Teaching students to write and read mathematics

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Students typically come to university without knowing how to write mathematics. Rather than treat it as a problem that individual members of staff have to deal with by themselves, we created half a module designed to explicitly teach students how they are expected to express mathematical concepts, ideas and proofs. By introducing this topic right at the start of students' university experience, we succeeded in created an improvement that lasted into students' second year. We also discovered along the way that students don't know how to read mathematics, and so we expanded our remit into 'study skills', rather than focusing purely on writing well.

Keywords: teaching; undergraduate; study skills; reading; writing.

Introduction

Upon the transition to university, many mathematics students struggle with writing, as they move from the more calculation-based mathematics of school to problems that require more thorough explanation. Being able to communicate their reasoning clearly is an important skill for students to develop, and one that deserves explicit attention. Students may receive feedback on their writing incorporated in feedback on homework, but this is often overshadowed by the feedback on the mathematical content and reasoning of their work. This paper discusses an attempt to address this by building opportunities for explicit feedback on writing into the first-year undergraduate mathematics curriculum. This takes the form of standalone workshops in which students discuss problems and write up solutions in groups, where the focus of their work and the feedback provided is on writing good-quality mathematics.

Literature review

Mathematical writing is a skill that presents many specific challenges, an overview of which can be found in Schleppegrell (2007). There are a number of writing guides available aimed at undergraduate students (e.g. Houston, 2009; Alcock, 2013), which address mathematical writing alongside other relevant study skills. However, as noted in Lew and Mejía-Ramos (2015), these are often based on the author's own experience with teaching undergraduates, rather than a systematic study of how students write mathematics.

In recent years, studies have begun to examine undergraduates' mathematical writing in more detail. In Lew & Mejía-Ramos (2015) the authors examine the common mistakes that students make, classifying these into fourteen categories that are divided into four broader areas that students struggle with: constructing complete and unambiguous mathematical sentences; making the flow of the argument clear; introducing variables; and using mathematical symbols. A similar categorisation was made in Guce (2017), where the frequencies of nine types of writing error were examined. The categorisation of Lew and Mejía-Ramos (2019) was used by the authors to compare the perspectives of students and mathematicians on these errors. In particular, this study showed that there was a disconnect between students and mathematicians regarding the use of academic language in proof writing, with students not recognising the need to use correct grammar and full sentences in mathematical writing. Additionally, the study compared students at different stages of their mathematical studies and did not find significant differences between them, suggesting writing skills do not improve as students progress through their degrees. According to Moore (2016), mathematicians consider fluency to be one of the four key-aspects of a well-written proof, but generally give this aspect little weight in mark schemes. This may lead students to pay less attention to this aspect of proof writing. The issue may be further compounded by the fact that mathematicians themselves may not have consistent expectations of students' proofs, as discussed in Lew & Mejía-Ramos (2020), which compared the expectations of mathematicians regarding mathematical writing in the contexts of textbooks, blackboards, and student work. This study also indicated that some of the errors perceived in students' writing were viewed as acceptable in blackboard proofs, suggesting that students may be used to seeing proofs in a classroom context that are less formal than the work they are expected to produce.

It is therefore important to make it explicit to students what is expected of them when writing mathematically. In order to do this at the University of Bristol, a new unit, Mathematical Investigations, was introduced in the 2018/19 academic year with a focus on independent study skills.

The workshop unit

The Mathematical Investigations unit is a compulsory unit for all single-honours mathematics students and runs throughout the academic year. The unit consists of two parts: one in which students regularly meet with their personal tutor and work on producing group reports on a variety of mathematical topics over a period of several weeks; and one where students attend self-contained workshop sessions each week and go through a worksheet in groups. It is these workshops that we used to focus on the issue of mathematical writing. Each workshop is a two-hour session and there are approximately 30 students and one instructor per workshop. The workshop component of the unit is pass/fail, where a student must attend and engage with 14 of the 16 workshops throughout the course of the academic year.

