Middle school students' errors in two-dimensional representations of three-dimensional shapes

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As a part of a design-based research study, the present study focused on seventh grade students' understanding of problems based on two types of polycubical shapes. The first of these asks students to draw two-dimensional representations (i.e., the orthogonal views from the front, left, right and above) of the given three-dimensional representations. In the second type, students were asked to construct polycubical shapes and then represent them isometrically in a two-dimensional environment where orthogonal views corresponding to type 1 −from the front, left, right and above− were provided. The current study found various types of common errors specific for both types of problems. Examples of these errors and their correct answers are illustrated in this paper. The next iteration of this study will be focusing on designing lessons to overcome such errors.

Assessment; middle school students' errors; 2D representations of 3D shapes; isometric drawing; orthogonal drawing; 12- to 13-year-old students

Introduction

Mathematics is one of the subjects which many middle school students have difficulty in learning. One way to understand student performance in mathematics is to use different international tests. The PISA Education Test is one of these which has been assessing middle school students' knowledge of mathematics, science and reading every three years since 2000. The results of PISA 2012 showed that 32 per cent of all students performed at level 1 out of 6 in mathematics (OECD, 2014). Although the general picture does not seem promising in mathematics, one way of supporting middle school students could be detailed research about particular topics in particular strands.

Geometry is one of the main strands of mathematics, others being algebra, numeracy, probability and statistics. Middle school geometry consists of the study of two-dimensional and three-dimensional shapes, their transformations such as rotations and iterations, and mathematical calculations related to the measurements of lengths, areas and volumes of shapes according to Euclidian geometry.

Three-dimensional shapes is one of those topics which students start to learn in their early education and continue to do so throughout their college education. Three-dimensional shapes include prisms, pyramids, cylinders and cones as well as random shapes constructed from combinations of those shapes. Students are required to study them by representing them in two-dimensional forms on paper. Two-dimensional representations of three-dimensional shapes have been problematic in mathematics learning for decades (Altun, 2013; Knuth, 2000; Kouba, Brown, Carpenter, Lindquist, Silver & Stafford, 1988; Usiskin, 1987). Thus, it would not be wrong to conclude that learning two-dimensional representations is straightforward neither for the teachers nor for the students.

Literature

Investigating students' knowledge of two-dimensional representations of three-dimensional shapes is an important topic as it is a difficult topic to understand and learn even for pre-service teachers and teachers (Gökkurt, Şahin, Erdem, Başıbüyük & Soylu, 2015). Many researchers conducted studies to investigate understanding of three-dimensional (3D) shapes. Ural (2011), for example, tried to determine pre-service primary teachers' criteria for dimension to investigate their knowledge of three-dimensional objects. The participants were given familiar three-dimensional objects and asked to specify the dimensions of the 3D objects provided to them and to generate a standard form for them. The findings indicate that none of the pre-service teachers could manage to establish consistent criteria for 3D shapes. Moreover, Bozkurt and Koç (2012) conducted a study to reveal pre-service teachers' knowledge of three-dimensional objects, specifically prisms. They found that pre-service teachers struggled to understand the concept. They concluded that pre-service teachers have insufficient knowledge to utilise mathematical language, and therefore, cannot provide well-rounded descriptions of three-dimensional objects.

While pre-service and in-service teachers' understanding of three-dimensional shapes is not satisfactory, students are expected to be successful in learning this topic. There are some studies in the literature which looked into students' understanding of two-dimensional (2D) representations of three-dimensional shapes, and some are specifically about polycubical shapes, which are three-dimensional shapes constructed from unit cubes. To illustrate, Pittalis and Christou (2013) aimed to investigate students' ability to interpret 2D representations of 3D shapes. They administered a test to 279 11- to 15- year-old students and interviewed 40 of them. Their results suggested two categories of student ability in interpreting 2D representations of 3D shapes: coding and decoding. Coding is defined as the 2D constructions or drawings of 3D shapes, while decoding refers to interpretation of 3D shapes based on their 2D representations to determine structural elements and geometric properties and then drawing different parts of them based on the interpretations. In the same year, Widder and Gorsky (2013) focused on secondary school students' learning process when they were asked to use a 3D computerised software tool (Cabri 3D) in order to visualise three-dimensional geometric objects (a cube, triangular prism, and square based pyramid). They found that students with high spatial ability used the tool less than those with low spatial ability. Moreover, students with high and low spatial ability had different purposes for using the tool. While students with limited spatial abilities used the program to discover the relationships, to see the structures and to calculate the measurements, students with well-developed spatial abilities used the tool only for the reflection of the structures, such as rotations, to see perspective drawings. A final and the most current example is Fujita, Kondo, Kumamura and Kunimune (2017). They developed an assessment framework for 2D representations of 3D shapes using the results of existing studies and their current study. To be more specific, the participants of the existing studies were 570 grade 7 to 9 students whereas the participants of their current study were 455 students from the same grades. They suggested this framework as a new form of understanding students' thinking processes when they solve problems on cubes in two dimensions. All these studies, the main findings of which are explained above, reported the need for further in-depth studies on students' understanding of 2D representations of 3D shapes. The present study explored middle school students' errors in two types of two-dimensional representations of polycubical shapes: orthogonal drawings and isometric drawings.

