# Mapping out different discourses of mathematical horizon

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'Knowledge at the mathematical horizon' refers to a particular domain of teachers' knowledge related to connections across mathematics. This construct has been used and elaborated in research. Nonetheless, 'knowledge at the mathematical horizon' is still considered a 'grey area' with different interpretations and meanings. In this paper, I report a preliminary commognitive analysis of a sample of papers about knowledge at the mathematical horizon attending to the use of the term in the related research. The aim of this paper is to investigate different narratives in relation to the construct and how these narratives might be linked to how knowledge at the mathematical horizon is conceptualised and operationalised into research. To conclude, I argue that a discursive approach might provide better insight about the nature and use of mathematical horizon in research and set the scene for further development of these ideas as part of mathematics teachers' discourses.

**Keywords: mathematical horizon; teachers' knowledge; commognitive analysis; research as discourse; literature**

## Introduction

Connections across mathematics are at the core of the discipline, and mathematics in school is not an exception. Some of the connections might be intended, included in the curriculum and supported by resources for the teacher. Yet, it is possible that discussion in the classroom might hint at unexpected links with mathematical ideas not included in the curriculum. The Mathematical Knowledge for Teaching (MKT) framework (Ball, Thames, & Phelps, 2008) seems to include a domain of teacher's knowledge that specifically addresses situations like that. In the literature, the terminology varies. The domain is more commonly referred to as 'horizon (content) knowledge' (Ball & Bass, 2009; Ball et al., 2008; Jakobsen, Thames, Ribeiro, & Delaney, 2012) or 'knowledge at the mathematical horizon' (e.g. Zazkis & Mamolo, 2011). To avoid confusion, I am using the term 'knowledge at the mathematical horizon' throughout the report. Knowledge at the mathematical horizon was first described as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p. 403). Over the years, researchers attempted to develop and describe knowledge at the mathematical horizon. This led to diverse discourses challenging its conceptualization and use in research. The idea seems to be the least understood among those described in the MKT framework. The aim of this report is to explore possible links between descriptions in research papers and the conceptualization and operationalisation of mathematical horizon into research.

## Commognition as a critical lens

According to Sfard (2008) cognition and communication are inseparable. In commognition, the theory developed under this new scope, discourses are "different types of communication, set apart by their objects, the kinds of mediators used, and the rules followed by the participants" (Sfard, 2008, p. 93). Discourses have four characteristics: word use, visual mediators, endorsed narratives and routines. Usually, the theory of commognition is employed to analyse mathematical discourses, but its potential does not stop there. The importance of definition and the use of metaphors in research is highlighted in Sfard's (2008) theory. Research is defined as the "discourse produced with the intention of creating endorsed narratives with which we can mediate and enhance our deeds" (Sfard, 2008, p. 301). In this report, I use the theory as a critical lens to analyse the researchers' discourses when describing and using knowledge at the mathematical horizon in research papers.

Specifically, I will look into the endorsed narratives that are the descriptions or definitions given for knowledge at the mathematical horizon in the papers. I will focus on the word use in defining and describing the notion and the routines, particularly how knowledge at the mathematical horizon is used to describe a phenomenon, how it is conceptualised in research and how it is operationalised in research design, the analysis and the interpretation of the findings. Although, visual mediators are also very interesting, their analysis goes beyond the scope of this paper.

## The papers

There are a number of papers that use the construct of knowledge at the mathematical horizon (Ball & Bass, 2009; Cho & Tee, 2018; Fernández, Figueiras, Deulofeu, & Martínez, 2011; Jakobsen et al., 2012; Wasserman & Stockton, 2013; Zazkis & Mamolo, 2011). This is a preliminary analysis that I exemplify with a small number of papers and not a complete literature review of the concept. Because of the limited space, I will only focus on three of the most cited ones (Ball & Bass, 2009; Jakobsen et al., 2012; Zazkis & Mamolo, 2011).

## Analysis

### *Word use and routines*

The first narrative is given by Ball and Bass (2009) as an attempt to clarify the concept introduced earlier as part of the MKT framework.

> We define horizon knowledge as an **awareness** [emphasis added] – more as an experienced and appreciative tourist than as a tour guide – of the large mathematical landscape in which the present experience and instruction is situated. (Ball & Bass, 2009, p. 6)

The keyword here is 'awareness'. According to the Cambridge dictionary, awareness means "knowledge that something exists or an understanding of a situation or subject at the present time based on information or experience". Using the word awareness to describe knowledge at the mathematical horizon could indicate that the focus in not on knowing specific characteristics of concepts but rather knowing about mathematics as a discipline.

