# The influence of context and numerical complexity on the tendency to focus on scalar relations when solving missing-value ratio items. A study involving lower secondary school pupils and PGCE mathematics students.

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In this paper I discuss responses to three pairs of missing-value ratio items presented in the form of parallel one-page written tests. The items varied in numerical complexity (involving ratios that ranged from 1:3 and 1:4 to 2:9 and 3:10). They also varied in the contexts used (recipe, geometric enlargement and currency exchange). Within each pair of items, the given numbers were the same but 'flipped', so what was a scalar relation of 1:3, say, in one version of the item, became a functional relation of 1:3 in the other version. I use these notions of scalar and functional, derived from Vergnaud, to classify responses, and to examine the influence of context and numerical complexity on the frequency of such responses and on overall item facility. From a teaching perspective, the findings suggest that it is *not* helpful to think of ratio concepts as forming a simple hierarchy.

**Keywords: ratio; scalar and functional relations; multiplicative reasoning**

## Introduction

If one examines how aspects of ratio are presented in school textbooks, one can see a widespread tendency to declare the multiplicative nature of the topic in advance and to focus on formal procedures. This means that pupils' informal methods are easily overlooked, rather than celebrated and developed, and that the limitations of their methods, including possible misconceptions, often go unaddressed. So, for example, when textbooks introduce the topic of enlargement, they commonly focus on the multiplicative procedure for enlarging a shape. They rarely explore whether pupils appreciate that the relationship between similar shapes *is* multiplicative, and why.

In the ICCAMS project (Increasing Competence and Confidence in Algebra and Multiplicative Structures), Jeremy Hodgen, Margaret Brown and I worked with classroom teachers to develop lessons, mainly at Key Stage 3 (11 - 14 year olds). The lessons involve tasks designed to elicit and address pupils' ideas. Consider, for example, the task in Figure 1 below (developed from an activity by Guy Brousseau - see Küchemann, 2017). This involves quite a complex ratio (4 : 7) and consequently many pupils adopt the strategy of adding 3 to the (horizontal and vertical) edges of each piece. This means that the top edge of the 6 by 6 tangram is increased by 3 units, but the bottom edge by 3 + 3 units, so that the 'enlarged' pieces don't fit.

Lesson

24A

Multiplicative Reasoning 12A

**Tangram**

This tangram consists of three pieces. We want a larger version of the tangram where the 4 cm length becomes a 7 cm length.

Work in a group of three. Choose **one** piece each.
Draw the larger version of **your** piece on 1 cm squared paper.
Carefully cut it out. Check that the 3 new pieces again fit together.

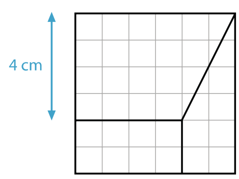



Figure 1: Tangram task for ICCAMS lesson 24A

We also devised mini-assessments that teachers use a day or two before embarking on an ICCAMS lesson, which are intended to harvest pupils' informal ideas and thus help teachers match the subsequent lesson to their pupils. Most of the mini-assessments consist of a task that leads to a roughly 10 minute whole class discussion. However, some are mini-tests, including a mini ratio test which is the subject of this paper.

One purpose of the Mini Ratio Test is to identify the informal (or indeed formal) strategies that pupils have at their disposal and, more fundamentally, to see whether they are in some sense aware that the test items involve ratio. Another purpose is to heighten teachers' (and our own) awareness of some of the factors that might influence which methods pupils choose to use.

Figure 2 shows version A of the mini test, in condensed form. Version B involves exactly the same numbers, but 'flipped'. Figure 3 shows a schematic (ratio table) representation of the two versions. The design of the items builds on the findings of the CSMS project (Hart, 1981; Küchemann, 1989) about how context and numerical complexity can influence choice of strategy and also, in particular, on the work of Vergnaud (1983) on the use of *scalar* and *functional* relations (see below).

