Understanding students' understanding of a mathematical concept through lesson play: A focus on metaphors

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This research is aimed at finding a novel way of investigating students' understanding of mathematics. Mathematics as a language of conceptual tools is full of representations, and students re-interpret the representations to understand mathematical concepts. A metaphor is an important aspect of learning mathematics, with regards to it beings a powerful linguistic approach for creating and extending meanings of a concept. In this research, I carry out script writing tasks with 6th grade Korean primary students. In these tasks, a prompt that includes a mathematical situation is provided, and students are asked to write a script in a dialogue format for the prompts, similar to writing a scenario for a role-play. I describe their scenarios and examine a concept of a fraction in the students' writings.

Keywords: metaphor; students' understanding; script writing; lesson play; fraction.

The personal or flexible understanding of mathematical concepts are found in many studies. In Marmur, Yan, and Zazkis (2020), with regards to the unfamiliar representations of a fraction (e.g. 6.5/10), some teachers accept the representation while other teachers reject it. According to Vinner (1983), students understood the concept 'function' as a correspondence rule, an algebraic term, or as a visual representation.

Students' prior understanding of a concept affects their further learning, since mathematical concepts are interrelated (Carpenter & Lehrer, 1999). Thus, a knowledge of students' understanding is crucial for effective teaching and learning in mathematics. Here, my questions became, 'What is the students' understanding of mathematical concepts' and 'How can I explore it?' Therefore, I aim to find a way of investigating students' understanding of mathematical concepts.

Understanding a Mathematical Concept

Carpenter and Lehrer (1999) stress that learning with understanding allows the connection between new knowledge and an individual's previous knowledge, in order to create rich and integrated conceptual knowledge structures. Hiebert and Carpenter (1992) define understanding as recognition of relationships between mathematical information, such as mathematical ideas, facts, and procedures. Adopting the definition of Hiebert and Carpenter, I characterise understanding in this research as a process of constructing relationships between mathematical concepts in one's mind.

I adopt a radical constructivist perspective, in which a mathematical concept is an abstracted idea in an individuals' mind (von Glasersfeld, 1995). Continuous mathematical experiences encourage individuals to keep reconstructing and evolving knowledge of a concept they have in their mind (von Glasersfeld, 2008), and thus mathematical concepts are never fully formed, determined, or actualised. Lakoff and Núñez (2000) regard a concept as a metaphorical representation of the experiential world. When we represent a concept abstracted, language plays an important role (von

Glasersfeld, 2008). We use language with an intention, and language represents the conceptual structure in our mind (von Glasersfeld, 1995). Individuals actively form the meaning of the language they use, and the result of individual interpretation with regards to the representations is taken as the meaning of a concept (*ibid*). Thus, language plays a role in reflecting individuals' understanding of a concept.

Metaphor

Abstract concepts are understood in terms of more concrete concepts via metaphor (Lakoff & Núñez, 2000). The focus of a metaphor is the conceptualisation of one domain with another domain, called "cross-domain mapping" (Lakoff, 1993, p.1). The conceptual cross-domain mapping consists of two domains; one is the 'target domain' that we want to understand, and the other is the 'source domain' that conceptualises the target domain. Conceptual metaphors are in our everyday language (Lakoff & Núñez, 2000). For example, 'take from' is a metaphor for subtraction.

The mathematical concept I investigate is a fraction; the concept of a fraction is related to concepts of 'number', 'division', and 'multiplication'. Davis (2020) explores the eight metaphors of numbers based on the metaphors of arithmetic in Lakoff and Núñez (2000). In addition, in Davis and Renert (2013), the meanings of division and multiplication are described. As 'metaphor' is a narrow category that is included in the broad category 'meaning', the metaphors of number from Davis (2020) and the meanings of division and multiplication from Davis and Renert (2013) can be a basis to explore metaphors of a fraction.