This indication is also supported by the way Ball and Bass (2009) describe an episode to exemplify knowledge at the mathematical horizon. The episode is about a teacher discussing with some students about even and odd numbers. One of the researchers' comments is:

> First, worth noting is that the episode is not only about even and odd numbers, but also centrally about mathematical communication, reasoning and proving . . . . (Ball & Bass, 2009, p. 8)

Acknowledging that the ideas communicated are part of a larger discourse seems to be very important in the researchers' routines. This could mean that even and odd numbers per se are not at the centre of knowledge at the mathematical horizon.

Based on Ball and Bass' (2009) descriptions, Jakobsen et al. (2012) developed a working definition of knowledge at the mathematical horizon:

> Horizon Content Knowledge (HCK) is an **orientation** to and **familiarity** [emphasis added] with the discipline (or disciplines) that contribute to the teaching of the school subject at hand, providing teachers with a sense for how the content being taught is situated in and connected to the broader disciplinary territory. HCK includes explicit knowledge of the ways of and tools for knowing in the discipline, the kinds of knowledge and their warrants, and where ideas come from and how "truth" or validity is established. HCK also includes **awareness** [emphasis added] of core disciplinary orientations and values, and of major structures of the discipline. . . . (Jakobsen et al., 2012, p. 4642)

In their definition, the word 'awareness' is more clearly connected to mathematics as a discipline and not to specific concepts; it specifically refers to the core disciplinary values and orientations. Moreover, the words 'orientation' and 'familiarity' could be interpreted as 'knowing about' mathematics but on a deeper level than 'being aware'. The phrase 'explicit knowledge of the ways of and tools for knowing in the discipline' supports the view of the expectation of more in-depth knowledge. Additionally, the choice of the word 'orientation' might be related to the authors' perspective on the mathematical horizon which will be discussed later.

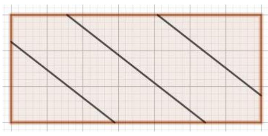

Figure 1: Adaptation of students answer

To illustrate how knowledge at the mathematical horizon might benefit teaching, Jakobsen et al. (2012) offer two vignettes. One of them is an episode where primary school students were asked to divide a rectangle in four equal parts. One of the students (Maria) divided the rectangle in the way shown in Figure 1. The student explained to the class that she knows that the parts do not look equal, but she claimed that she could make them equal by squeezing the lines closer together. The student's idea is correct and can be proven. The authors explain that when a line slides across a figure the area on one side can be thought of as a continuous function going from 0 to the whole area of the figure. Based on the intermediate value theorem there will be a line that cuts the shape exactly in half. Repeating this for the two new shapes results in four shapes having equal areas. They continue:

> Experiences with the concept of continuity and different ways of thinking and talking about continuity would provide a teacher with resources for hearing mathematical ideas in Maria's talk — ideas related to major structures and developments in the discipline. . . . Understanding the formalisms related to continuity can add precision to a teacher's thinking. Having language to talk about it casually yet with integrity can position a teacher to draw students' nascent attention to important mathematical ideas . . . . (Jakobsen et al., 2012, p. 4638)

This quote depicts the authors' interpretation of how knowledge at the mathematical horizon could help a teacher hear the student's idea and act accordingly. The concept of continuity is treated as an important mathematical idea appearing in many seemingly unrelated situations and not as a characteristic of formally defined functions. This might require a deeper understanding of continuity, possibly also at a meta-level, which is consistent with the use of the words orientation and familiarity. Referring to formalism separately might suggest that they do not consider it part of knowing about continuity, rather, as being subsequent. Finally, being able to address the idea casually but with integrity seems to be part of knowledge at the mathematical horizon for the researchers.

The last narrative is an attempt of Zazkis and Mamolo (2011) to extend the idea of knowledge at the mathematical horizon.

> We consider **application of advanced mathematical knowledge** [emphasis added] in a teaching situation as an instantiation of teachers' knowledge at the mathematical horizon. More explicitly, a teacher's use of the mathematical subject matter knowledge acquired in undergraduate studies is recognized as an instantiation of knowledge at the mathematical horizon when such knowledge is applied to a . . . teaching situation. (Zazkis & Mamolo, 2011, p. 9)

Advanced mathematical knowledge, in this narrative, is defined as knowing university mathematics (Zazkis & Leikin, 2010). University mathematics includes learning of formal definitions and paying attention to characteristics of advanced concepts. Definitely, university mathematics is part of the discipline and students at university level may come across key ideas and structures but it is worth noticing that what they learn is usually constrained by the curriculum and the objectives of the modules.

