A literature review on rigour in mathematics education

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In this paper, I review the existing definitions of rigour in the literature within the realm of mathematics education. From the review, I suggest that there are four categories of discussion of rigour: mathematical definition of rigour, rigour in curriculum, rigour in teaching, and learner's rigour. The four categories will be elaborated on respectively. In addition, in order to investigate how learners act out rigour in mathematical practice, I attempt to form a definition of learner's rigour, with the thought that different views of rigour may be at the heart of cultural differences in mathematics teaching and learning (e.g., between China and England).

Keywords: learner's rigour; mathematical rigour; rigorous proof; rigorous curriculum; rigorous learning; academic rigour

Introduction

There are a variety of phrases used in relation to rigour in the literature within the realm of mathematics education: 'mathematical rigour', 'rigorous mathematics', 'rigorous problem solving', 'rigorous mathematical thinking', 'rigorous proof', 'rigorous classroom', 'rigorous curriculum', 'rigorous learning'. However, there is no consistent definition, and rigour can be applied and interpreted in various ways even when used in the same phrase. What is more, rigour is not always clearly defined. The purpose of this review is to categorise existing definitions of rigour relating to mathematics education. Drawing on the relevant literature, four main categories can be drawn, which are mathematical definition of rigour, rigour in curriculum, rigour in teaching, learner's rigour. Although there can be overlaps in reference to rigour in curriculum, teaching and learning, I intend to discuss them respectively in this writing so that comparisons and contrasts can be made where appropriate. In the following sections, each category will be elaborated on respectively.

Mathematical definition of rigour

The mathematical definition of rigour generally refers to a rigorous proof or rigorous arguments, whereas what exactly accounts for a rigorous proof varies in the eyes of different beholders (see for Kitcher, 1981; Marfori, 2010). Definitions are not always provided where the phrase 'rigorous argument' or 'rigorous proof' appears in the literature, but traditionally an argument is deemed to be rigorous when the conclusion is derived from premises by gapless inferences, where logical validity is its main characteristic (Kitcher, 1981). This definition of rigour draws criticism that it sacrifices meaning and relevance in practice (Freudenthal, 1973; Hanna, 1995; Duncan & Range, 2013). Thus, apart from logical validity, rigour should also be relevant and meaningful to learners in the context of mathematics education (Freudenthal, 1973; Hanna, 1995). Specifically, Freudenthal (1973) refers to mathematical rigour as a measuring-rod of mathematics and the mathematics taught at schools, which is a criterion for judging whether a result and the process to get to that

result are correct or not. Hanna (1995) affirmed that rigour is a question of degree and it is a pragmatic choice of the level of rigour in practice, which resonates with Freudenthal's (1973) idea that it is teachers who are to judge whether the mathematics taught is rigorous or not for their pupils and to develop their pupils' rigour to a level which they believe suits their pupils. As a result, the mathematical definition of rigour is closely related to rigour in curriculum, rigour in teaching and learners' rigour, which will be elaborated on respectively in the following sections.

Rigour in curriculum

Before I proceed with further elaboration, there is a need to make it clear that the meaning of the term 'curriculum' in the literature can vary a lot and it depends on how the authors use it in the writings cited. It may refer to national or state mandatory curriculum or the day-to-day curriculum in class which can incorporate elements of teaching and learning. Moreover, it may be defined broadly involving curriculum goals, teaching strategies, learning experiences, and assessment systems. No explicit definitions or explanations of the term 'curriculum' will be provided as curriculum is not the primary focus of this paper.

One aspect of rigour in curriculum is the role and position of formal proof in curriculum, where rigour is in a sense of formal proof. Hanna (1995) asserted that the role of formal proof and rigour should not be downplayed in curriculum because it is an indispensable tool for promoting students' mathematical understanding. Hoyles and Healey (1999) suggested making formal proof an explicit focus as it contributes to better engagement in argument and deduction. And there is little doubt that the differences in the role and position of proof in curriculum in different countries can shape their students' approaches to proving differently, which may even differentiate students' capacity to evaluate the validity of arguments of their own or others. Nevertheless, there are few comparative studies on the role and position of proof in curriculum in the literature.

A second aspect of rigour in curriculum often appears in the literature as 'academic rigour' referring to high proficiency standards whereas the phrase 'academic rigour' can be defined differently (see for example Draeger, Hill, Hunter & Mahler, 2013). When academic rigour denotes high proficiency standards, the differences of existing relevant definitions of academic rigour mainly lie in the different aspects that they believe students should be up to high proficiency standards. Aspects can be content knowledge, critical thinking skill, reasoning skill, higher-order thinking skill, or capacity to manage rigorous content (Marzano & Toth, 2014; Quint, Thompson & Bald, 2008; Strong, Silver & Perini, 2001), where rigorous content is defined as content that is "complex, ambiguous, provocative, and personally or emotionally challenging" (Strong, Silver & Perini, 2001, p. 7).

