Mathematical definitions: what works and what doesn't?

Dominika Majewska

Cambridge Mathematics

The Cambridge Mathematics (CM) team has developed the *CM Define It* app – a survey tool which collects information about existing definitions of mathematical key words. The tool is aimed at professionals in the mathematics education community, including teachers, academics, researchers, and curriculum and resource developers. Survey respondents are presented with a key word and up to five definitions that address all learners, and which are taken from international sources. They are asked to rate the presented definitions on a five-point rating scale, according to how suitable they are for the learners with whom they work. They may then choose to provide justification for their ratings, including how accessible and accurate the definitions are for the learners they work with. The aim of this survey tool is to inform the CM team of what makes for a 'successful' definition of a mathematical term, which may eventually inform the glossary layer in the Cambridge Mathematics Framework (CMF) – a flexible and inter-connected map describing the mathematics learning of 3-19 year-olds. This paper will present the *CM Define It* app, the theory behind its development and its structure, functionality and aims. It will also describe the changes made to the survey tool based on user feedback following a pilot study.

Keywords: CM Define It; survey tool; users; mathematical definitions.

Introduction, background and aims

The Cambridge Mathematics team has carried out an extensive review of the mathematics education literature. This highlighted a lack of, and a need for, a holistic and accessible glossary of mathematical vocabulary and definitions. To understand this gap better, the team developed a glossary survey app called *CM Define It*. This paper will present the background and development of the survey app tool and will demonstrate the importance of collating information to develop a single source of mathematical key terms and definitions.

Background research

Language has been defined as "the words, their pronunciation, and the methods of combining them used and understood by a community" ("Language", 2013, as cited in Riccomini, Smith, Hughes, & Fries, 2015, p. 236). Language and communication are very important elements involved in learning, understanding and applying mathematics. The ability to communicate using mathematical language requires several factors, such as a strong mathematical vocabulary, numerical fluency and proficiency, and comprehension skills (Riccomini et al., 2015).

Research has demonstrated the importance of language in the development of mathematical skills. For instance, Seethaler, Fuchs, Star and Bryant (2011) investigated

the cognitive predictors of whole-number versus rational-number computational skills. Among many predictors, oral language appeared to support early whole-number development. The researchers suggest that language can support the development of rational-number skills, possibly by encouraging the development of conceptual understanding of calculations with fractions, percentages and decimals. Monroe (1998) suggests that understanding mathematical vocabulary gives access to mathematical instruction and concepts.

Research has also indicated that knowledge of mathematics vocabulary can predict mathematical attainment. van der Walt (2009) explored the extent to which performance in study orientation and knowledge of mathematical vocabulary can predict mathematics performance in students in grades 4 to 7. The study found that knowledge of basic mathematical vocabulary predicted learners' performance in mathematics.

In addition to the importance of mathematical vocabulary, Rubenstein and Thompson (2002) (cited in Riccomini et al., 2015, p. 238), propose 11 categories of difficulty that students may face when learning mathematical vocabulary. These include:

- meanings of terms being context-dependent (e.g. the term 'foot' meaning 12 inches or the bottom of the bed)
- meanings being more precise (e.g. 'product' meaning the solution to a problem in multiplication or a product of a company)
- terms holding meaning specific to mathematical contexts (e.g. polygon, parallelogram, imaginary number)
- terms holding multiple meanings (e.g. side of a triangle vs. side of a cube)
- technical meanings that are specific to different disciplines (e.g. 'cone' meaning the shape or food)
- everyday homonyms (e.g. pi vs pie)
- words that are related and different at the same time (e.g. circumference and perimeter)
- challenges with translated terms (e.g. mesa vs table)
- irregularities in spellings (e.g. obelus vs obeli)
- concepts being verbalised in different ways (e.g. 15 minutes or quarter past)
- using informal terms instead of mathematical ones (e.g. diamond vs rhombus)

This demonstrates that although language is crucial in developing mathematical skills, there are many barriers to understanding mathematical language. It is important for those who work with students to recognise these difficulties and understand the implications they might have.

Furthermore, definitions have a very important role in mathematics, but how they are created and used is different to definitions in "everyday language" (Edwards & Ward, 2008, p. 223). Edwards and Ward differentiate two kinds of definitions: *extracted* definitions and *stipulated* definitions. Extracted definitions are extracted from examples of actual usage (Landau, 2001, as cited in Edwards & Ward, 2008), whereas stipulated definitions come from consciously creating meaning between a word and an object (assigning a name to an object) (Robinson, 1962, as cited in Edwards & Ward, 2008, p. 224). Landau (2001, as cited in Edwards & Ward, 2008, p. 224) suggests that definitions come from "expert advice" and should make communication easier and accurate for those who use "the language of science".

