How to make practice more perfect? How to make practice more productive?

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Although practice is regarded as a crucial component for promoting procedural fluency, it is always stereotyped as mechanically repeating steps and being over-simplified to 'More practice makes perfect'. This phenomenon might result from the incomplete understanding of conceptual and procedural knowledge. Therefore, it is necessary to reposition the role of procedural learning by introducing deep procedural learning. Deep procedural knowledge refers to the cognitive understanding of the computational processes and flexible use of computational strategies. Purposely designed productive practices, which aim at developing higher-order thinking when practising essential procedural skills, are expected to prompt the deep procedural learning. To evaluate the progress of procedural learning by using productive practice, theoretical framework about the relationship between deep procedural learning and mathematical thinking is introduced in this paper.

Keywords: deep procedural knowledge; productive practice; mathematical thinking

Introduction

According to the National Curriculum in England (Department for Education, 2013), the main objectives of mathematics learning are achieving procedural fluency, promoting conceptual understanding and enhancing critical thinking. There is a growing concern about effectively carrying out practices (Codding et al., 2011). Nonetheless, the didactic potential of practice is often inadvertently reduced to a mechanical one; most of the practice-related studies are about the way of implementation, but rarely discuss the fundamental design of practice from the whole mathematical development point of view. Consequently, there are always doubts about the actual function of the practices in mathematical understanding. Lehtinen et al. (2017) mention that the role of practice in mathematics education became questionable after the constructivist epistemology was advocated in the field of mathematics education. Practice is considered to be a method for automatising skills with shallow understanding, whereas constructivism leads to a deeper conceptual learning. Thus, educators always focus on developing methods to improve conceptual learning but overlook the potential of procedural learning and practice. The aim of this study is to introduce productive practice and to present a theoretical framework for explaining how productive practice can deepen procedural knowledge in theory.

The understanding of conceptual knowledge and procedural knowledge

From the mathematics education point of view, procedural and conceptual knowledge should not be placed in an opposing position (Schneider & Rittle-Johnson, 2011; Star, 2005). The importance of practice is being diminished, probably because of the incomplete understanding of procedural knowledge. Star (2005) argues that both conceptual knowledge and procedural knowledge are not being described thoroughly; that is, conceptual knowledge is defined in terms of in-depth quality – particularly emphasising the richness of connection, but procedural knowledge is defined in term of superficial quality – particularly focusing on the sequences of action but ignoring the heuristic procedure which requires deeper knowledge to justify choices. He points out that both types of conceptual and procedural knowledge contain their own knowledge quality, i.e., superficial one and deep one (Star, 2005).

> The term conceptual knowledge has come to encompass not only what is known (knowledge of concept) but also one way that concepts can be known. Similarly, the term procedural knowledge indicates not only what is known (knowledge of procedures) but also one way that procedures (algorithms) can be known. (Star, 2005, p.408)

Superficial procedural knowledge is referred to knowledge of symbols, syntax, and steps for completing the tasks (Hiebert & Lefevre, 1986) which “is not rich in connections” (Star, 2005, p.407); whereas deep procedural knowledge concerns the quality of the connections within procedures which is related to having planning knowledge, for example, when and which particular procedure(s) should be used (Star, 2000). Star (2005) emphasises that making wise choices of using certain strategies can indicate sophisticated cognitive understanding of the computational process. Conclusively, superficial procedural knowledge would be the common usage of procedural knowledge, like following the procedure correctly; while deep procedural knowledge would be associated with comprehension, flexibility and critical judgement (Star, 2005). The educators’ incomplete understanding of procedural knowledge leads to shallow procedural learning.

Theoretical framework

Flexibility on choosing strategies, which is categorised as deep procedural knowledge, is abstract to measure as it cannot be simply determined by the students’ answers because completion of a task can be the result of choosing an effective strategy or the result of blindly repeating the process of trial and error. Flexibility involves careful thinking which reflects the ability of planning strategies for approaching and solving new problems (Star, 2000). Therefore, a framework with clear indicators of students’ thinking should be developed for describing and explaining the process of procedural learning. Tall (2009) accentuates that Mason’s framework is suitable in explaining the process of mathematical thinking and deep procedural learning because it can capture the moment that critical ideas are generated when students are tackling the productive practice. Mason et al. (2010) define three phases of mathematical thinking: Manipulating, Getting a sense of pattern and Articulation. Manipulating is an entry stage when students encounter unfamiliar but well-designed tasks. At this stage, students are expected to actively explore the meaning of the tasks and try specialising the particular examples. After enough exploration, the intention of conjecturing about the relationship of the variables in the tasks indicates that the students are moving to the second phase – Getting a sense of pattern. During this phrase, students’ conjecture may be vague; so, further checking and justifying happen naturally afterwards until they recognise the pattern lying between the variables and are ready to experience the process of generalising. This recognition makes the conjecture clearer to their own self. To elevate to the phrase of Articulation, the student applies his/her conjecture, then continuous to tests his/her statement of generalisation and convinces not only himself/herself but also the others. This process of convincing crystallises their thought, allow students to achieve a more efficient utilisation with their generalisation and reach the phrase of Articulation. According to Mason et al. (2010), this process of mathematical thinking does not end at the stage of articulation; on the contrary, achieving articulation can become an element of new manipulation.