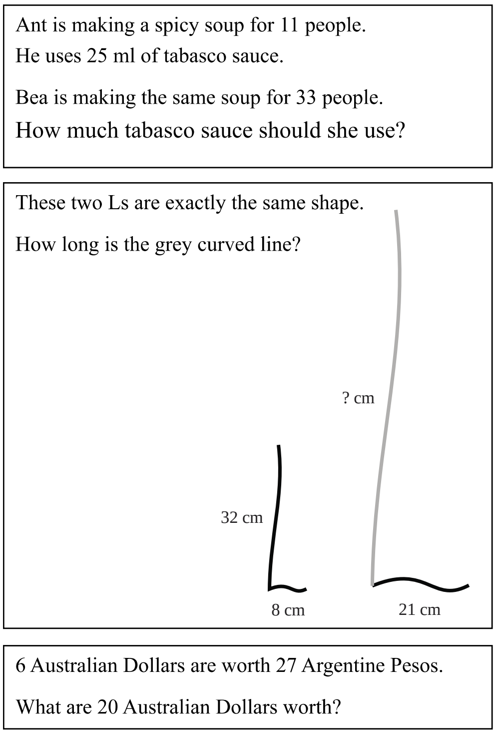



Figure 2: Mini Ratio Test (Version A)

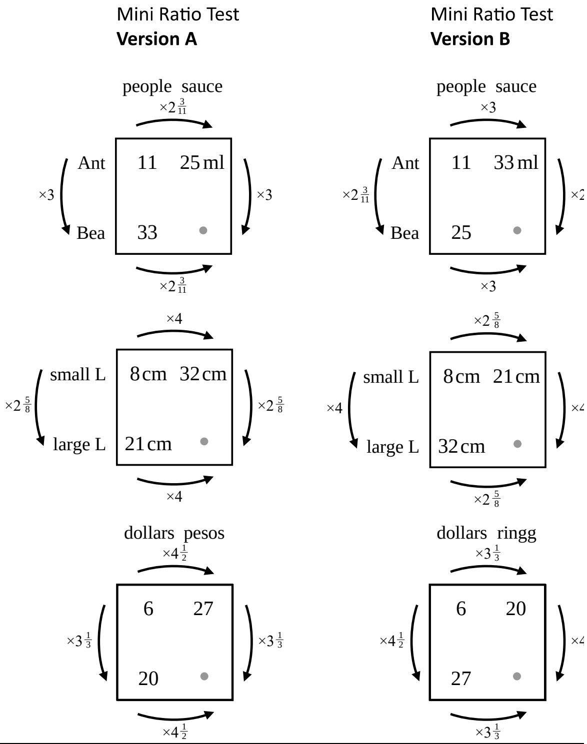
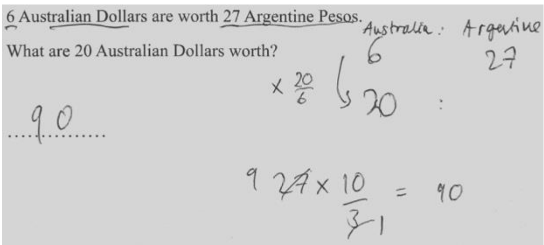



Figure 3: Mini Ratio Test (A and B, schematic)

## Responses to the Ratio Mini Test

We recently administered the mini test to 35 PGCE mathematics students, with 17 students given Version A and 18 given Version B. Not surprisingly, all the PGCE students answered all the items correctly. Figures 4a and 4b show two response to Item 3, Version A. Here the students have found (multiplicative) relations between the two amounts of Dollars (×20/6 in Fig 4a, or ÷3, ×10 in Fig 4b) and applied these to the given number of Pesos. Of course, it is also possible to find comparable relations (eg ×27/6 or ÷2, ×9) between the original number of Dollars and Pesos, which can then applied to 20 Dollars to find the new number of Pesos. Vergnaud (1983) describes methods like those in Figure 4 as involving *scalar* relations, since the relations are between quantities in the same 'measure space', while the alternative methods involve *functional* relations. Other researchers have described the same phenomena, though sometimes using different terminology, eg *internal* and *external* (eg Freudenthal, Noelting) or *within* and *between* (eg Lybeck, Lamon).