Ball, Thames and Phelps (2008, as cited in Mosvold & Fauskanger, 2013, p. 44) state that one key task that teachers need to do is to choose and develop useable definitions. Zazkis and Laikin (2008) argue that the knowledge of mathematical definitions held by teachers can affect many aspects of their practice, including their instructional decisions, explanations given to students, and the guidance and mathematical discussions they offer. Ball et al. (2008, as cited in Mosvold & Fauskanger, 2013, p. 49) identify "choosing and developing useable definitions" as a challenge that is very specific to the role of teachers. They suggest that it extends past the ability simply to copy or recite mathematical definitions, to their ability to understand differing definitions and to know which definitions are accurate and useful. In a study of 15 teachers, Mosvold and Fauskanger (2013) found that most teachers suggested that knowing definitions is an important element of teachers' knowledge (p. 51), supporting the emphasis of Ball et al. on the ability to choose and develop appropriate definitions. Furthermore, Leikin and Zazkis (2010, as cited in Mosvold & Fauskanger, 2013) proposed that it is important for teachers to be able to match definitions and their teaching of them to students' needs and motivations.

The above demonstrates the importance that language and definitions play in the learning of mathematics. Ball et al. (2008, as cited in Mosvold & Fauskanger, 2013), and Mosvold and Fauskanger (2013), clearly highlight the crucial role that teachers play in supporting students' understanding of mathematical definitions and language. This in turn suggests that their own knowledge and understanding of mathematical definitions are important factors that can affect students' learning.

Aims

Literature review searches carried out by the CM team highlighted the lack of a single, internationally appropriate and widely agreed source of key terminology in mathematics. This paper presents the importance of investigating how the mathematics education community views existing definitions of mathematical key terms, and how this has been addressed by the development of a survey research tool called *CM Define It* – an app which has been used to collect opinions about mathematical definitions from mathematics education professionals internationally. Ultimately, the wider aim of the *CM Define It* app is to use the collected data to inform the glossary embedded in the CMF. However, the aim of this paper is to present the structure and development of the app as a survey tool.

Design and survey methods

In a general sense, *survey* refers to "systematic data collection about a sample drawn from a specified larger population" (Schwarz, Groves, & Schuman, 1998, p. 143). There has been an increased use of virtual communities and the Internet to collect information and conduct primary research (Matheson, 1991, and Wright, 2002, as cited in Wright, 2005). Over time, the availability and ease of the use of technology to conduct survey research has greatly increased, making online survey research quicker and simpler than before (Wright, 2005). Conducting survey research online has many advantages, including automated data collection, the ability to reach those in distant locations, and collecting data from a much wider and more representative pool of participants (Wright, 2005). In educational settings, apps have been used for a range of tasks, such as monitoring behaviour, communicating with families and providing subject-specific training (Bouck, Satsangi, & Flanagan, 2016). This highlights that using online surveys is a widespread method of conducting research. It also justifies the

decision to develop an online survey app to collect data on current mathematical definitions.

The *CM Define It* survey app

In order to find out what the mathematics education community thinks about existing definitions of mathematical key words and what they would like to see in successful definitions, the Cambridge Mathematics team developed the *CM Define It* app. The app is a survey research tool that collects ratings of existing definitions of mathematical key words. The data collected through the survey app will enable the CM team to develop an understanding of what the mathematics education community perceives as sound definitions of mathematical terminology. It may shine light on whether certain definitions and sources are preferred, whether certain definitions and sources are preferred by specific groups (for instance, those working with novice learners) and what makes definitions successful. In the long-term, data collected through the app may help to inform the glossary embedded in the Cambridge Mathematics Framework.

The app can be accessed as a weblink or downloaded from the Android, Apple and Windows app stores. Upon registration, users are asked to provide demographic information, including occupation, first language and country of residence. Once registered, users are presented with a key word and up to five definitions of that key word taken from international sources, and are asked to rate the presented definitions on a scale of 1-5 stars, with 1 star holding the lowest rating and 5 stars holding the highest. The definitions are taken from respectable, international and credited sources.

According to Ball and Bass (2002), mathematical definitions are not useful if they do not contain information that is appropriate for the prospective user and their knowledge. To address this, users select the category/ies of learners (novice, intermediate and advanced learners) they work with the most. Novice learners are defined as "learners who are developing their early knowledge of core mathematical concepts, for instance, young children or novice learners". Intermediate learners are "learners who are building on previous knowledge and refining their understanding, for instance younger teenagers", whilst advanced learners are identified as "learners who are comfortable with many core mathematical concepts, and are studying or starting to study more advanced or specialised topics, for instance older teenagers or trainee teachers" (Cambridge Mathematics, 2020). Although these address different levels of learners, from primary to further education, we did not label the categories based on the British system and we did not attach specific age ranges to the categories as the app may be used by professionals from different countries. Recognising that teachers and other professionals may work with different learners and different levels of definitions follows the work of Ball et al. (2008, as cited in Mosvold & Fauskanger, 2013).