A third aspect of rigour in curriculum is indicated by different conceptions of rigour in the literature from a perspective of how to assess the level of proficiency and how to achieve the high standards required. There are test-based conception of rigour, performance-based conception of rigour, conceptions of rigour integrating elements of curriculum, teaching and learning that often occurred in the literature as 'rigorous curriculum' (Bintz & Moore, 2010; Ainsworth, 2011; McTighe & Wiggins, 2012). A test-based conception of rigour suggests assessing rigour via standardised tests and imposing harsh sanctions against schools that fail to comply with the standards. However, McTighe & Wiggins (2012) argued that instead of a test-based conception of rigour, a performance-based conception is necessary for curriculum design as the

qualities of performance that must be attained to live up to standards should be illustrated and therefore worthy performance tasks of sufficient richness to assess students' level of rigour should be constructed and incorporated into curriculum. The performance-based conception overlaps with rigour in teaching as it is teachers who are to devise or choose tasks that they think are worthy for assessing the quality of their students' performance and to make judgements. To make academic rigour an attainable curriculum goal, 'rigorous curriculum' incorporates aspects of teaching and learning to provide essential supports and resources (Ainsworth, 2011; Duncan & Range, 2013). Bintz and Moore (2010) defined a rigorous curriculum as an alignment of national standards, appropriate textbooks, pacing guides and benchmark assessments. Ainsworth (2011) gave a definition of a rigorous curriculum as:

> "an inclusive set of intentionally aligned components – clear learning outcomes with matching assessments, engaging learning experiences, and instructional strategies – organised into sequenced units of study that serve as both the detailed road map and the high-quality delivery system for ensuring that all students achieve the desired end: the attainment of their designated grade- or course-specific standards within a particular content area" (p. 8)

In this view, rigour has a broader view as "reaching for a higher level of quality in both effort and outcome" (p. 8). Although this definition combines aspects of curriculum, assessment, and teaching and learning in order to ensure the achievement of rigour, an underlying assumption appears to be that such a rigorous curriculum can be un-problematically devised and implemented for all teachers and students whereas this is highly doubtful in practice due to the complexity of practice when taking into account factors of teachers and students. Also, what seems problematic in this definition is that 'engaging learning experiences' should be an ideal outcome of an implemented curriculum rather than a component in a curriculum.

Rigour in teaching

As mentioned above, some aspects of rigour in teaching go hand in hand with rigour in curriculum. Teachers assess students' level of rigour or quality of performance by carefully chosen or constructed performance tasks (McTighe & Wiggins, 2012). Also, Ainsworth (2011) claimed that, in order to design a rigorous curriculum, teachers should offer students "precise learning targets, meaningful and relevant lessons and activities, and multiple opportunities to succeed" (p. 8). Some researchers define rigour with reference to factors around teachers that they believe are significant in achieving the curriculum goal of high academic standards. Blackburn (2018) defined rigour as teachers "creating an environment in which each student is expected to learn at high levels, each student is supported so he or she can learn at high levels, and each student demonstrates learning at high levels". Raising teacher expectations and complexity of content along with providing appropriate teacher support are recommended to be ways of increasing rigour (Blackburn, 2013).

Apart from the close relation with curriculum goals, rigour in teaching is also seen as promoting high-level learning, or rigorous learning. In this respect, Bintz and Moore (2010) delineated qualities of rigour in teaching, which include creating engaging learning experiences, developing open-ended lessons to provoke students' curiosity, creating a comfortable classroom environment where students are willing to take challenges, designing learning experiences that are relevant to students, pushing students to go beyond knowing the what but also knowing the how and the why, and

offering rich mathematical tasks to promote students to be thoughtful problem posers and solvers.

My sense is that the different views on rigour in teaching result from different understandings of what rigour refers to and of the qualities of rigour. For instance, when rigour is referring to the qualities of mathematics delivered and received in a class, it often denotes its precision and accuracy, adequacy, appropriateness.

