Secondary mathematics teachers' analogies to function: An application of structure-mapping

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This study examines the analogies used by 26 secondary mathematics teachers when they respond to five open-ended items about function. The argument is that the descriptions present in the responses of teachers provide information about how teachers view and construct functions. The study presents the main analogies used by the participant teachers and examines ways in which they are structurally mapped to function by applying the *structure-mapping* theory. Among the analogies discussed are the *child-mother linkage* and *machine/factory* analogies. The paper suggests the use of analogy as a tool in teacher knowledge investigations.

Keywords: analogy; secondary mathematics teachers; structure-mapping; the concept of function.

Introduction

Functions are special kinds of mathematical relations. The features of *univalence* and *arbitrariness* distinguish functions from other mathematical relations (e.g., Cooney et al., 2010). Uniqueness indicates that, for the relation between the two sets (domain and range) on which the function is defined, each element of the domain maps exactly one element of the range. This feature thus only allows one-to-one and many-to-one relations but not one-to-many. Arbitrariness means a function does not "have to be defined on any specific sets of objects; in particular, the sets do not have to be sets of numbers" (Even, 1993, p. 96), and even it does, it does not "have to exhibit some regularity, be described by any specific expression or particular shaped graph" (Even, 1993, p. 96). While uniqueness is explicit in function definitions, "arbitrariness is less visible as a criterion in definitions of function" (Steele et al., 2013, p. 455). Any mathematically valid conception of function yet "must include univalence and not exclude arbitrariness" (p. 455), and these features about function should be explicitly addressed in the teaching and learning of functions (Cooney et al., 2010). Teachers should know about uniqueness and arbitrariness characteristics of functions (Nyikahadzoyi, 2015; Steele et al., 2013) and possess a wide-ranging repertoire of examples that best illustrate them (Nyikahadzoyi, 2015). Research studies, however, show that both preservice and practising teachers sometimes have an incomplete understanding of function (e.g., Hatisaru, 2020; McCulloch et al., 2020).

The research reported in this paper comes from an ongoing investigation of secondary teachers' *mathematical knowledge for teaching* (MKT) (Ball et al., 2008) about the concept of function. The paper aims to explore the occurrence of analogies to function in the participant teachers' responses to five open-ended questionnaire items and describes some of the popular analogies that the study uncovered. The research question that guided the study is: *What analogies are evident from the responses of teachers in a questionnaire on the concept of function?*

Structure-Mapping in Analogy

An analogy is a mapping of knowledge from one domain (the base) to another (the target) which conveys that two domains share relational structure independently of the objects in which the relations are embedded (Gentner, 1989). The *structure-mapping* theory posits that such commonalities convey that the attributes holding among the base objects also hold among the target objects (Gentner et al., 2001). A continuum exists between *analogy* and *abstraction.* In both, preliminary relations are matched, and a relational structure is mapped from base to target. If the base representation includes concrete objects whose individual attributes must be left behind in the mapping, the comparison is analogy. As the object nodes of the base domain become more abstract, the comparison becomes abstraction (Gentner, 1989). A comparison with neither attribute nor relational overlap is considered as an anomaly, and therefore, anomalous comparisons share no significant attribute or relational commonalities (Gentner, 1989).

In the learning process, analogies are used to make unfamiliar concepts familiar. As school students do not necessarily have the background knowledge to learn the concept of function, one of the useful ways to assist their learning is for teachers to make a link between the unfamiliar concept and the knowledge which students have already hold. This link can be an analogy which allows function to be more easily connected with students' prior knowledge enabling them to develop an understanding of it. For example, *a function is (like) a machine* analogy defines a mapping from the object of machine to function. As it is the domain being explicated, function is called the target. As it is the domain that serves as a source of knowledge, machine is called the base. Assume that the illustration of the base domain is stated in terms of object nodes such as *transform* and predicated such as *a machine transforms things*, and the target domain has object nodes such as *evaluate, change*. The analogy maps the object nodes of machine onto the object nodes of function. The essence of the analogy between a machine and function is that both receive some input and give an output.

Various other examples that can be used to explain concepts related to function are proposed. Sand (1999), for instance, presents a *mail carrier* analogy to explain *uniqueness* and *arbitrariness*. By comparison with a mail carrier, it is inferred that domain and range sets of functions can be other than number sets, i.e. letters and mailboxes. The relational similarities between a mail carrier and function are as follows: each letter is placed exactly in one mailbox, i.e. each domain value is mapped to exactly one range value. Every letter is delivered, i.e. every domain value is mapped to range values, and one letter cannot be placed in two mailboxes, i.e. one domain value cannot be mapped to more than one range value (uniqueness). Several letters may be sent to one mailbox, i.e. several domain values may be mapped to one range value (many-to-one mapping), and some mailboxes might remain empty, i.e. some range values may remain unmapped (upper boundary for the range).

