A design based research project: Exploring pedagogies that develop abstract and algebraic thinking within secondary school mathematics

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Within the current Welsh educational system, 'real life' problems have become a focus within Mathematics lessons. Pupils are given a worded or visual problem which often needs to be translated into a mathematical form to solve. The OECD refer to this process as *mathematizing*. It is this process of mathematizing that often causes difficulties for pupils. Pupils have to abstract during the translation, meaning they have to locate and express the mathematical structure within the problem. Only after abstraction is the problem in a form which is able to be manipulated and solved. This research is the 1st iteration in a design based research project that covers four intervention lessons. This project aims to develop specific pedagogies which support pupils to focus on mathematical structure and think abstractly and algebraically. This paper seeks to justify and evaluate intervention tasks and suggests possible changes for the next cycle of intervention.

Keywords: Abstract thinking; algebraic thinking; mathematical structure; problem solving

Introduction

There is a wide range of literature which discusses problem solving in mathematics and the sills needed to be an effective problem solver (Kilpatrik 1985; Schoenfeld 1992; Tanner & Jones, 2000). Skills such as metacognition are explored explicitly and suggestions are given for ways to improve and development pupils' metacognitive skills. The importance of abstraction is often referred to within problem solving literature but not always explicitly explored in terms of how to develop the skill. OECD (2015) argue that often, the hardest cognitive element of mathematical problem solving is the point of mathematization. Pupils are given a worded or visual problem which needs to be translated into a mathematical form to solve. Pupils have to abstract during the translation, meaning they have to locate and express the mathematical structure within the problem. It is in this process of mathematizing that the problem is abstracted. Sfard (1991) suggests that abstraction in mathematics is achieved when an individual is able to view mathematics structurally rather than operationally. This paper looks at the 1st iteration of a design based research project. The project aim was to design and evaluate classroom based tasks and activities that improve a pupils' ability to think abstractly and algebraically within mathematics. The interventions were all designed with a focus on mathematical structure and draw on the body of literature surrounding mathematical structure, generalising and abstraction.

Abstract thinking

Abstraction is often seen as a higher order cognitive function and something that many school pupils struggle to develop (Sfard, 1991; Kilpatrick, 1985; Mason, 1989). If we take the stance that mathematical thinking is hierarchical, then we may view processing skills e.g. a pupil being able to add two numbers together in any order, as a lower order skill. The pupil is able to work operationally and complete a series of additions. The ability to generalise the operational additions and represent the underlying mathematical structure as an abstracted relationship into a form such as $x+y=y+x$, can be seen as a higher order skill. In this example, the pupils have been able to attune to the underlying mathematical structure. Sfard (1991) suggests that mathematics can be seen as either structural, operational or both and that abstraction occurs when pupils are able to view mathematics structurally rather than operationally. Operational knowledge is the ability to do something with the mathematics and structural is the ability to see it as a static object. If we again consider the equation $x+y=y+x$, a progression of abstraction would move a pupil from seeing this as a generality to seeing instead that addition is commutative. In abstraction, pupils have shifted from seeing the expression as an expression of generality to seeing it as an object or property (Mason, 1989).

Intervention lessons

This research project covered four intervention lessons. As this research was within a real educational context, focused on designing and testing an intervention to forward a theoretical agenda and had multiple iterations, design based research methodology was used (Anderson & Shattuck, 2012). The research took place in a large secondary comprehensive school with a currently high achieving year 8 class (12 to 13 years old). Within the research project, pupils were given opportunities to engage with a range of tasks specifically designed to focus their attention to mathematical structures. A socio-constructivist stance was taken and mathematical discussions were a key element to the intervention lesson. This research paper will focus specifically on the tasks within the 1st lesson of the intervention.

Task 1

One of the key symbols within mathematics is the equals sign '='. Sfard (1991) argues that pupils who think operationally rather than structurally often see the equals sign as a command to do something, rather than a statement of equivalence. Jones & Pratt (2012) suggest that many pupils see the equals sign as an instruction to write the answer. This limited view of the equals sign can lead to difficulties when expressing and dealing with generalities, especially in algebraic form. The first task was designed as a starter activity and pupils were asked to write down what the equals sign means. Pupil responses to the task can be seen in Figure1 below.

= means the 'answer'
It means the answer or what the answer is to something
The answer to the question. It is a symbol put after a question.
Equals or the answer to a question
The equal sign is the connection between the sum and the answer
The answer. Final part of a sum
I think it mean, the asnwer to something
The answer
Shows that it is the answer to the question
I think it means what the answer to something is
Its the end of a sum to answer it
when a sum like 1+1 and then you have an equal sign, so you work it out or it means 1+1 is equal to 2

The same as
It means that its equivalent to something in the sum
The same as
The same as
It means that the sum is the same as the answer e.g. 2+2 =4 2+2 means 4 so that means it is the same as it
Equals sign means that the number either side are equal to each other
The same as
The equals sign = means that whatever equation is equal to whatever number on the side of the equal sign
When something is equal to something else. Shows ways to write the same number
It means that something is the same as the other thing. For example: 20-17 is the same as 19-16 0r 20-17=19-16
it shows that soemthing is equal to soemthing else
When something is equal to something else.