When doing productive practice, mathematical thinking process is expected to happen during acquiring the abstract operational conceptions, such as the law of arithmetic operations. Sfard (1991) suggests that the correlation between the development of operational conceptions and mathematical thinking is remarkably strong. Thus, she advocates a framework especially for describing the relationship between the mathematical thinking processes and operational origins of mathematical objects. Her framework has three phrases, i.e., interiorisation, condensation and reification. Interiorisation represents a stage of getting to know with process; condensation represents a stage of automatisation which shows the ability of thinking particular processes as a whole and students can easily output the result without too much procedural thinking; reification represents “an ontological shift” and “ability to see something familiar in a totally new light” which means students are enlightened to see the relationship and make use of it (Sfard, 1991, p.19).

Sfard’s and Manson’s frameworks complement each other perfectly; for example, there is a strong evidence that when one is in the condensation phrase, he/she is ready for making a conjecture or has reached the stage of Getting a sense of pattern. This is because students who can finish the task automatically can consider the tasks as a whole, rather than as separate drill and practice exercises. Then they can start seeing the relationship of different variables within the whole tasks in which they make some conjectures. These are explicit external indicators for investigating the progress of the students’ mathematical thinking development. Combining the theories of specific cycles in pattern recognition and computational concept development (Figure 1) enable a consistent and concrete interpretation of the structure and quality of students’ responses during productive practice. This framework can be used for understanding the students’ mathematical deep procedural learning progress when completing productive practices.

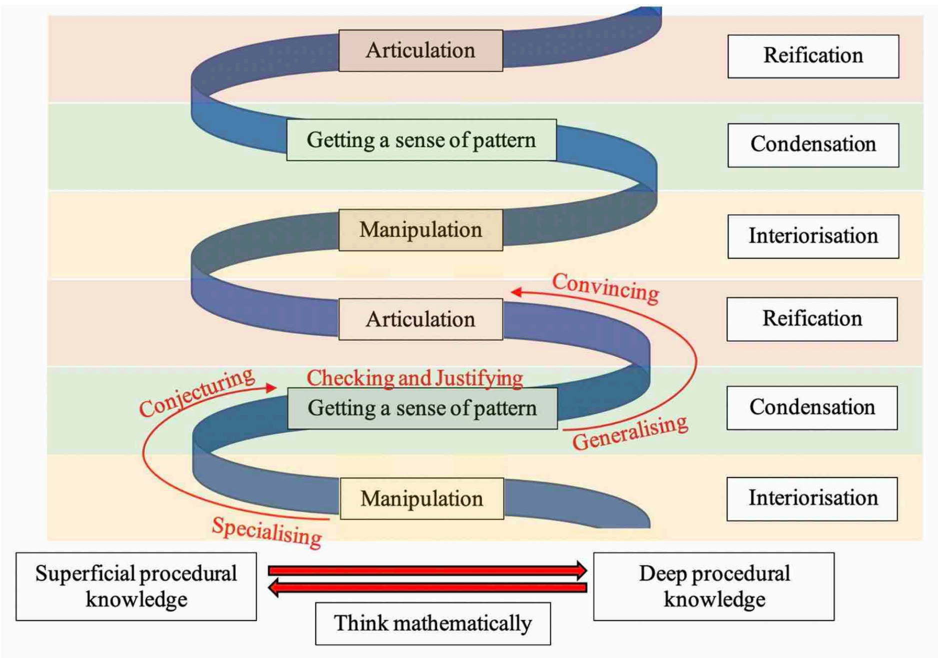

Figure 1: Theoretical framework

How to achieve deep procedural understanding?

To achieve deep procedural understanding, the role of practice must be redefined. When considering an effective design of practice for a complete mathematical learning, the view of mathematics as a science of pattern plays a prominent role (Steen, 1988; Wittmann, 2006; Fujita & Hyde, 2013). Steen (1988) explains that most of the mathematics relates to patterns seeking, disclosing the relationships among patterns by generalising mathematical theories and then applying the discovered patterns for predicting. It is crucial that the mathematics learning should provide opportunity for developing logical thinking, for enhancing ability of abstraction and generalisation, and for encouraging the habit of looking for patterns (Cockcroft, 1982). Therefore, the role of practice should not remain in the concept of 'drill and practice', but should purposely create opportunities for developing higher order thinking and understanding during the training of essential procedural skills. To have a clear view of practicing, Wittmann and Müller (2017) categorise different types of practice, namely Introductory practice, Basic practice and Productive practice.