Users can view the previous week's ratings for all categories of learners or by selecting a specific category. They can also check specific feedback; for instance, what proportion of the audience thought the definition was technically accurate.

Survey app adjustments

Since the launch of *CM Define It* in October 2018, feedback from pilot studies has informed adjustments to the app. One of the adjustments includes the option to receive push notifications on mobile phones, reminding users to complete weekly ratings. Users can also go back and rate definitions for another category of learner and can change the rating they gave to the definitions that week. Users cannot change ratings given to definitions in previous weeks.

An important change is that although all users must provide a basic rating of each definition, further feedback is optional. Previously, users had to rate the following criteria on a five-star scale:

- whether the definition is technically accurate,
- whether the definition emphasises key points,
- whether the definition is accessible for its intended audience,
- whether the definition is sufficient for the intended audience, and
- whether the definition added to or clarified participants' own understanding.

Another new feature of the app is the introduction of an incentive. After giving the general rating for each definition, users are presented with a unique mathematical key term. If respondents want to find out its definition, they are asked to rate the above criteria on a 5-star rating scale. This makes giving further feedback an option and encourages users to provide additional feedback on the presented definitions and their ratings.

How collected data will inform the Cambridge Mathematics Framework

Data collection through the *CM Define It* survey app will stop at the end of December 2020, which means that the survey app will have been collecting information for over a year. The Cambridge Mathematics team will be preparing to explore the data collected to decide how it can inform the glossary embedded in the CMF.

In order to decide how the collected data can inform the Framework's glossary, the team will:

- meet to discuss the data,
- identify what questions the data can help to answer,
- examine the data for trends (such as whether specific groups of professionals prefer specific sources of definitions or which sources are rated the highest),
- discuss how identified trends and findings can inform the glossary in the CMF,
- decide on a protocol for applying findings to the glossary in the CMF, and
- write a report explaining the decisions made about how the data will inform the Framework glossary.

The final report that will appear on the Cambridge Mathematics website will explain what decisions were made with regards to using the data collected and why such decisions were made.

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References

Ball, D. L., & Bass, H. (2002). Toward a practice-based theory of mathematical knowledge for teaching. In E. Simmt & B. Davis (Eds.), *Proceedings of the 2002 Annual Meeting of the Canadian Mathematics Education Study Group* (pp. 3–14). Edmonton, AB: Canadian Mathematics Education Study Group.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407.

Bouck, E. C., Satsangi, R., & Flanagan, S. (2016). Focus on inclusive education: Evaluating apps for students with disabilities: Supporting academic access and success. *Childhood Education, 92*(4), 324–328.

Cambridge Mathematics. (2020). *CM Define It*. https://cmdefineit.org/

Edwards, B., & Ward, M. (2008). The role of mathematical definitions in mathematics and in undergraduate mathematics courses. In M. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (p. 223-232). Mathematical Association of America. doi:10.5948/UPO9780883859759.018

Monroe, E. E. (1998). Using graphic organizers to teach vocabulary: Does available research inform mathematics instruction? *Education, 118*(4), 538–542.

Mosvold, R., & Fauskanger, J. (2013). Teachers' beliefs about mathematical knowledge for teaching definitions. *International Electronic Journal of Mathematics Education, 8*(2-3), 43-61.

Riccomini, P. J., Smith, G. W., Hughes, E. M., & Fries, K. M. (2015). The language of mathematics: The importance of teaching and learning mathematical vocabulary. *Reading & Writing Quarterly, 31*(3), 235–252.

Schwarz, N., Groves, R. M., & Schuman, H. (1998). *Survey methods.* In D. T. Gilbert, S. T. Fiske, & G. Lindzey (Eds.), *The handbook of social psychology* (pp. 143-179). New York: McGraw-Hill. Retrieved from https://www.researchgate.net/profile/Norbert_Schwarz2/publication/232562918_Survey_methods/links/55db515d08aed6a199ac5d02/Survey-methods.pdf

Seethaler, P. M., Fuchs, L. S., Star, J. R., & Bryant, J. (2011). The cognitive predictors of computational skill with whole versus rational numbers: An exploratory study. *Learning and Individual Differences, 21*(5), 536–542.

van der Walt, M. (2009). Study orientation and knowledge of basic vocabulary in mathematics in the primary school. *South African Journal of Science and Technology, 28*(4), 378–392.

Wright, K. B. (2005). Researching internet-based populations: Advantages and disadvantages of online survey research, online questionnaire authoring software packages, and web survey services. *Journal of Computer-Mediated Communication, 10*(3), https://doi.org/10.1111/j.1083-6101.2005.tb00259.x

Zazkis, R., & Leikin, R. (2008). Exemplifying definitions: A case of a square. *Educational Studies in Mathematics, 69*(2), 131–14.