The focus of the workshops is reading, writing, and discussing mathematics, and so the content of the worksheets is generally material that they have already seen (with a reminder of key facts provided within the worksheet) or self-contained topics that they will not be assessed. Part of the initial unit aim was to support students in learning analysis, an area that many students find difficult upon transitioning to university, and so many of the worksheet topics were chosen to complement the topics that students would be encountering in the first year analysis course. At the end of the session, students hand in a group submission to their instructor, who provides formative feedback at the start of the next session.

The structure of the workshops

Writing workshops

The writing workshops are designed to be worked on in groups of 1-5 students (we call 1 a group to prevent avoidable stress, and in general any lone students were happy to join groups after a few weeks). They begin with an introduction giving a bit of relevant history of the topic and some background to remind students of relevant facts from lectures. There are then a series of questions designed to guide them through the steps needed to prove a theorem, in a similar way to the approach described in Alcock, Brown, & Dunning (2015). They are intended to be self-contained and end with a result that is substantial enough to be interesting in its own right. The breakdown into steps facilitates this and students are encouraged to discuss each step and write an answer collaboratively.

Reading workshops

In the 2019/20 academic year, we also introduced several workshops which focused more on reading skills. Students are often expected to read mathematics independently, but may not realise how to do so effectively; in particular, Weber & Mejia-Ramos (2014) found that students may not realise the length of time they are expected to take to understand a proof, nor that they may be expected to provide additional justifications for steps themselves. We attempted to address this in our reading workshops. The students are given some time (typically 30-45 minutes) to read through a proof in pairs, discussing and annotating the proof as they work through it. The pairs are then combined to create groups of four, in which students discuss and answer proof comprehension questions. These questions were written based on the model developed in Mejia-Ramos, Fuller, Weber, Rhoads, & Samkoff (2012).

Before the first reading workshop, the workshop instructors spoke to the students about how to read mathematics and provided them with self-explanation training, based on the self-explanation training booklet developed by the Loughborough University Mathematics Education Centre (2020). This is a technique that has been demonstrated to help students with proof comprehension (Hodds, Alcock, & Inglis, 2014), and students were instructed to keep this training in mind as they worked through the proofs provided. In later reading sessions, a brief reminder of the principles of the self-explanation training was provided.

Our findings

Initial thoughts

Students typically got to grips quickly with the point of the workshops; by Christmas we felt that the students had a solid grasp of writing in full sentences. The concept of writing an introduction and a conclusion to give their work direction, as well as including signposting within their work was harder.

In the second year of running the unit several sheets were modified to add an extra question designed to have students reflect on their work. For example, we asked students to prove that there is a rational number between any two irrational numbers. The sheet guided students to construct an algorithm to produce this rational, using a few steps, so we asked them to summarise what this process did in a specific example. Another example (on mechanics) asked students to suggest a conjecture to explain the 'coincidence' that two a priori different problems gave them the same result.

About halfway through the Spring term we decided to challenge the students by removing the breakdown into steps and only giving some general hints and a diagram. This meant that students had to develop an idea which led them to decide what steps to try and then make sure they were rigorous. Most students were happy to get on and explore the mathematics for themselves. We were all pleasantly surprised that the students didn’t feel the need to ‘have their hand held’ and were happy to experiment with different approaches until they found the correct one. In fact, several students appreciated the fact that they felt they had developed the idea themselves and made it work rather than following steps to implement an idea that was given to them.

During the first year of running the workshops, it became increasingly apparent that students were often struggling with the reading aspect of the workshops; for example, many students would not read the background information carefully. This led to the introduction of the reading workshops in the next year. In the initial reading workshop students struggled with the task, especially when they had to provide additional justifications for steps in the proof that were not explicit. This may have been due to the novelty of the situation, as when we ran another reading workshop several weeks later it was much more successful. Students seemed more willing to talk to each other, and particularly to ask questions about the mathematics to each other, and the teachers. The workshop instructions made it more explicit that students would be expected to fill in some 'gaps' in the proof, and some of these steps were used in the proof comprehension questions to ensure that students attended to them. As instructors, we also improved our ways of answering questions that students had, to ensure that we led them through understanding the proof themselves.