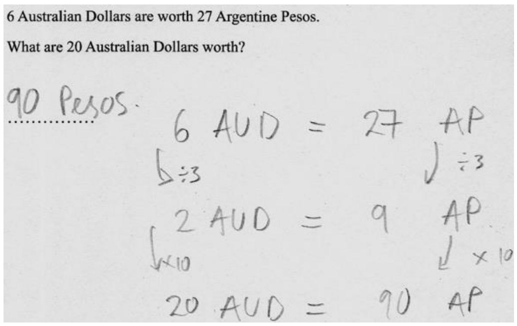
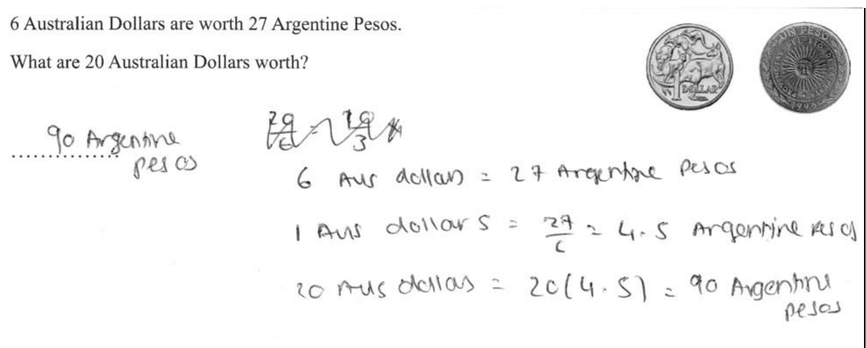




Figure 4a: Item 3 (Version A), Scalar ×20/6 Figure 4b: Item 3 (Version A), Scalar ÷3, ×10

Vergnaud argues that pupils will choose scalar over functional relations if the relations are of the same numerical complexity. This suggests that students are more likely to give responses like those in Figure 4 than ones that map Dollars to Pesos. This makes sense: the multiplier 20/6 (or 10/3) that maps Dollars to Dollars is a pure number, whereas the multiplier 27/6 is not, if one stays rooted in the context. It represents something more complex, namely a rate: '27/6 Pesos per Dollar'. The situation is even less intuitable if one uses an intermediate step, for example by going from 6-and-20 to 3-and-10 (÷2) to 27-and-90 (×9). The numbers are perfectly sound, but what do they mean? If one tries to stay in context, the 3 and 10 might be thought of as referring to dollars but then how do they fit the story? And note that we still have to make the switch from Dollars to Pesos to get to 27-and-90.

Scalar responses similar to those in Figure 4a and 4b were given by 12 of the 17 PGCE students on Version A of Item 3 and by 12 of the 18 PGCE students on Version B. This compares to just 3 students and 1 student giving functional responses on Versions A and B respectively. In the remaining 7 responses, it was difficult to determine whether they were functional, scalar or entirely formal. Several of these involved an intermediate step involving 1 Dollar, which might thus be classed as involving some form of the unitary method, as in the example in Figure 5a for Version A of Item 3. Here it looks as though the student has adopted a scalar approach (÷6, ×27); however it is just possible that the student was seeing 27/6 functionally, as the rate that maps 6 Dollars onto 27 Pesos, and that subsequently maps 20 Dollars onto 90 Pesos. In Figure 5b we are simply presented with a formal expression. It is possible that this too stems from a scalar approach (with 27 being multiplied by the scalar multiplier 20/6), but we can't be sure.



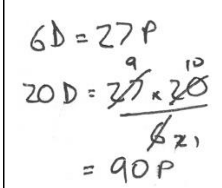




Figure 5a: Item 3 (Version A), Unitary method, possibly scalar Fig 5b: Item 3 (A) Formal expression

A few years ago, during the development of the ICCAMS lesson materials, we also gave the Mini Ratio Test to just under 150 pupils in Key Stage 3 (mostly Year 8 pupils). Each pupil received one or other version of the test, distributed randomly within their mathematics class. The pupils used similar methods to the PGCE students, but only about 25% answered Item 3 correctly. Interestingly, the pupils who used a fractional multiplier almost always tried to evaluate it, in contrast to the PGCE students, most of whom would leave it as an expression (eg 20/6) to be operated on later. And rather than writing the multiplier as a mixed number (eg 3⅓), the pupils would attempt to write it as a decimal (eg 3.33) or as a whole number and a remainder (eg 3 r 2), which often led to difficulties when applying it to, say, 27.