Resnick, O'Connor and Michaels (2007) regarded mathematical rigour in teaching as one dimension of effective teaching, which refers to explicating conceptual underpinnings of mathematics, discussing procedures clearly and accurately, identifying the adequacy in students' explanations and justifications, clarifying connections among facts, conceptions and procedures. They argued that rigour in teaching depends on the quality of teachers, especially teachers' subject matter knowledge. Furthermore, Michaels, O'Connor and Resnick (2008) rephrased the conception of rigour as providing authoritative knowledge when necessary in classroom talk and steering conversation towards academically correct concepts by a knowledgeable teacher. Michaels et al. (2008) illustrated that maintaining mathematical rigour in class can be a quite complex process as teachers have to decide when and where to replace less rigour with more rigour, to ensure students get correct knowledge as well as making sense of that knowledge. Rigour stated above in teaching appears to be aiming for students to attain deep, accurate and meaningful mathematical knowledge. However, it is unclear whether there are generally accepted criteria for the level of rigour within a country and across nations, for instance, criteria for accuracy and adequacy in learners' responses, expected in class and in learners. According to some cross-national research, it seems that criteria can be quite different across nations (Ma, 1999; Miao, 2015). For instance, a mathematical expression 1/3 = 4 with a diagram showing the whole is 12, instead of 1/3 * 12 = 4, was adopted by the teacher for convenience, which is deemed to be an unacceptable expression in a Chinese classroom (Miao, 2015). How learners' mathematical understanding and performance can be affected by possible differences in the level of mathematical rigour in teaching is worth further investigation.

Learner's rigour

In this section, along with delineating different types of rigour relating to learners that are present in literature due to different understandings of rigour, I will attempt to devise a definition of learner's rigour for research.

Firstly, rigour is interpreted in a sense of mathematical rigour, which is defined by Freudenthal (1973) who asserted that there are different levels of rigour which learners need to pass through as their mathematical rigour advances over time. Teachers play a significant part in developing students' rigour as it is the teachers who are to judge whether the mathematics taught and the way students work on problems is rigorous or not (Freudenthal, 1973). Correspondingly, learner's rigour acquired/internalised can be investigated by the actions a learner takes at different levels to validate their mathematical working, and by the criteria they adopt for judging whether certain mathematical responses in a specific context are correct or not (the criteria conforming to teachers' instruction, or making sense to the learner, or both).

Secondly, Kinard and Kozulin (2008) defined rigour as a quality of "thought that reveals itself when students' critical engagement with material is driven by a strong, persistent, and inflexible desire to know and deeply understand" (p. 4). It

consists of different elements (Kinard, 2006; Kinard and Kozulin, 2008), of which the fundamental elements were delineated as:

- sharpness in focus and perception
- clarity and completeness in definition and delineation of critical attributes
- precision and accuracy (Kinard, 2006, p. 256)

Furthermore, a definition of rigorous thinking was provided, which incorporates all the fundamental elements of rigour as well as "the depth of comprehension and understanding" (p. 4). Illuminated by this definition, actions learners take to ensure precision and accuracy, completeness, and clarity for good communication of reasoning and ideas can be incorporated into the definition of learners' rigour.

Thirdly, rigour is used in the phrase of 'rigorous learning'. Although there are different definitions on rigorous learning, elements such as meaningfulness and active engagement are commonly considered to be necessary for learning to be rigorous. Schnee (2008) considered it to be "deep, critical and inquiry-based learning" (p. 64) while Bintz and Moore (2010) regarded rigorous learning as "the extent to which learners effectively and efficiently act on meaningful problems" (p. 295) with the belief that all learning deals with problem solving. Bintz and Moore (2010) summarised that qualities of rigour include active engagement, curiosity and inquiry, confidence, meaningfulness, critical thinking and being thoughtful problem posers and problem solvers. Draeger et al. (p. 278) argued that "learning is most rigorous when students are actively learning meaningful content with higher-order thinking at the appropriate level of expectation in a given context", while, in addition, Strong, Silver and Perini (2001) defined rigorous learning as learners' work on rigorous content by employing such strategies or skills such as structuring, comparing and contrasting, metacognition, etc.. Despite the fact that these definitions vary a lot, they all indicate expected ways of learning mathematics by different researchers.

Finally, I suggest a definition of learner's rigour as actions one takes at different levels to validate one's mathematical working (e.g., actions on ensuring accuracy) and the criteria one adopts for judging whether certain mathematical responses in a specific context are correct or not (e.g., criteria conforming to teachers' instruction, or making sense to the learner, or both). It indicates a quality of doing mathematics and reflects rigour in teaching.

Conclusion

Rigour, on the one hand, has been defined in various ways across the relevant literature. On the other hand, conceptions of rigour seem to be existing in every aspect of mathematics education. However, rigour has never been investigated from the perspective of learners' mathematical performance. In other words, how students act out rigour in practice. In order to do this, forming a definition of learner's rigour is indispensable. An initial definition of learner's rigour has been devised in this paper, what should be done next is to operationalise the definition in practice to inform an instrument assessing learner's rigour. One interest I have in this work is a thought that perhaps different views of rigour are at the heart of some cross-cultural differences in practices of mathematics teaching and learning (e.g., in China and England).

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