A new property of flexibility in equation solving: Making connections

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Algebra involves various activities including transformational activities that are mainly about solving equations. Fluency (or flexibility) in these activities is important. Several researchers have proposed conceptualisations of flexibility in equation solving. This paper makes a reflection about flexibility in equation solving that contributes to the extension of Star and Seifert's operationalisation. Examples are used as context for the reflection. The need for another property of flexibility, namely making connections, is suggested to deepen investigations of students' flexibility in equation solving and its provision in teaching.

Keywords: algebra; connections; equation solving; flexibility

Introduction

Making connections across mathematical and other learning areas recently has been a focus of school curricula across the world including in Australia and the USA. The desire for connections in mathematics is yet not new. The National Council of Teachers of Mathematics (NCTM) called for opportunities for students to experience the connections and interplay of various mathematical topics in their Curriculum and Evaluation Standards for School Mathematics document in 1989. This desire has persisted in the associated updated documents (e.g., NCTM, 2014) because students develop key competencies as a result of experiencing connections in mathematics. Students link conceptual and procedural knowledge, recognise equivalent representations of the same concept, use the connections among mathematical topics and see mathematics as an integrated whole, and use mathematics in daily life activities (Coxford, 1995).

Algebra learning plays an important role for students in college level studies. Success in algebra, nevertheless, is an ongoing concern for educators as students' algebra learning outcomes are sometimes poor in both national (e.g., National Assessment Program – Literacy and Numeracy [NAPLAN]) and international assessments (e.g., Trends in International Mathematics and Science Studies [TIMSS]). Algebra involves various activities including representational, transformational, and generalizing and justifying activities (Kilpatrick et al., 2001). Transformational activities are rule-based activities which are chiefly about collecting like terms, factoring, expanding, substituting, simplifying expressions, and solving equations (e.g., Find x if $4(x+3)=2x+19$) (Kilpatrick et al., 2001). Fluency (or flexibility) in these activities is important, but usually students do not truly see the aim and structure of the procedures (McCallum et al., 2010). For instance, only 15 percent of Year 9 Victorian students (in Australia) gave the correct answer to the transformational question: $2(2x-3)+2+?=7x-4$ (Sullivan, 2011).

Several researchers have proposed operationalisations on flexibility in equation solving (see Newton et al., 2010). One of the most relevant of these is the one proposed by Star and Seifert (2006). In this operational definition, flexibility is knowing multiple solution procedures to a problem and having the capacity to generate new and more efficient procedures to solve it (Star & Seifert, 2006). Even though this definition has had an impact on the research on flexibility (e.g., Rittle-Johnson & Star, 2007; Star & Rittle-Johnson, 2008; Xu et al., 2017), there are calls for a more comprehensive account of flexibility (e.g., Ionescu, 2012) given its contribution to efficient problem solving (e.g., Ionescu, 2012; Newton et al., 2010)

The current paper offers a reflection about flexibility in equation solving that contributes to the extension of the definition by Star and Seifert (2006). To this end, examples of equation solving are included as a context for the reflection that there is a need for another property in the definition, namely the aspect of making connections, to deepen both the investigations of flexibility in equation solving in students and its provision or fostering in teaching. Connections are used because they are fundamental in the teaching and learning of mathematics and enable students to perform mathematical tasks. Importantly, when performing transformational activities, students make a number of procedural connections.

Mathematical Connections

A mathematical connection is an exact relationship between two or more mathematical ideas (Businskas, 2009). Several researchers have sought to understand the concept of connection and identified types of mathematical connections. One of the initial conceptual models for investigations in this field was proposed by Businskas (2009). In this model seven types of connections are suggested: alternate representations; equivalent representations; common features; inclusion; generalisation; implication; and procedures. Other researchers have extended Businskas' model, and recently, Rodríguez-Nieto et al. (2020) refined the existing conceptualisations and suggested an extension of the current types of mathematical connections by including metaphors. Equivalent representations are the focus of this paper.

Two major registers of representations are defined: treatments; and conversions (Gagatsis & Shiakalli, 2004), which are also identified as equivalent and alternate representations, respectively (Businskas, 2009). Conversions are transformations of representations that consist in changing the register without changing the objects being denoted, while treatments are transformations of representations, which take place within the same register in which they have been formed (Gagatsis & Shiakalli, 2004). For example, the graph of a parabola is an alternate representation (conversion) of the function $f(x)=ax^2+bx+c$ because the two representations are from two different registers (i.e. algebraic and graphic). While $3+2$ is equivalent to $2+3$, or $f(x)=ax^2+bx+c$ is equivalent to $f(x)=a(x-p)^2+q$ since both are from the same registers.