Item 1 involves a much simpler direct multiplier (×3) than the direct multipliers in Item 3 (20/6 or 3⅓, and 27/6 or 4½). It is perhaps not surprising, therefore, that 13 of 17 PGCE students who tried Version A of Item 1 gave a scalar response involving ×3. (Of the remaining 4 students, 2 gave a functional response, 1 used the unitary method and 1 gave only a formal expression). On Version B of Item 1, the ×3 multiplier now involves a functional relation since it is a rate (×3 ml per person). Interesting, only 2 students gave a clear-cut functional approach (such as 11×3 = 33, 25×3 = 75) with another 3 students using a unitary method that might well have been functional. Slightly more students (4) gave a clear cut scalar approach, which involves the more complex multiplier 25/11, as in Figure 6a, with another 7 students using a perhaps-scalar version of the unitary method, as in Figure 6b.

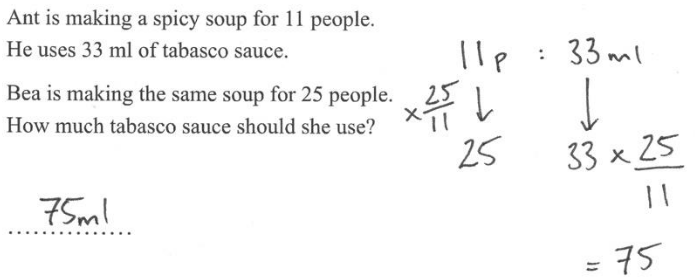



Figure 6a: Item 1 (Version B), Scalar ×25/11

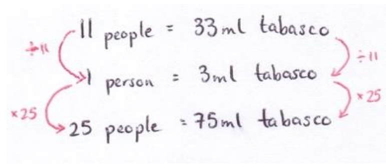



Figure 6b: Item 1 (Version B), Unitary method, possibly scalar

Not surprisingly, the school pupils found Item 1 to be markedly easier than Item 3. However, there was a substantial difference in their performance on the two versions of the item. The success rate was 91% (70 out of 77 pupils) for Version A, but only 51% (38 out of 74) for Version B. This strongly supports Vergnaud's claim that pupils favour scalar over functional relations.

Item 2 involves a geometric enlargement of an L-shape. Figure 7 (below, next page) shows the two versions, along with a scalar response to each. The item involves the simple multiplier ×4, which serves as a functional multiplier in Version A and a scalar multiplier in Version B. This distinction again had a marked effect on the methods used by both the PGCE students and the school pupils. On Version A, only 9 PGCE students used the functional multiplier ×4, with the remaining 8 students opting for the more complex scalar multiplier ×21/8. In contrast to this, 16 of the PGCE students used the now scalar multiplier ×4 on Version B. [Of the remaining 2 students, one used some version of the unitary method while the other wrote only a formal expression.]

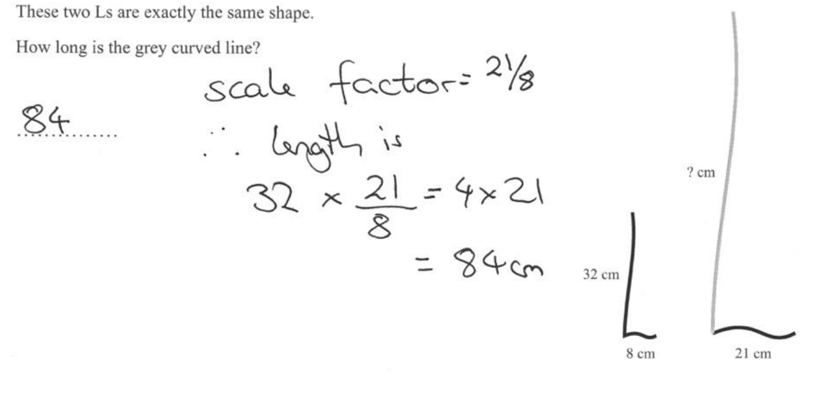



Figure 7a: Item 2 (Version A), Scalar ×21/8

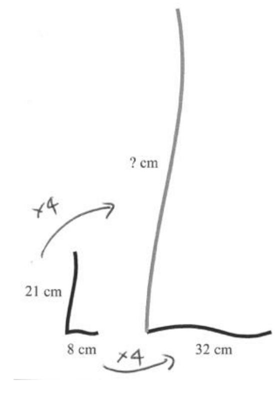



Figure 7b: Item 2 (B), Scalar ×4

The school pupils' performance on Item 2 again showed a marked difference in performance on the two versions, with success rates of 36% and 75% for versions A and B respectively. Further, a substantial minority of pupils (22%) opted for a simple additive approach on Version A (resulting in a value of 45 cm rather than 84 cm for the length of the grey line). Again, these frequencies illustrate the general preference for using a scalar rather than a functional relation.

