A logical perspective on Cuisenaire and bar modelling

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A novel approach is proposed to the analysis of the expressive powers of Concrete, Pictorial and Abstract representations of arithmetic. Drawing on the theory of formal mathematical logic, it is shown that precisely defined syntaxes and semantics can be introduced for Cuisenaire rods and part-whole bar models. By interpreting sentences expressed in these representations as sentences in the first-order theory of arithmetic, it is possible to rigorously study the potential and limits of these representations. It is shown that different approaches to bar modelling vary depending on the semantic content given to geometric bar length, and the implications of this observation are studied to reveal the relative power of these representations for expression of word problems and their subsequent solution within the representation.

Cuisenaire rods, bar models, part-whole model, formal logic, abstract algebra

Introduction

This paper uses the tools of mathematics, specifically the ideas of formal logic and abstract algebra, to rigorously study the class of mathematical sentences expressible, provable and disprovable with the use of Cuisenaire rods and part-whole bar models.

Bar models are often presented as a way to express word problems to aid mathematical reasoning (Ng and Lee, 2009). This paper considers this concept of *expression* of problems formally, in the sense that we explore how bar models can be translated into sentences in the formal mathematical language of first-order arithmetic. The Concrete, Pictorial, Abstract (CPA) method (Mudaly & Naidoo, 2005) is reviewed, as are the basic ideas used in the meta-mathematical study of arithmetic (Hájek & Pudlák, 1993). This theoretical approach leads to insight into exactly which logical sentences are expressible within the bar model, deepening our understanding of both the power and the limits of problem solving using bar models. Several distinct syntactical part-whole bar model representations have been observed in practice, and this paper clarifies the differences between these syntaxes, their applicability, and their power in problem solving.

The bar model is contrasted with the use of Cuisenaire rods. While superficially similar, it becomes apparent through the formal logical framework presented that there are several key differences. In particular, the Cuisenaire syntax presented is sufficiently restrictive so that only true sentences of first-order arithmetic are expressible, unlike some variants of the bar model. With reference to this result, conclusions are drawn on the appropriate level of child development suitable for working with these two CPA approaches, the relative need for direct instruction when working with these two approaches, and the potential for additional CPA methods.

Background

The results of this work draw on two main bodies of prior work: in meta-mathematics and in pedagogy and child development. This section briefly reviews the context and necessary prior knowledge in these two fields.

The CPA method

Both Cuisenaire and bar modelling fit within the Concrete to Pictorial to Abstract (CPA) approach to learning, also referred to as the Concrete to Representational to Abstract (CRA) method (Witzel, Riccomini, & Schneider, 2008; Mudaly & Naidoo, 2015), an adapted version of Bruner's work on the 'enactive, iconic and symbolic' stages of learning (Bruner, 1966):

- *Enactive stage* - action is led by the learner, involving them in handling and manipulating concrete materials in order to gain understanding (Orton, 2004).
- *Iconic stage* - images, pictures and drawings allow the learner to consider, organise and represent their learning, as a route to abstraction.
- *Symbolic stage* - language and symbols are recognised as representing concepts and show the learner's ability to understand abstract concepts.

As pupils begin to develop their understanding of number, calculations can be modelled pictorially in different ways. One of these is the part-whole model which can show the inverse relationship between subtraction and addition. When moving towards abstraction, these same relationships can be modelled using symbols. Koleza (2016) recognises that diagrams are an efficient strategy in teaching and learning mathematics as they clearly show the relationship between quantities in a story and limit abstraction therefore aiding the problem-solving process.

Formal Arithmetic

The mathematical study of arithmetic is usually conducted within the formal setting of first-order logic - see (Boolos, Burgess, & Jeffrey 1974) for a more detailed exposition.