> Introductory practice aims at making students familiar with a new topic…Basic practice refers to the extended practice of a small set of skills which must be mastered automatically…Productive practice is a kind of magic wand: It integrates the practice of skills with the exploration and explanation of patterns, with the solution of problems and with application. (Wittmann, 2019, p.21)

These three types of practice have their own unique functions and serve different learning purposes, so they are both important in the learning process and they are irreplaceable. Among these three types of practices, productive practice widely adopts the philosophy of viewing mathematics as sciences of patterns (Wittmann, 2019) and it is highly possible to enhance both superficial and deep procedural knowledge, especially able to catalyse the ability of selecting between different strategies.

Productive practice

Wittmann (2019) emphasises that productive practices are mathematically rich and well-structured small tasks. These practices aim to integrate skills practising with mathematical investigation which involves varied kinds of cognitive activities. These activities can provide unique opportunities for the students to explore and explain the mathematical patterns based on their operational experience while also providing enough practice opportunity (Wittmann, 2019). Two examples of productive practices from 'Handbuch produktiver Rechenübungen. Band 1: Vom Einspluseins zum Einmaleins' (Wittmann & Müller, 2017) are shown below:

Schöne Päckchen (Pretty Packages) as an example

Pretty Packages (Figure 2) contains exercises and pattern exploration. As the questions are deliberately arranged in roll, it is easier for students to recognise the patterns. While students are having plenty of practice time on two-digit addition, they also have the chance to explore the relationship between the purposely arranged questions and the answers. Through the discovery of patterns, students can understand the concept of particular arithmetic laws (in this case, the associative law) and solve the questions effectively. Sets A, B and C are examples of three different patterns: Set A has one fixed addend and a flexible addend that either ascends or descends; both addends in set B increase or decrease by 1 together every time; and one of the addends in set C increases and the other addend decreases accordingly.

A	B	C
12+15=?	12+15=?	12+15=?
12+16=?	13+16=?	13+14=?
12+17=?	14+17=?	14+13=?
12+18=?	15+18=?	15+12=?
:	:	:

Figure 2: Schöne Päckchen (Pretty Packages)

Number pyramid as another example

Number pyramids are commonly used in school for practicing addition and subtraction. Every two adjacent number bricks in the same row are added and their sum is written on the upper brick which connects to these two bricks (Figure 3). Number pyramid itself is based on Pascal's triangle, which is a source of rich mathematical properties in mathematics. It can be used as a productive practice because it is flexible and the numbers in the bricks can be arranged deliberately to create different circumstances for exploring. Series of number pyramids can provide opportunities for students to explore mathematical patterns, generate higher order thinking and experience some advanced level of mathematical contents. For example, in a series of three rows number pyramids, if all the numbers on the bottom bricks remain unchanged but the middle one increases by 1, then the sum of the top brick (red one) will increase by 2 (Figure 4). Students do not need to explain the pattern in terms of algebraic expression. During the process of dealing with the actual numbers and finishing a collection of addition exercises, they can discover the pattern of the given numbers at the bottom and observe how this might affect the later answers.

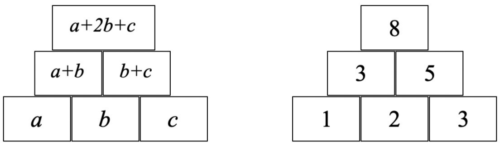

Figure 3: Number pyramids

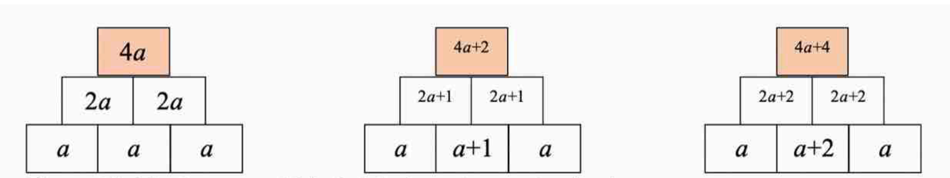

Figure 4: The bottom middle bricks keep increasing by 1

Final remarks

Students should have enough opportunity to get familiar with essential mathematical skills by doing productive practices; meanwhile, they could discover and describe the pattern behind the tasks, then they can make conjectures, test it and justify it. In order to explore the students' developmental process from superficial procedural learning to deep procedural learning with productive practice, a study of using number pyramids in year two addition and subtraction learning will be conducted (involves students aged around 6 to 7). The theoretical framework mentioned in previous section is used for disclosing the mathematical thinking process, thereby revealing the progress of deep procedural learning. Furthermore, both the design of productive practice and the theoretical framework will continuously be refined and developed throughout the study.

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