Data and Analysis

Data for this paper comes from a larger study investigating secondary mathematics teachers' MKT about the concept of function (Hatisaru, 2020). The paper focused on the analogies used by 42 teachers (31 female and 11 male) who were voluntarily involved in the study. The teachers taught in fifteen different high schools in Ankara, Turkey. Data were collected through *The Function Concept Questionnaire* (Hatisaru, 2020). The questionnaire calls for three types of responses: *anticipated; desired;* and *assessing understanding* (see Table 1). Relevant to this paper, *anticipated* items involved having the teachers envision how students might mathematically approach function. This involved the teachers' expectations about how students *might* define and exemplify function in their own words, the array of students' definitions and exemplifications, both correct and incorrect. The *desired* items involved that the teachers stating how they think students *should* define and exemplify function. The *assessing understanding* item required that the teachers create questions that could reveal students' understanding of function. The teachers could give as many descriptions as they would want to. The teachers were designated codes (T1, T2, and so on).

Table 1: The function concept questionnaire items used in this paper.

Type of response	Item
Anticipated:	
definition	1. Imagine that you have asked your students to define the concept of function in their own words. How do you think they will define what a function is? Please write a few examples. (Adapted from Breidenbach et al., 1992; Even, 1993)
exemplification	2. Imagine that you asked your students to give some examples of functions. What kind of examples do you think they would give? Please write a few examples. (Adapted from Breidenbach et al., 1992)
Desired:	
definition	3. How do you think students should define the concept of function? Please write them.
exemplification	4. What kind of examples do you think students should give for functions? Please write them.
Assessing understanding	5. What questions would you ask your students to find out if they understand the concept of function? (Adapted from Cooney, 1999)

In Hatisaru (2020), a deductive content analysis was implemented to analyse the teachers' descriptions and exemplifications for the concept of function. Both abstract (e.g., *"[Function is] A relation such that each element of the domain has a unique image in the range"* T4, Item 3) and concrete-based (e.g., *"[I would ask] Whether the correspondence between children (domain) and mothers (range) is a function or not"* T17, Item 5) domain comparisons were analysed. The teachers' responses were coded as: a set of ordered pairs such that no two pairs have the same first entry but different second entries; a mapping from one set to another; taking inputs to outputs; or a rule assigning x to $f(x)$ (Cooney et al., 2010). In the present paper, the data were reviewed, and the *concrete-based* domain comparisons analysed further to answer the research question. These comparisons were noted and tallied. Some of the teachers provided more than one comparison, and each was recorded: *"We usually use mail and mailbox [the mail-mailbox analogy] examples. Child and mother [the child-mother linkage analogy]"* (T15, Item 4). The total number of concrete-based domain comparisons to define and exemplify function for this study was 61 (*f:* frequency) (27%), representing total of 26 teachers.

Teachers' Analogies

Two kinds of domain comparisons were identified in the teachers' responses: *analogy* and *anomaly* (see Table 2). Also, a third category *(other)* was evident which contain a word problem and a few responses referencing real-life based descriptions, but no further detail was given, such as: *"I would give examples from daily life conveying the notion of function"* (T13, Item 5). In the analogies representing function as *input-output machines*, intended inferences concern mainly the relational structure, such as *a function receives some input and gives an output, just as a factory or machine*, but not *function is large, like a factory*. Another relation that is present in the base that can be matched with the target is *a function does not have to work only with numbers, like a machine or factory* (arbitrariness). That is clearly indicated in T26's comparison between a function and an olive oil factory in Item 4: *"[The function is like] Producing olive oil (output) by processing olives (input) in the factory"*. This group also involved the *schooling* and *bus* analogies where the teachers made comparisons between a function and bus or formal schooling. The intended inference seems to be, *a function receives inputs and give an appropriate output like schools or buses receive students or passengers and educate them or carry them from one point to another.* Among them, while in ten analogies the domain and range have concrete, realistic values (e.g., input: fruits, olives, and wheat; output: juice, olive oil, and flour), in two, the domain and range have sets of numbers.

Table 2: The concrete-based domain comparisons in teachers' responses (*f*=61).

Kind of comparison	Anticipated: definition (20)	exemplification (20)	Desired: definition (2)	exemplification (13)	Assessing understanding (6)
Analogy (46)	Machine (10) Postman (1) Factory (2) Dancing (1) Fork (1)	Machine (4) Child-mother (5) Factory (2) Postman (1) Restaurant (1) Bus (1) Schooling (1) Backgammon (1)	Cars (1)	Machine (4) Child-mother (4) Mail-mail box (1) Room allocation (1) Unable to being in two places (1)	Child-mother (3)
Anomaly (9)	Noun (4) Sets (1)	Noun (3) Mirror (1)	-	Noun (2)	-
Other (6)			Real-life based (1)	-	Real-life based (2) Word problem (1)