In terms of routines, Zazkis and Mamolo (2011) seem to focus on characteristics of specific concepts rather than more general ideas contrary to the other two papers. For example, the main episode discussed in the paper is around an activity where primary school students had to identify the number of triangles formed by the diagonals in a regular hexagon, in which the students' answers varied. The authors then claim:

> The teacher, though she had not yet determined the number of triangles herself, immediately knew that both answers were incorrect. She recognized rotational symmetry of order 5 in the figure and, as such, she knew that the number of triangles should be a multiple of 5. (Zazkis & Mamolo, 2011, p. 10)

They describe how knowing about a characteristic of a specific concept, rotational symmetry, could help the teacher determine if the answers were correct. According to Zazkis and Mamolo (2011) this knowledge came from a university course. Although they previously indicated that knowing advanced mathematics is an example of knowledge at the mathematical horizon their routines seem to focus on specific advanced concepts, which is contradictory to the other narratives. They continue:

> With this understanding in mind, she helped students identify different kinds of triangles and where, with each triangle-shape found, there were 5 of the same kind. She led students to catalogue different shapes and account for them systematically. (Zazkis & Mamolo, 2011, p. 10)

Zazkis and Mamolo (2011) do not go into the details of how the teacher aided the students to find the different types of triangles. Since they do not discuss what knowing about rotational symmetry can add to the practice, in comparison to knowing the strategy to solve the problem, the application in the classroom seems coincidental.

### *Different perspectives and the metaphors used*

The word horizon is used figuratively, possibly to indicate the idea of the connection between mathematics in general and mathematics taught in school. Back in the 20's, Felix Klein was the first who acknowledged these connections. He talked about the gap in the double transition of teachers between university and school mathematics and proposed that knowing elementary mathematics from an advanced perspective will help teachers close this gap (Klein, 2004).

In these three papers, the researchers position knowledge at the mathematical horizon comparative to Klein's idea. The metaphors the researchers use to describe knowledge at the mathematical horizon seem to line up with the different perspectives found in the papers. The following table summarised this observation.

| Standpoint | Metaphors used |
|---|---|
| **Elementary perspective on advanced mathematics** | • "peripheral vision" (Ball & Bass, 2009, p. 1)<br>• "a view of the larger mathematical landscape" (Ball & Bass, 2009, p. 1)<br>• "mathematical environment surrounding the current 'location'" (Ball & Bass, 2009, p. 6)<br>• "an orientation" (Jakobsen et al., 2012, p. 4642) |
| **Advanced perspective on elementary mathematics** | • "where the land appears to meet the sky" (Zazkis & Mamolo, 2011, p. 9)<br>• "the higher one stands, the farther away the horizon is and the more it encompasses." (Zazkis & Mamolo, 2011, p. 10) |

*Table* 1: Standpoints and metaphors used

Ball and Bass (2009) and Jakobsen et al. (2012) adopt a standpoint complementary to Klein's. For them, knowledge at the mathematical horizon is a kind of elementary perspective on advanced mathematics. The researchers' discourse includes analogies between the literal horizon in a landscape and the mathematical horizon. The word 'orientation' that Jakobsen et al. (2012) use in their definition can be interpreted as "the position of something in relation to its surroundings" (according to the Cambridge dictionary) which might indicate a hidden metaphor there. In all these metaphors, there is an underlying assumption that the person is fixed in a 'location' (i.e. elementary mathematics) looking to the horizon (i.e. advanced mathematics).

On the other hand, Zazkis and Mamolo (2011) visualise knowledge at the mathematical horizon as one being able to approach elementary mathematics from an advanced perspective. To support their ideas about advanced mathematics, they use a physical property, that the higher above sea level one stands the horizon seems to be further away. Corresponding to this property they suggest that the more advanced mathematics one knows, the further away is the limit of one's knowledge.

## Discussion and conclusion

To sum up, knowledge at the mathematical horizon seems to be conceptualised and operationalised differently by the researchers. A commognitive analysis of the papers could help to rigorously distinguish and/or group together different discourses pertaining to what knowledge at the mathematical horizon is.

In this particular sample of papers, the focus of 'what is in the mathematical horizon' seems to change depending on the word use. For Ball and Bass (2009) as well as for Jakobsen et al. (2012) the horizon seem to include the connections spanning across mathematics, whereas, for Zazkis and Mamolo (2011) the horizon is mainly the advanced mathematics taught at university as the limit of what the teacher knows. In all three cases, the metaphors used seem to be consistent with the standpoint of the researchers. It is worth mentioning that the extensive use of metaphors has been stated in the past (Jakobsen, Thames, & Ribeiro, 2013). Therefore, it is interesting to wonder what the implications of that are. Could it be that the word 'horizon' is actually clouding our understanding of the notion? Could the researchers be talking about different ideas but calling them by the same name?

Finally, considering that teachers in the UK have different mathematical backgrounds, how could mathematical horizon be conceptualised and operationalised in the UK context? Is it important for the teachers to know advanced mathematics or to know about the discipline? Their diverse experiences might contribute to further understanding the notion.

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