Number from Davis (2020)	Count, Rank, Amount, Size, Length, Location, Reification, Numeral
Division from Davis & Renert (2013)	Grouping, Repeated subtraction, Number line hopping, Fraction, Inverse of multiplication,
Multiplication from Davis & Renert (2013)	Repetitive addition, Repetitive grouping, Expanding, Folding or Splitting, Scaling, Skip counting, Stretching a number line or linear function, Making areas

Table 1. Metaphors of a number and meanings of division and multiplication

Playwriting Task

For the investigation of students' understanding of a fraction, I carried out a 'Lesson Play' as a small-scale research; this is a pilot study for my further research. Lesson play refers to script writing about an imaginary lesson scenario and consists of a dialogue between a teacher-character and student-characters (Zazkis & Zazkis, 2013). Lesson play has previously been designed for use in teacher education to develop teaching by considering instructional strategies and interactions between a teacher and students in mathematics classrooms. In my research, however, I applied lesson play to primary students as playwriting for imagined role-playing. It is a new trial in that lesson play is conducted with primary students. The participants are sixth grade (11-12-year-old) Korean primary students and the number of participants is eight.

In lesson play, mathematical situations that involve student errors, misunderstanding, or difficulties called prompts are presented to participants who are then asked to write a dialogue format script of the ensuing situation (Zazkis, Sincliar, & Liljedahl, 2013). Dialogic playwriting has been conducted to clarify mathematical understanding (e.g. Lakatos, 1976). If students write the scripts of lesson play, they will use their knowledge relating to the mathematical concepts to address the situations in the prompts. In addition, through the choice of characters and particular statements in the dialogue, a writer's perspective on the concepts can be examined. Straight after playwritings, I interviewed the students to clarify their intention for their scripts. Lesson plays and interviews were carried out via online meetings.

Prompts

The four prompts below (Transcript 1-4) were provided to the participants. I created the prompts based on students' difficulties in dealing with fraction situations such as equivalence, a mixed number, and representations of a fraction. The written prompts were shared with students via an online program.

Students are finding a fraction between two given fractions in mathematics class. Teacher: Mike, what fraction did you find between ½ and ¾? Mike: I found 2/3. Teacher: How did you find it?

Transcript 1. Prompt 1

A teacher shows four number-cards (35, 2/9, $1\frac{4}{7}$, 5.6) to students. Teacher: How many numbers are there?

Transcript 2. Prompt 2

Prompt 2 as above was used for four students, but for the other four, a different version of prompt 2 with the changed number cards (35, 8, 2/9, $2\frac{4}{7}$, 5.6) was used. A one-digit number (8) is added to compare the responses when there is both a two-digit number and a one-digit number. Moreover, the natural number part of a mixed number is changed to number two instead of number one to examine the difference in dealing with a mixed number depending on the difference of the natural number part.

A teacher is teaching a fraction. The teacher shows two fractions 5/6 and 7/8 to explain what a fraction is. Tom suddenly raise his hand. Tom: Teacher, I have a question. Teacher: Yes, Tom. Tom: 5/6 and 7/8 are equal? Teacher: Why do you think that?

Transcript 3. Prompt 3

Students are talking about a fraction in mathematics class. Mary: Hmm… I think 2/0.7 is not a fraction. Jane: I don't think so. 2/0.7 is a fraction.

Transcript 4. Prompt 4

The students' scripts arising from the prompts will later be qualitatively analysed by focusing on metaphors presented in their scripts. The analysis will not exclusively focus on the language itself but instead focus on identifying students' knowledge that underlies their language.

Students' Scripts

Students took approximately 20 minutes to write a script in response to each prompt. Based on interlinear morphemic gloss - which translates one language into a different language in each morph or its meaning (Lehmann, 2004) - I translated the students' scripts into English using three steps; 1) word by word translation, 2) direct translation, 3) natural English translation. This translation is to help readers to identify the grammatical structure and meanings of the language I want to translate (Korean). Transcript 5 is the example of the translation. In a direct translation as step two, some sentences do not make sense or there can be some missing parts in a sentence.

	선생님:	왜	그렇게	생각하지?
Step 1	Teacher:	why	like that	think
Step 2	Teacher:	Why do think like that?		
Step 3	Teacher:	Why do you think that?		