Flexibility in Equation Solving

The notion of flexibility is colloquially defined as the ability to change according to particular circumstances (Star & Seifert, 2006). In mathematics it has a specific meaning, and I would like to define it based on Star and Seifert (2006). Think about the equations $4(x+1)=8$; $4(x+1)+2(x+1)=12$; and $4(x+1)+3x+7=8+3x+7$ and review the solutions to these given by two hypothetical students that are presented in Table 1. What do you observe?

Table 1: Hypothetical Student Solutions to the Equations (adapted from Star and Seifert, 2006, p.281).

Equation	Student A	Student B
$4(x+1)=8$	$4(x+1)=8$ $x+1=2$ $x=1$	$4(x+1)=8$ $4x+4=8$ $4x=4$ $x=1$
$4(x+1)+2(x+1)=12$	$4(x+1)+2(x+1)=12$ $6(x+1)=12$ $x+1=2$ $x=1$	$4(x+1)+2(x+1)=12$ $4x+4+2x+2=12$ $6x+6=12$ $6x=6$ $x=1$
$4(x+1)+3x+7=8+3x+7$	$4(x+1)+3x+7=8+3x+7$ $4(x+1)=8$ $x+1=2$ $x=1$	$4(x+1)+3x+7=8+3x+7$ $4x+4+3x+7=8+3x+7$ $7x+11=3x+15$ $4x=4$ $x=1$

In these cases, both students solve the equations correctly. According to Star and Seifert (2006), however, Student A creates more innovative solutions to the problems and completes all three equations using three different solution procedures. Student A divides by 4 as a first step in the first equation; combines the like terms $x+1$ first in the second equation; and recognises and cancels the like terms $3x+7$ in the final equation. However, Student B uses the same (standard) algorithm on all problems: expand; combine; subtract from both; and divide. Student A's solutions are more efficient on all three equations; that is, they require fewer steps. Like Student B, Student A has the knowledge of standard algorithms, but Student A has the additional capacity to use them in non-standard ways in performing particular types of tasks.

For Star and Seifert (2006), flexibility in mathematics, then, can be defined as having knowledge of multiple solution procedures to a problem, a sense of when each way is most efficient, and the capacity to invent or innovate creative new procedures. Qualities such as being able to create multiple and efficient solutions to a given problem make students more flexible thinkers and problem solvers. Star and Seifert (2006) identify "a flexible solver as one who (a) has knowledge of multiple solution procedures, and (b) has the capacity to invent or innovate to create new procedures" (p.282) accordingly and consider these two as indicators of flexibility in equation solving. The present paper suggests extending this operationalisation. It is proposed to add a third property to the above definition of a flexible problem solver, namely: (c) (a flexible thinker) has the ability to make connections between mathematical ideas and concepts.

In equation solving one of the main big ideas is equivalence. Examples of mathematical understanding in terms of algebraic expressions and equations include:

> Algebraic expressions can be named in an infinite number of different but equivalent ways. For example: $2(x–12)=2x–24=2x–(28–4)$
>
> A given equation can be represented in an infinite number of different ways that have the same solution. For instance, $3x–5=16$ and $3x=21$ are equivalent equations; they have the same solution, 7 (Charles, 2005, p.14).

In order to become flexible, students must make contact with equivalent representations of the given expression or equation.

Consider the equation below used by Star and Seifert (2006, p.286) to assess flexibility, and typically solved in mathematics classes using algebra: $4(x+3)=16x$. Figure 1 presents a number of different solutions to the equation. Based on Star and Seifert's operationalisation, Student B (less flexible thinker) would solve the equation following standard algorithms (e.g., Solution 1), but how would Student A, who is a more flexible thinker, solve it? Would Student A see the equation as $x+3=4x$ instead and perform it accordingly, such as Solution 2? Or would Student A represent the equation by equating zero and perform the equation accordingly, such as Solution 4?

Solution 1	*Solution 2*	*Solution 3*	*Solution 4*
$4(x+3)=16x$	$4(x+3)=16x$	$4(x+3)=16x$	$4(x+3)=16x$
$4x+12=16x$	$4(x+3)=4\times 4x$	$4\times \blacksquare = 4\times \Delta$	$4(x+3)-16x=0$
$12=12x$	$x+3=4x$	$\blacksquare=\Delta$	$4x+12-16x=0$
$x=1$	$3x=3$	$x+3=4x$	$12-12x=0$
	$x=1$	$x+3=x+3x$	$12=12x$
		$3=3x$	$x=1$
		$x=1$	

Figure 1: Multiple Solution to the Equation *4(x+3)=16x*.