## Context

Items 1 and 2 can both be solved by using a simple multiplier (×3 or ×4 respectively). However, as we have just seen (and is shown in collated form in Table 1, below, on the next page), the school pupils achieved higher success rates on the two versions of Item 1 than on the corresponding versions of Item 2. This fits with other studies (eg Hart, 1981) that have found that a recipe context is generally easier than geometric enlargement. This might, in part, be due to a curriculum effect - we probably devote more time to the everyday context of recipes than to the more abstract context of enlargement. However, it might also be due to some of the characteristics of the two contexts.

Consider, for example, the method of *rated addition*, which allows students successfully to solve some ratio items while still treating multiplication as repeated addition rather than scaling. This method fits a recipe context more comfortably than it does enlargement. We can solve Version A of Item 1 in this additive way: 11 people require 25 ml; another 11 people will require another 25 ml, and another 11 people will again require another 25 ml, ie 11+11+11 people require 25+25+25 ml. However, this doesn't really work for the L-shapes: the attempt (see Figure 8, below, on the next page) to 'add together' 4 of the small Ls to make the large L in Version B of Item 2, is not entirely convincing.

On the other hand, it is easier to see that a simple additive approach does *not* fit in the recipe context than with enlargement: if 11 people require 25 ml, it might well seem strange to add just 22 ml for an extra 22 people. However, if both segments of the small L-shape in Version B of Item 2 are increased by 24 cm, we still get an L-shape that is substantially taller than it is wide and which, at first sight, might therefore still appear to be 'the same shape'.

We can't so readily compare the currency-exchange context in Item 3 with the other two contexts because of the item's greater numerical complexity. It seems likely that a version with a simple multiplier like those in the other two items, would behave more like Item 1 than Item 2. However, this needs to be tested.

| | Simple scalar mulitplier (×3 or ×4) Correct | N | Simple functional mulitplier (×3 or ×4) Correct | N |
|---|---|---|---|---|
| Item 1 | **91 %** | 77 | **51 %** | 73 |
| Item 2 | **75 %** | 74 | **36 %** | 74 |

Table 1: Item 1 and 2 facilities (KS3 pupils)

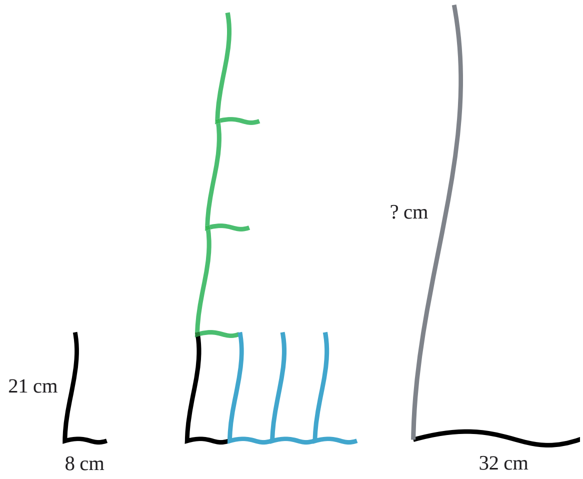



Figure 8: 'Adding' L-shapes

## Conclusion

The findings reported in this paper show that responses to ratio items can be influenced by the item's context, its numerical complexity and by whether the simpler multiplicative relations are scalar or functional. This gives further support to the view that 'ratio' is not a simple binary concept that pupils either do or do not 'get'. Nor, it would seem, does it form a simple, linear hierarchy where understanding can be achieved in a carefully planned sequence of small steps. Rather, it would seem more fruitful (better fitting) to view ratio as a network of ideas woven into a complex 'multiplicative field' (Vergnaud, 1983). This suggests that we need to find effective ways to help pupils repeatedly to explore ratio contexts and to articulate and scrutinise their ideas so that their understanding can continually be developed and re-structured.

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