Dual Function
Something what is the same as or the answer to a sum
To me the equal sign means either the asnwer to a question (4+5 = 9) or that something is the same as something else. 4x5 = 10+10
It means the answer to a sum or if something is the same as something else
it represents the asnwer to a problem or it could mean that something is equivalent

Other
The equals sign means equal to (translation from symbol to words)
I think that it is where 2 things are equal. For example a man and a woman (non mathematical example)
The equal sign is the operation that we do to find the x value

Figure 1: Pupil responses to the starter task

The responses from pupils were grouped into four separate categories: ‘The answer’, ‘the same as’, ‘dual function’ and ‘other’. ‘The answer’ appeared to represent pupils who had an operational view of the equals sign and saw it as an instruction to calculate an answer (Jones & Pratt, 2012). ‘The same as’ appeared to represent pupils who viewed the equals sign as some form of statement of equivalence. The ‘dual function’ appeared to represent pupils who viewed the equals sign as an instruction and a representation of a relationship but that these were two different things.

Development of task 1

It could be argued that the responses given in Figure 1 by the pupils were definitions that they have learnt and may not correlate to the way in which they work mathematically. For the 2nd iterative cycle each group of pupils will be mapped through the remainder of the lesson to see if there is any correlation between their definition or view of the equals sign and the way in which they work with the follow on tasks. E.g. there needs to be further analysis undertaken to see if ‘the answer’ group, initially work more operationally than structurally (Sfard, 1991).

Task 2

Task 2 is a linking task and aimed to ensure all pupils engage with the equals sign as a statement of equivalence. Research suggests that pupils need to be exposed to tasks that explicitly teach the equals sign as a statement of equivalence (Li, Ding, Capraro & Capraro, 2008) as not all pupils automatically attune to it. Task 2 draws on the work of Hewitt (2001) and uses number sentences such as $4 + 5 = \_\_ + 3 = 12$. Hewitt suggests that some pupils see no problem in having $4 + 5 = 9 + 3 = 12$ as an incorrect number sentence. This links back to the idea that pupils see the equals sign as a command to write the answer (Jones & Prat, 20012). The task along with examples of pupil work can be seen in Figure 2 & Figure 3 below.

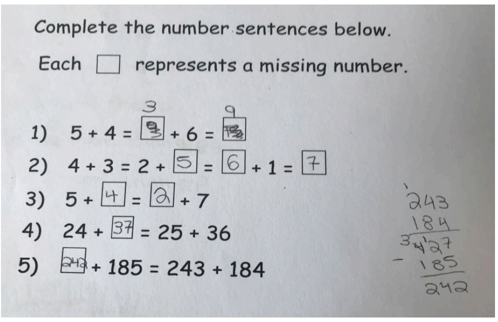

Figure 2: Example of pupil work

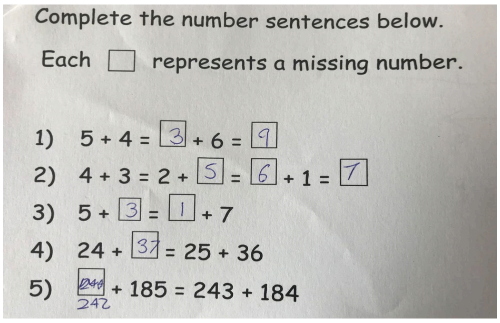

Figure 3: Example of pupil work

All pupils correctly identified the missing values but we can see from Figure 2 that some pupils had initially written 5 + 4 = 9 + 6 = 15 but had then corrected themselves. It is not clear why the pupils corrected themselves. Pupils could have looked over and seen their partner had a different answer or it could be that completing the 2nd question alerted pupils to the fact that they had answered the 1st question incorrectly. No pupils appeared to have answered the 2nd question incorrectly, which could be due to the unknown value not being directly after the equals sign.

Question 5 was designed to see how many pupils would work operationally and calculate an answer, such as in Figure 2, and how many pupils would use the relationship between the values to work more structurally and balance the equation to produce an answer. Figure 3 shows a correct answer but does not explain the thinking by the pupil. Pupils can often think that the less workings you have to write, the better at mathematics you are (Tanner & Jones, 2000). The pupil in Figure 3 could have done the same calculation as the pupil in Figure 2 but may not have shown any workings. The class were asked how they calculated question 5 and only one pair of pupils volunteered a structural solution.

Development of task 2

The 2nd iterative cycle needs to be able to better capture pupils thinking. An additional question at the end of the task will provide pupils with multiple solutions for question 5 and pupils will be asked which solution best reflects their thinking. The 1st iterative cycle suggested that most pupils worked operationally rather than structurally, although evidence is weak (the addition of the follow on task will strengthen this in the 2nd iteration). Pupils will also have an additional task where they have to work structurally to find missing values like in question 5. This additional activity will provide pupils with a range of questions which vary slightly in numerical value but are set within a background of invariance, the use of variation theory to underpin the task will support pupils in attuning to the mathematical structure (Marton, 2015; Al-Murani, Kilhamn, Morgan, & Watson, 2019).