The Study

This paper presents the common errors of 199 7th grade middle school students in two types of two-dimensional drawings. Four classes of seventh grade students were observed for four 40-minute lessons, while they were learning orthogonal and isometric drawings of polycubical shapes. Observational study findings were presented elsewhere (Saralar & Ainsworth, 2018), so this paper will solely focus on worksheet results with the aim of describing some common errors.

All students were asked to complete a worksheet at the end of their lessons. To complete the worksheet, the students were given two lesson hours (80 minutes) with a ten-minute break in between. Worksheets had 10 questions in total, each type of 2D representation having 5 questions. The first five questions were about constructing orthogonal views and the second half of the questions were about constructing isometric drawings of the given polycubical shapes. Sixteen students were interviewed, during which they were asked questions about their reasoning in and strategies for replying to the questions on the worksheets.

Students' most common mistakes in orthogonal drawing were redrawing the 3D shape or a part of it, drawing the squares at the back onto another row/column diagonally, drawing the part only at the very front, swapping the left and right views and drawing the view upside down. Drawing only one of the views as 3D (mostly the front view), swapping left and right views, not using the isometric paper properly and drawing a 2D shape or combining the given views as 2D or 3D were found to be the most common errors in the isometric drawings. The following tables illustrate examples of each common mistake (see Table 1 and Table 2). While the first columns of the tables show descriptions of the errors, the second columns of the tables show the provided 2D representations, correct answers and sample student mistakes. Small drawings are the correct answers for the illustrated student mistakes for each question.

Hereby it is important to note that while some students were consistent with making the same mistake in their answers to all questions, some others were not consistent in their errors. Moreover, some students made more than one type of error in their answers, and this made it difficult to categorise them. Such answers were coded more than once to correspond to all of the categories they fit into.

Conclusion

To conclude, 7th grade middle school students have difficulty in constructing two-dimensional representations of three-dimensional shapes. There were significant differences in the number of total correct responses to the questions on orthogonal views (66%) and isometric drawings (34%). Students found it challenging to construct both orthogonal drawings and isometric drawings –even though they performed better in orthogonal drawings. In this design-based research, it was important to describe these mistakes so that it would be possible to design lessons to overcome such errors.

Helping 7th grade middle school students in constructing orthogonal and isometric drawings might seem like a minor step in helping students' overall mathematics performance. However, design-based research can bring practical and theoretical solutions to students' errors in orthogonal and isometric drawings (Bakker & van Eerde, 2014; Cobb, Jackson, Dunlap, English & Kirshner, 2014). It can be considered the first step toward overcoming a larger problem which is students' low performance in mathematics in national and international exams.