It is important to recognise that recorded mathematics deals in symbols and their manipulation. These may be abstract symbols like '=' or '52', or concrete symbols like a red Cuisenaire rod. Symbols can be put together, following certain rules, to make 'well-formed formulae' (commonly abbreviated as wff). As an example, 1 + 2 = 4 is a wff, because it follows correct rules for constructing an arithmetic sentence (even if it expresses something untrue), whereas + + = 1 is not a wff – we cannot even begin to understand the truth of what it might express because *syntactical rules* have not been followed.

First-order logic studies those wffs including a variety of logical symbols, such as equality (=), disjunction (∨), universal and existential quantification (∀, ∃). It is standard to enrich these basic logical symbols with additional symbols useful for expressing arithmetic sentences. For example, we may add the symbols **0**, **1**, and **+**. A wff can be assigned meaning (*semantics*) or *interpreted* by assigning meaning to each of its components. For example, the standard interpretation of 2+3 is the number five, the standard interpretation of 1 + 1 = 2 is true.

This framework of formal languages allows us to express non-trivial sentences, such as $\forall x. \exists y. x = y + y \vee x = y + y + \mathbf{1}$, which states "every number is either odd or even". Given that we may express both false and true statements in standard arithmetical syntax (1 + 1 = 2, 1 + 1 = 3), it is necessary to provide some rules that allow us to only derive true statements from other true statements; these are known as *rules of inference*. Possibly the most well-known such rule is *modus ponens*, which states that if I know that B follows from A and I also know that A is true, I may conclude the truth of B. We shall return to the rules of inference as they apply to bar models later in this article.

Cuisenaire rods

We may now apply the ideas of formal arithmetic to Cuisenaire rods. Firstly, we define a syntax allowing us to create terms and sentences with rods. We define a *term* to be either a single Cuisenaire rod, or the end-to-end composition of two existing terms. We define a sentence to be one term under another, arranged so that the left edges of the left-most rods are vertically aligned, as illustrated in Figure 1.

Figure 1: Terms (left and centre) and a sentence (right) in Cuisenaire Syntax.

We *interpret* Cuisenaire wffs as expressing sentences in the formal arithmetic syntax of the previous section, i.e. we consider the interpretation of a term (sentence) in Cuisenaire to be a term (sentence) in first-order arithmetic. Firstly, individual rods get mapped to a numeral between 1 and 10, following the Cuisenaire universal colour system (Ollerton, Williams & Gregg, 2017). Secondly, horizontal composition (abutment) of Cuisenaire rods is interpreted as the + symbol. Finally, a Cuisenaire sentence constructed from an upper term t_1 and a lower term t_2 is interpreted as denoting one of $t_1 < t_2$, $t_1 = t_2$, or $t_1 > t_2$, depending on whether the upper right-hand edge is to the left, in line with, or to the right of the bottom lower right-hand edge.

It is worth noting two interesting properties of this interpretation. Firstly, composition of terms naturally has no notion of which rods are composed first: taking the composite term in Figure 1 (centre), for example, it is impossible to tell whether the red rod was added after the green and purple rods were composed, or whether the purple rod was the last to be added. This corresponds to the associativity of addition being "baked in" to Cuisenaire syntax itself, *i.e.* $(a + b) + c = a + (b + c)$, which mathematically follows from Cuisenaire terms forming a free semigroup. Secondly, it trivially follows from the proportionality of Cuisenaire rods that the standard interpretation of any first-order arithmetic sentence generated from interpreting Cuisenaire sentences in this way is *always true*. This is in stark contrast to standard arithmetic syntax: we may *write* 1 + 2 = 4, even if it is false.

This mathematical property can be exploited: Gattegno (1965) believed that pupils can learn many things from *unstructured* exploration of the rod, such as: rods of the same length are the same colour, any rod length can be made from white rods, and that many lengths can be made by combining other rod lengths. His approach was to demonstrate arithmetic truth visually, for example when finding numbers adding to a given total, pupils can self-correct, as it can be seen immediately whether combined rods sum to more or less than the desired total. This was a deliberate feature of the rods as highlighted by Gattegno (1965) when listing Cuisenaire's philosophy of active teaching; he describes the child (who uses Cuisenaire rods) as being able to check his own results and rely on his own criteria for correcting his mistakes. When pupils handle concrete materials, they gain understanding and make links with past learning, supporting Bruner's 'enactive' stage of learning.