Task 3&4

Task 3 & 4 were interconnected and gave pupils the opportunity to work operationally before moving onto more abstracted tasks. Pupils often learn to view mathematics structurally as objects through practice and development of operational processes (Sfard, 1991). Task 3 was a discussion surrounding 5 +__ = __ + 7. The discussion surrounding possible unknown values allowed pupils to attune to the mathematical structure and describe the relationship between the unknown values. Pupils then moved onto describing the relationship between p and q in the equation $5 + p = q + 7$.

The operational start appeared to scaffold pupils and enabled them to make the link to the more abstracted equation. Kieran (2016) argues that pupils will only be able to understand using algebra to express generalities and relationships if they are first able to understand the relationship themselves.

The 4th tasks was designed to build on task 3 and support a shift towards working and thinking structurally. Within task 4, pupils could initially work operationally as a support but a final answer needed to be presented as a relationship between p and q. Variation theory was used to underpin the task and support pupils in attuning to mathematical structure. Examples of pupil work can be seen in Figure 4, Figure 5 & Figure 6 below.

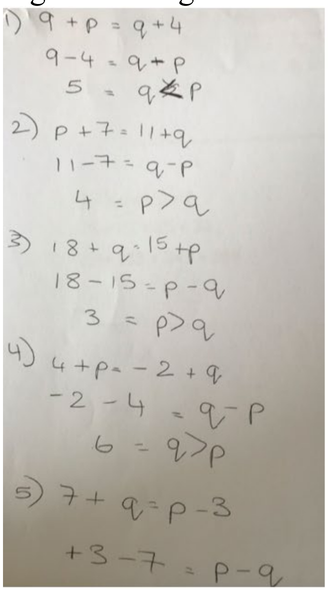

Figure 4: Pupil A

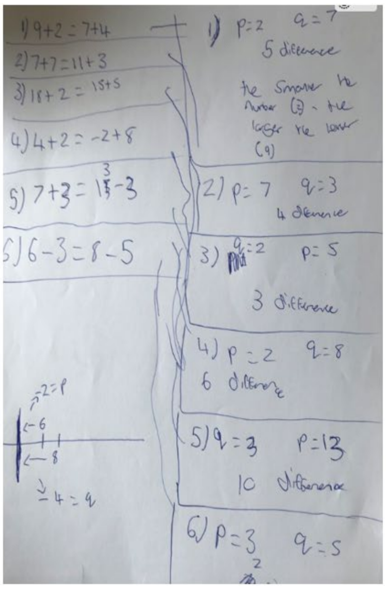

Figure 5: Pupil B

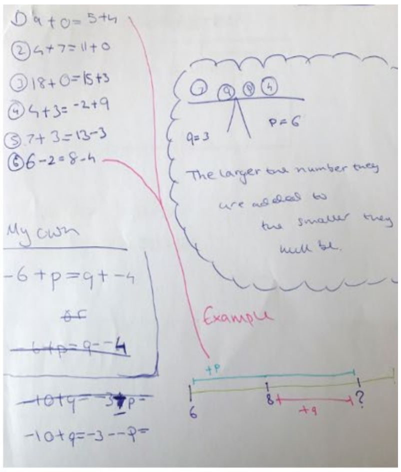

Figure 6: Pupil C

Pupils A, B and C all appeared to work operationally in the first instance. Pupil A attempted to rearrange the equations but possibly incorrect algebraic knowledge lead to a mathematically incorrect solution line. Pupil A had started to make a shift and identified if p < q or q < p but there is a step missing in their thinking. Pupil B was able to correctly substitute values into the equation and found the difference between p and q but then appeared stuck at the operational stage and only produced a numerical value as the answer rather than a relationship. Pupil C wrote correct number sentences but didn't seem to be able to make links between their specific number sentences and the initial abstracted question.

Pupils appear to be at different stages and making a range of progress towards thinking structurally. There were a number of pupils who were able to correctly identify the relationship between p and q and who had created a process or method. Some pupils explained that you find the difference between the given values and then work out if p or q is bigger. Question 6 (*6-p=8-q*) was added to stop pupils who had developed a 'quick' operational method. Christiansen & Walther (1986) warn that during a task, pupils sometimes do something different to what the teacher thinks they are doing. Tasks need to be designed to offer cognitive opportunities (Watson, 2003) and need to force pupils to move beyond a superficial or operational method.

Development of task 3&4

Early on in the lesson, pupils appeared to be able to move from the operational to the structural but this later appeared superficial and was only specific to the example.

Some pupils appear to need additional support in making the shift between thinking operationally and structurally. Some of these issues could be centred on pupils' apparent difficulty with algebraic notion and this needs exploring further. Pupils also needed longer to explore and engage with 5 +__ = __ + 7 and $5 + p = q + 7$ before moving on.

Limitations & next steps

This is a small scale research project. The participating class was a currently high achieving group and although the class did highlight some of the difficulties pupils face when trying to make a shift to a more structural view of mathematics, there may be wider issues not identified. The class in the next iterative cycle are currently working at a lower level which will help to broaden the scope of the project.

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