In fact, how more flexible thinkers would solve the problem is ambiguous and implies the need to involve other properties of flexibility, namely establishing connections. The terms in the equation, and accordingly the equation itself, can be represented by its various equivalents. The equivalent representation type of mathematical connections is extensively identified in Solutions 1 through 4.

Here is another equation: *5(x+3)+10x=35+5x* (Star & Seifert, 2006, p.286). Some solutions to the equation are presented in Figure 2. While in Solution 1 the standard algorithms are performed, in Solution 2 each term is divided by 5, and in Solution 3 the like terms are cancelled. According to Star and Seifert's definition, a less flexible thinker is supposed to give Solution 1 where standard procedures are followed step by step. How more flexible thinkers would solve the problem again is ambiguous. In solving the equation, procedural types of connections are established, and examples are provided in Figure 2. The equation is represented in different but equivalent forms in each solution.

Solution 1	*Solution 2*	*Solution 3*
$5(x+3)+10x=35+5x$	$5(x+3)+10x=35+5x$	$5(x+3)+10x=35+5x$
$5x+15+10x=35+5x$	$\frac{5(x+3)}{5}+\frac{10x}{5}=\frac{35}{5}+\frac{5x}{5}$	$5(x+3)+5\times 2x=5(7+x)$
$15x+15=35+5x$	$x+3+2x=7+x$	$x+3+2x=7+x$
$10x=20$	$3+2x=7$	$x+3+2x=4+3+x$
$x=2$	$2x=4$	$2x=4$
	$x=2$	$x=2$
		OR
		$5(x+3)+5\times 2x=5(7+x)$
		$x+3+2x=7+x$
		$3x+3=7+x$
		$2x=4$
		$x=2$

Figure 2: Multiple Solution to the Equation *5(x+3)+10x=35+5x*.

The examples provided here show that, even though abilities such as being able to create multiple and efficient solutions to a given problem are important flexible behaviors in equation solving, there should be other qualities. These examples imply that linking concepts with broader big ideas is one mechanism involved in flexibility. Making connections must, therefore, be considered as a property of flexibility. As such, in addition to having knowledge of standard algorithms to perform relevant tasks and using that knowledge in non-standard ways to do a better job, more flexible thinkers have the capacity to connect procedures with the broader big ideas or concepts (e.g., equivalence).

Final Remarks

This paper makes a reflection on Star and Seifert's (2006) conceptualisation of flexibility in equation solving. The reflection is by no means exhaustive. Its main purpose is to contribute to the development of the current conceptualisation. Specifically, it makes evident that the two properties of flexibility in equation solving in Star and Seifert's operational definition need to be extended. One way of doing this is to incorporate an important quality of flexibility, making connections, which is fundamental for understanding mathematical concepts. Thus, the incorporation of making connections as an additional property of the definition of flexibility in equation solving is proposed. It is considered that this extension will allow a better analysis of flexibility in relevant investigations and will contribute to teaching and learning processes, especially when making connections is considered as a medium frequently used by mathematics teachers during their teaching (e.g., Hatisaru, 2020; Rodríguez-Nieto et al., 2020).

The paper opens potential routes to study the issue further. Firstly, this paper suggests that when mathematical connections are made in transformational activities, they represent a property of flexibility. Studying whether and how this is so may provide a more profound way to define and analyse flexibility in transformational activities and may also help to foster the use of mathematical connections in teaching. It is important to fine-tune this new property into more specific types of mathematical connections in transformational activities. Importantly, it is seen that, as Ionescu (2012) indicates, a number of variables play a role in flexibility. As well as the ability to make mathematical connections, knowledge in the relevant content domain (Zazkis & Mamolo, 2011) may influence flexibility, or this knowledge may impact both on the ability to make connections and on flexibility. The complex relationships among these three variables need further research. Future work could also study how the mathematical connections made by teachers in teaching transformational activities influence their students' flexibility in those activities and the connections that the students make. Finally, it is believed that these types of future reflections or investigations would expand the understanding of the concept and properties of flexibility in performing mathematical tasks, as the current paper only focuses on a small number of equation solving examples.

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