Part-whole bar models

There are various sentences that Cuisenaire rods do not let us express. Amongst these, sentences involving variables are of significant interest from a pedagogical perspective, as they are typically used for modelling word problems. To address this gap, bar models have become a popular pictorial representation. Our discussions with primary school teachers suggest that there are a variety of diagrammatic approaches advocated under a general heading of 'the part-whole bar model'. This section addresses the differences.

Various approaches to drawing part-whole models

Perhaps the most straight-forward form of pictorial bar model we have seen used in practice is to draw bars with lengths *proportional* to the number to be represented, typically using squared paper. Formally, we may define terms and sentences for this model in a very similar way to those for Cuisenaire rods; the only change is the definition of a term: a proportional bar model *term* is either (i) a single axis-aligned rectangle of unit height containing a decimal numeral, with length proportional to that integer or (ii) the horizonal composition of two existing terms.

Note that the bar model allows bars representing unbounded numerals, in contrast to Cuisenaire numerals, which are limited as white (1) to orange (10). As a result, we may express sentences such as 12 + 8 = 20 in bar models, which are unrepresentable using Cuisenaire rods (which could, however, express 10 + 2 + 8 = 10 + 10, for example). The price we pay for this greater expressivity is that children must be confident with the decimal number system (place value) in order to use bar models, putting them largely outside the scope of the EYFS curriculum in England. The proportional bar model inherits the other key pedagogical properties of Cuisenaire discussed above (associativity of addition in syntax, coincidence of expressibility and truth.) However, proportional models cannot represent *unknown values* or *variables*; in order to draw the model, we must *know* what the values are – this makes it impossible to draw proportional bar models expressing word problems, yet it is certainly possible to *check* the solution of a word problem using a proportional bar model. In practice, we observe that teachers tend to relax proportionality when faced with word problems.

Taking the relaxation of proportionality to its logical conclusion, we may study a variant of the bar model in which length of individual rectangles is arbitrary. We refer to this as a *topological bar model*. A topological bar model *term* is either an axis-aligned rectangle of unit height containing a decimal numeral, or an axis-aligned rectangle of unit height containing a variable name, or the horizontal composition of existing terms.

This definition allows unknowns to be represented. For example, in Figure 2, we present two models, for "$20 = x + 2$" and for "$x = y + y$"; note that if unknowns are taken to range over positive integers, then the latter expresses that x is an even number without needing to know *which* even number – this is inexpressible in the proportional bar model or in our Cuisenaire syntax.

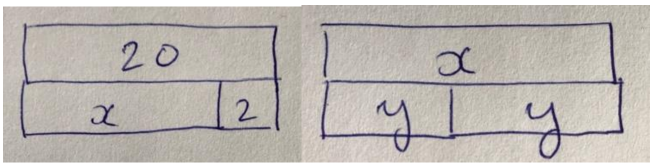

Figure 2: Topological bar models allow unknowns.

We observe that – in practice - teachers often intuitively aim for a solution between the two bar models discussed above: they try to ensure that longer bars should represent larger values. However, there is a distinct problem with this approach: unlike *either* the proportional bar model or the topological bar model, such "order-preserving" bar models are not compositional. For example, we may construct a bar model representing $x + y = 100$, incorporating the knowledge that $x > y$. Separately, we may construct a bar model representing $u + v = 50$, incorporating the knowledge that $u > v$. It would seem reasonable to be able to combine these two models to conclude that $x + y + u + v = 150$, yet from the bar models alone we do not have information on the relative magnitudes of x and y compared to u and v, so simply composing the bars together may not result in a valid conclusion when expressed in the bar model, even if the premises were true.