The first author has tutorials with both single honours and joint honours students, and the difference between the two groups becomes noticeable by approximately four weeks into term, which is around when the second homework for other modules is set or due. The joint honours students are told to write in complete sentences, but the time it would take for the change to become habit cannot be given, as the focus of tutorials is the mathematics. The quality of mathematics itself seems roughly unaffected, although our general impressions are that there seem to be more typos and missed special cases (such as qualifying that $b \neq 0$ when dividing by b) by the students who were not doing the workshops.

The difference made by the workshops is also evident in another part of our first-year curriculum: our A-level revision sessions, which are support sessions for students who may be struggling with A-level content. Both single and joint honours students attend the sessions, and as the weeks progress a noticeable difference starts to emerge. Early on, when a student asks for help with a question and shows their work so far, this work is often a labyrinthine mess of vague algebraic statements in no particular order. A great deal of pointing and explaining is required before the would-be-helper can even determine what the student has tried so far and thus give some advice on what to try next.

Amongst most of the joint honours students this state of affairs is as true in the final revision session as it was in the first. But amongst the single honours students – those taking the workshops – things start to change after just a few sessions. Their works in progress may not become suddenly beautiful, but they are at least structured. It is clear what they are trying to prove and each line follows from the previous one, sometimes even with an explanation as to how. Some of this may of course come from a greater mathematical maturity developed as the students see how their lecturers present mathematics. But the joint honours students attend many of the same lectures, so the workshops must take some of the credit.

Student feedback

Towards the end of the 2019/20 academic year, students were asked to fill out a feedback questionnaire for the unit, which included questions about how the workshops had helped to develop their reading and writing skills. From 72 responses, 93% agreed or strongly agreed that "the workshops have improved my mathematical writing skills", and 85% agreed or strongly agreed that "the workshops have improved my mathematical reading skills".

Additionally, students were asked for their comments regarding what they liked and what could be improved about the workshops. The main improvements students suggested were having slightly shorter sessions and broadening the range of topics covered in the workshops. Positive comments generally reiterated that students felt the workshops had improved their writing skills. For example, one student noted that "[the workshops] have helped me think about how to write and communicate mathematics efficiently", and another that "they help to switch from the style of A level mathematics to university mathematics". Overall, responses to the feedback questionnaire suggested that students saw the benefit of the workshops and indicated that the workshops were meeting the stated aims.

Other lecturers' thoughts

The improvements we see in the students over the course of the first year last longer than just during the time they are doing the workshops. Second year lecturers have commented that the students are more careful in the layout of their work, which has advantages such as better following students' methods of work and being able to spend less time on trying to teach second years how to correctly present their work.

The Head of School and Education Director have also commented that this part of the module seems to be the favourite part of the first-year curriculum.

Conclusions

It is not to be inferred from the above report that the Mathematical Investigations unit has arrived at some perfect solution for the problem of teaching undergraduates how to write mathematics – far from it. At only two years old the unit is still very much in its infancy, and will no doubt continue to evolve for many years to come. The introduction of the reading workshops in the most recent academic year was one such evolution, and the individual worksheets will also change based on our experiences with the students. However, for all that, informal evidence suggests that the workshops are already making a real difference.

Future advances will not only be informed by our anecdotal experiences with the students, but will also hopefully be guided by research projects carried out into the efficacy of the unit and ways to improve it. Plans for these have been put on the back burner, however, due to a more pressing concern: the move online.

The Covid-19 outbreak has forced almost all aspects of University teaching to transition to an online environment. Achieving this with the Mathematical Investigations unit is no small challenge. A typical workshop sees groups of students huddled around desks, chatting animatedly and gesturing at scribbled bits of mathematics on ever more cramped bits of paper. Replicating this online would be all but impossible. We are still in the process of determining how exactly we can deliver the workshops online, and will no doubt be making changes to our approach even after the new academic year begins. Breakout rooms in Zoom may provide some of the same functionality as desk-huddling, but ultimately the social aspects of the workshops will have to be sacrificed, at least in part, to concentrate on the educational aspect of teaching the students how to properly read and write mathematics.

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