade levels. Therefore, the 6th and 7th graders may have forgotten what they have learnt in the 5th grade level. Moreover, the correct reasoning of the 8th grader reflects the availability of content regarding the surface area in the 8th grade level.

In the second task, when the cubes were removed, a regular-shaped prism turned into an irregular-shaped object. Familiarity with regular-shaped prisms before and having more experience in regular-shaped prisms might have led to the 6th grader's difficulty and inability to recognize newly appeared faces in this task. Moreover, the 8th grader could not use the volume formula with irregular-shaped objects since he could not determine which numbers to be substituted in the formula. This might stem from learning the volume formula as an algorithm that only applies to regular-shaped prisms (Vasilyeva et al., 2013). The 8th grader's formula-based approach lacks an understanding of the conceptual foundations of that approach. Thus, it can be concluded that there should be more content regarding the volume and surface area of irregular-shaped objects in school mathematics.

The reason behind the misconception of Irmak might be related to handling surface area and volume measurement as unrelated domains (Tan Sisman & Aksu, 2016). Thus, in order to improve students' reasoning skills, they should be asked to find a family of prisms having constant volume or surface area and discover the pattern of maximum-minimum values regarding surface area or volume. In this way, the volume and surface area of prisms can be contrasted, and the distinction between them can be highlighted.

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