Proofs in the topological bar model

The topological bar model, by allowing variables, is a powerful representation. However, this results in the possibility of constructing false sentences; as a trivial example, one can easily draw a bar labelled "1" aligned with a bar labelled "2", denoting "1 = 2".

The ability to express false statements requires a precise understanding of which manipulations of bar models are and are not allowable to generate new true statements from old true statements – *rules of inference* – as well as an understanding of which truths we might take for granted *(axioms / axiom schemata)*. Thus, unlike in the Cuisenaire case, unstructured exploration does not help in the discovery of number facts. An example axiom schema and rule of inference is shown in Figure 3, along with a proof that $x + 1 = 5$ leads to the conclusion $x = 4$, through first applying the axiom schema and then the rule of inference.

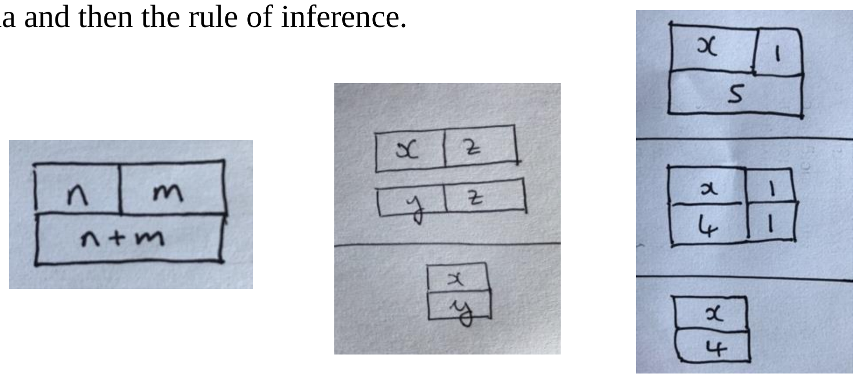

Figure 3: An axiom schema (left), rule of inference (centre), and proof (right) in the topological bar model.

Bar models in development

Case (1985) developed a framework to assess children's developing spatial awareness. Around the age of six children begin to demonstrate an ability to show spatial awareness in their drawings by including a line in their drawings which provides a reference for objects, referred to as the *uniaxial* stage, representing the beginning of children spatial awareness. Neale (2017) also suggests that pupils have a developing awareness of geometric insight in relation to bar length and number value around the ages of six and seven, based on their ability to construct bar models of addition and subtraction calculations and their ability to problem solve. This is in line with Bruner's 'iconic' stage of learning where pupil's use images, pictures and drawings to consider, organise and represent their learning. Therefore bar modelling supports the 'iconic' stage of learning allowing pupils to make links in their learning, but requires both spatial / topological awareness and an existing understanding of place value.

Conclusion

It has been demonstrated that Cuisenaire rods and proportional bar models have a restrictive syntax, resulting in the truth of every interpretation as a first-order logic sentence. In contrast, the topological bar model is far more flexible, but requires careful consideration over which rules of inference are correct – geometric reasoning on a topological structure by children can easily lead to incorrect conclusions. All the models discussed in this paper have associativity "baked in" to the syntax. While Cuisenaire allows the construction of "truth through play", topological bar models, suitable for algebraic word problems, require careful manipulation to avoid drawing incorrect conclusions, as well as requiring a certain level of maturity with place value, in order to express the numerals required.

Neither model naturally supports multiplication, though multiplication by a fixed numeral (e.g. $3 \times x$) rather than general multiplication (e.g. $x \times x$) is possible – formally, this corresponds to Presburger Arithmetic (Boolos, Burgess, & Jeffrey, 1974). Large parts of first-order logic are not representable in any of the models presented, e.g. disjunction / negation. We believe the theoretical framework presented can be adapted to analyse a variety of manipulatives and pictorial representations in mathematics education.

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