Transcript 5. Three steps translation

For efficient reading and space management, this article only presents the last step, natural English sentences.

For Prompt 1

Most of the students' fictional characters focused on explaining the calculation process of finding a common denominator.

> Mike: If we make denominators of ½ and ¾ to common denominator 12, ½ become 6/12 and ¾ become 9/12. Then a number between these two fractions is 8/12. [student 2]

Students expressed a fraction comparison as '8/12 is a fraction that is in-between 6/12 and 9/12.' and '5/6 is ahead of 7/8 on the number line'. Through the use of the metaphorical expression 'between' and 'ahead of', I found out that students understand a fraction in terms of 'location' for fraction comparisons.

Judging from the accurate and well-explained calculation process students described, I could diagnose that the students have understood the prompts and successfully solve math problems about a fraction. However, the students' scenarios for this prompt tended to focus on explaining the calculation process rather than exploring individual thoughts about a fraction.

For Prompt 2

Table 2 summarises students' writings of the character that reflects their own view about how many numbers there were and what they think the numbers are. I can find the character representing their own view through interviews. The four number-cards (35, 2/9, $1\frac{4}{7}$, 5.6) and the five number-cards (35, 8, 2/9, $2\frac{4}{7}$, 5.6) were provided to students 1 to 4 and students 5 to 8, respectively.

	Students' responses	Numbers
S1	There are two numbers.	2/9, 35
S2	Natural number, fraction, and decimal number can be a number. Because a natural number, decimal number, and fraction are composed of numbers.	35, 2/9, $1\frac{4}{7}$, 5.6
S3	There are nine numbers. Because we asked 'numeral' not 'number'	3, 5, 2, 9, 1, 4, 7, 5, 6
S4	There are nine numbers.	3, 5, 2, 9, 1, 4, 7, 5, 6
S5	The answer is one. 'Numeral' refers to magnitude, and 'number' refers to a symbol. In other word, 'numeral' means an Arabic number from 0 to 9, and 'number' means integer number, rational number, etc. Thus, 2/9, 35, $2\frac{4}{7}$, 5.6 are number, and only 8 is a numeral.	8
S6	There are five numbers.	35, 8, 2/9, $2\frac{4}{7}$, 5.6
S7	There are five numbers. All is both a number and numeral.	35, 8, 2/9, $2\frac{4}{7}$, 5.6
S8	There are five numbers.	35, 8, 2/9, $2\frac{4}{7}$, 5.6

Table 2. Students' writings about a 'number'

Students' responses to a 'number' varied. While student 3 and 4 separately counted every single digit on each card, the other students counted the number on the cards as one entire number. In the interview, student 1 stated 2/9 is a number but $1\frac{4}{7}$ is not a number. She believes 1 refers to a whole figure and $1\frac{4}{7}$ means a whole figure and some more, and thus $1\frac{4}{7}$ is not a number. Students 3, 5, and 7 distinguished 'numeral' from 'number', but their definitions of 'numeral' and 'number' are different from each other. The metaphor of a number as numeral seems to make the students confused in understanding a concept of a number and a fraction.

For Prompt 3

Five students approached a fraction with a part-whole perspective.

> Teacher: A fraction is a number that represents a part of the whole. [Student 5]

The students regard a fraction as 'size' and 'amount'. They explained fractions as how big and much they are. In order to explain the part-whole approach, visual representations were used. Following figures showing a rectangle or circle that is equally divided were commonly used.

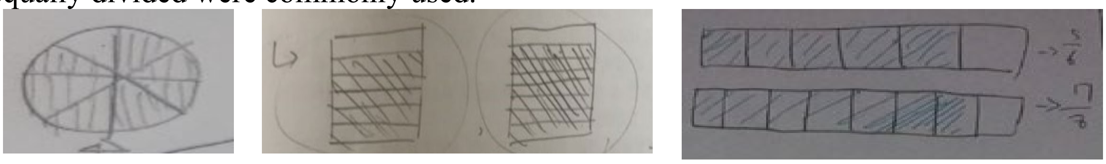

Figure 1. Students' drawings

For Prompt 4

Three students regarded 2/0.7 as a fraction whereas four students rejected the representation as a fraction. One student wrote that 2/0.7 can be a fraction and not be a fraction depending on perspective. A fraction was explained in relation to division, but the expressions that explain its relevance are various.