Within the analogies presenting function as a correspondence, the *child-mother linkage* analogy is the most popular, and the presentation of information as a biological link between a child and mother is consistent with Sand's (1999) account of the *mail carrier* analogy in teaching about function. In this comparison, similarly, the relations have priority over attributes of the objects. Such as, *the function maps each element of one set to exactly one element of a second set* (uniqueness), just as the (biological) linkage between a mother and her child, but not *the function is functionality of things, individuals, and similar, like the function of a mother for her child*. In this analogy, a function *f* is defined by its domain *(the set of mothers)*, range *(the set of their children)*, and definition *(f(mother)=her child)*. An example can be *f(Mother A)=Child A*. If students are asked: *Why you did not mention about Mother C?* they may say: *Mother C does not have a child!* Here, they suggest an upper bound for the range of this function. *How about f(Mother A)=Child A and f(Mother A)=Child B?* Students may say: *Mother A has two children: Child A and Child B*! That is, the *linkage* allows many-to-one mapping *(Mother A can has more than one child)* just as functions do. These commonalities are clearly explicated in the response of T37 in responding Item 4: *"Let us say the domain set is children, the range set is mothers. Every child has a mother. [A child] Cannot has more than one mother [uniqueness]. But not all women have to have a child [upper bound for the range]"*.

The other analogies forming uniqueness and upper bound for the range commonalities are the *room allocation* and *restaurant* analogies (T11 and T19). Like the *child-mother linkage* analogy, both analogies are nonliteral comparisons between the base (hotels and restaurants) and target (function) domains, involving relational carryover. T11's *room allocation* analogy maps domain of a function to a group of students camping, range of the function to rooms in a hotel, uniqueness feature to *every student must be allocated in a room, but a student can't stay in two different rooms*, and the upper bound for the range feature to *some hotel rooms may vacant.* T19's *restaurant* analogy, which is also presented in the mandatory textbook, maps domain to a group of students in a restaurant, range to the list of dishes that they can order, and uniqueness to the condition that each student can order exactly one dish.

The most common *anomaly* in the responses of teachers were those describing function as what something is used for, or the actions and activities assigned to or required or expected of a person. Illustrative examples are: *"[Students think] Function. The function of technological machines"* (T19, Item 1) and *"The function of a telephone; the function of a teacher in the classroom"* (T41, Item 2), where there need to be no analogical matches, merely a lexical look up. It is important to note that seven of those occurrences were the teachers' anticipation of student function descriptions and exemplifications (see Table 3). Two teachers, however, wrote that students *should* exemplify function (Item 4) as: *"Related to real life, devices they use, computer, phone"* (T1) and *"One can lose their life functions. A tool may have various functions. f(x), x"* (T25). Based on their responses, it seems these two teachers would be satisfied with student understanding viewing function as if a noun, through its everyday meaning. In the absence of interviews, the teachers' unsatisfactory thinking, though important, is difficult to discern from the current data. Therefore, I have made no analytical claims about them.

The teachers' anticipation of student (mis)thinking also included a *mirror* and *sets* (unsatisfactory) comparisons (*"Students think that function is a topic related to sets. For instance, the set of blond students in the class"* T23, Item 1). A mirror or sets and function, in fact, convey little relational commonalities, and as there might be only common object attributes between two (e.g., functions have domain and range sets), such a match can be considered as a mere appearance match.

Conclusion

The study's conclusion is not a claim about participant teachers' MKT regarding the concept of function, because all is known is their responses to five open-ended questionnaire items. Nor can the study generalise about the country based on 26 teachers using a convenience, voluntary sample which probed qualitatively in detail. The study rather claims to give a window into the teachers' conceptions. It does this through analysing the analogies in the responses of the teachers as a mechanism to make conjectures about how the concept of function is conceived and taught in the classroom, although the study is mindful of how curriculum materials mediate this.

The value of the study is addressed from the perspective of the productivity of the conceptual and methodological approaches to the issue of teacher MKT, and from the perspective of the study's contribution to the current understanding of analogy itself. On the latter issue, the present study contributes significantly to the investigations of analogies, because it deals in detail with analogies in the responses of teachers to the concept of function, a matter not typically addressed. On the former point, the study shows that the current approach is promising.

The analogical reasoning analysed in this study shows that the notion of analogy has power for representing something about teachers' construction of function. The reasoning used by most of the teachers contains major and recurring features that are used to describe function. Many of the teachers refer to function as if it is a unique case of a relation, and, for some of them, this feature seems to be caught up with their view of understanding or not understanding the concept of function: *"Students can think that they cannot be in somewhere else [at the same time] when they are here in the classroom [now]"* (T23, Item 4). The responses of many teachers reify features of functions like, uniqueness, arbitrariness, and upper bound for the range, and all employ varieties of the familiar conduit analogy when responding about functions. Also, there have been interesting varieties of a popular *child-mother linkage* analogy for reasoning about aspects of function. So, at the least, this study has revealed that teachers' repertoire is rich in analogy, and that the analogical arguments can be seen to relate to the concept of function. Further, the study raises questions about the role of analogy in the teaching of functions and draws attention to the importance of accounting for the conception of functions if not directly used in studies of teachers' MKT about function.

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