> Jane: Because 2/0.7 is eventually from division. [student 1]
>
> Hee: I think a fraction is a thing that simplifies a division equation. [student 2]
>
> Teacher: Fraction = number ÷ number [student 7]

Conclusion

The students described various points of view by featuring numerous characters in their script, and in most cases, the teacher-character reflects the students' thoughts. If there was no teacher-character in a scenario, I was able to find the characters that represented their thoughts through further interviews. Individuals unconsciously choose the words they want to write (Zazkis et al., 2013). In this regard, apart from the choice of a character that represents the students' perspective, students represent their knowledge of a fraction through their own expression.

I found out the variety of students' metaphors about a fraction. Students understood a fraction in terms of size, amount, location, number, numeral, and division. However, before understanding abstract mathematical concepts via metaphors, students have the difficulty in interpreting the metaphors used (English, 2013). Students who are successful with the calculation had confusion about what a number is. In terms of location as a metaphor of a fraction, for example, a number line is linked as a metaphorical representation. Since students have a tendency to regard a number line as “stepping stones” (*ibid*, p. 8), students can think there is no number between two whole numbers (e.g. 2 and 3) or between two pseudo-successive rational numbers (e.g. 2/4 and 3/4). Students’ personal understanding of a number can affect the construction of a concept of a fraction in their mind.

Through this small-scale research, I can see the possibility that lesson play can be applied to students as a novel way in which researchers can investigate students’ understanding. In further research with more students, I expect students’ metaphors and understanding can be more deeply examined through script-writing tasks.

References

Carpenter, T. P., & Lehrer, R. (1999). Teaching and learning mathematics with understanding. *Mathematics classrooms that promote understanding*, 19-32.

Davis, B. (2020). Number work: Teachers as experts who can think like novices. *For the Learning of Mathematics Monograph 1,* 49-54.

Davis, B., & Renert, M. (2013). Profound understanding of emergent mathematics: broadening the construct of teachers’ disciplinary knowledge. *Educational Studies in Mathematics*, *82*(2), 245-265.

English, L. D. (Eds.). (2013). Mathematical reasoning: Analogies, metaphors, and images. New York: Routledge.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-98). New York: Macmillan.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery.* Cambridge: Cambridge University Press.

Lakoff, G. (1993). The contemporary theory of metaphor. In A. Ortony (Ed.), *Metaphor and thought* (2nd ed., pp. 202-250). Cambridge: Cambridge University Press

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from*. New York: Basic Books.

Lehmann, C. (2004). Interlinear morphemic glossing. In G. Bookij, C. Lehmann et al. (eds.), *Morphologie. Ein internationales Handbuch zur Flexion und Wortbildung*, 2, (pp. 1834-1857). Berlin: de Gruyter.

Marmur, O., Yan, X., & Zazkis, R. (2020). Fraction images: the case of six and a half. *Research in Mathematics Education*, *22*(1), 22-47.

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology*, *14*(3), 293-305.

Von Glasersfeld, E. (1995). *Radical constructivism: a way of knowing and learning*. London: Routledge.

Von Glasersfeld, E. (2008). Learning as a constructive activity. *Antimatters, 2*(3), 33-49.

Zazkis, R., Sinclair, N., & Liljedahl, P. (2013). *Lesson Play in Mathematics Education. A Tool for Research and Professional Development*. New York: Springer.

Zazkis, R., & Zazkis, D. (2013). Exploring mathematics via imagined role-playing. *Proceedings of the International Conference for Psychology of Mathematics Education.* Vol. 4. (pp.433-440). Kiel, Germany.