Table I. Orthogonal Drawing Errors

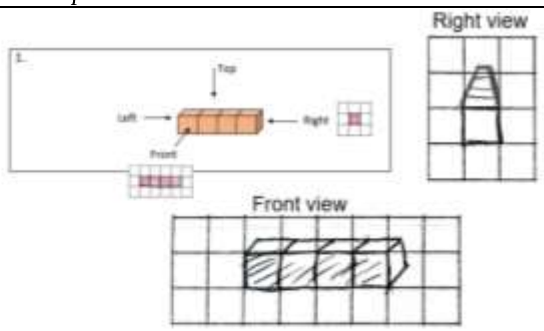
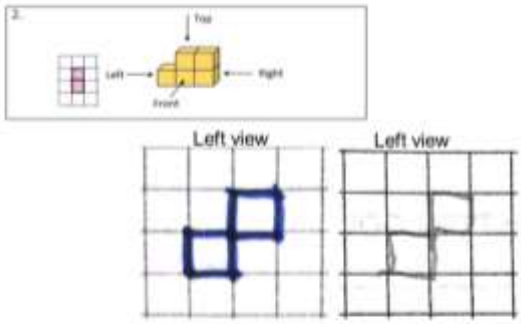
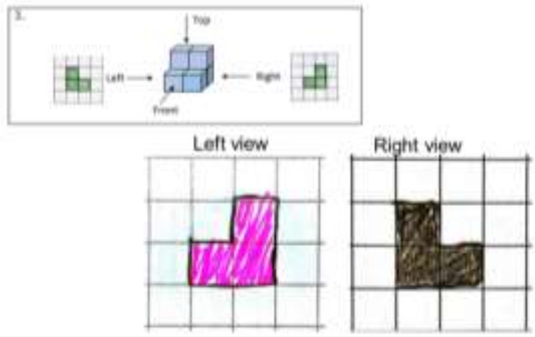
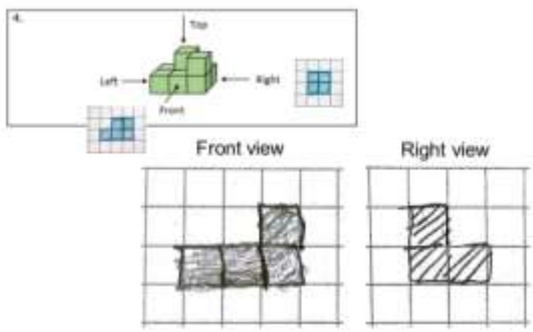
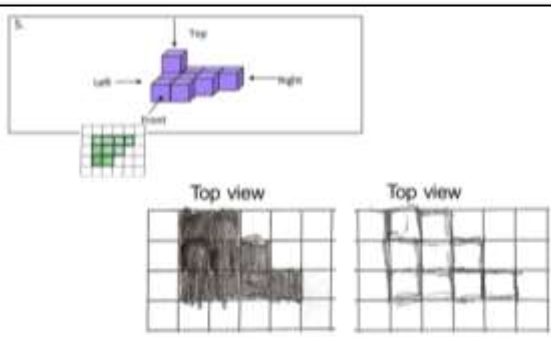
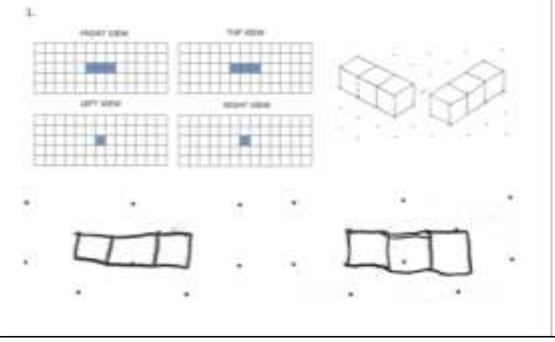
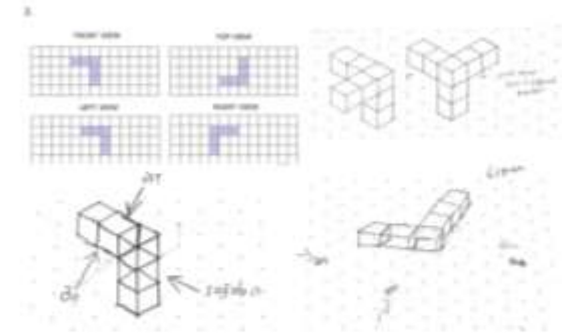
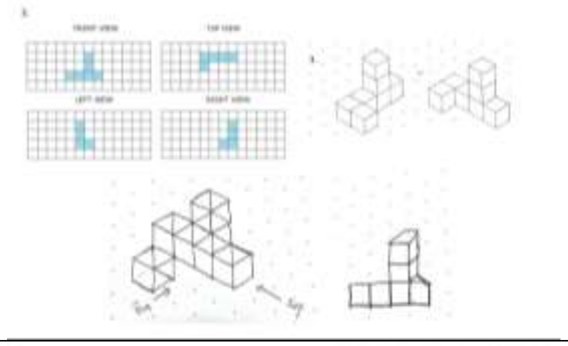
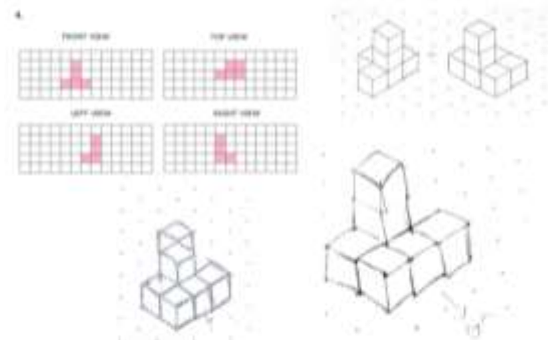
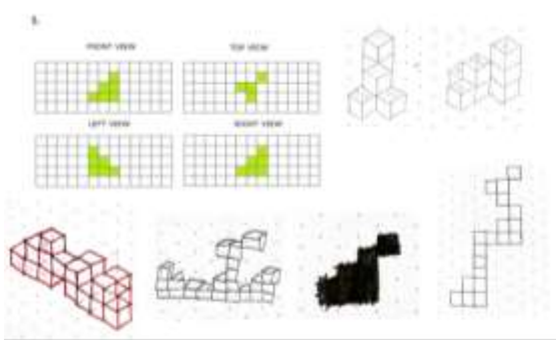
Description	*Example student mistakes and the correct answers*
Redrawing the 3D shape or a part of it	
Drawing the squares at the back onto another row diagonally	
Swapping the left and right views	
Drawing the part only at the very front	
Drawing the view upside down	

Table II. Isometric Drawing Errors

Description	*Example student mistakes and the correct answers*
Redrawing a 2D shape	
Drawing only one of the views as 3D	
Not using the isometric paper properly	
Swapping the left and right views	
Combining the orthogonal views as 2D and 3D	

Reference list

Altun, M. (2013). *Ortaokullarda matematik öğretimi* (Vol. 2). Bursa: Aktuel.

Bakker, A., & van Eerde, H. A. A. (2014). An introduction to design-based research with an example from statistics education. In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Doing qualitative research: Methodology and methods in mathematics education* (pp. 429–466). Berlin: Springer.

Bozkurt A., & Koç, Y. (2012). İlköğretim matematik öğretmenliği birinci sınıf öğrencilerinin prizma kavramına dair bilgilerinin incelenmesi. *Educational Sciences: Theory & Practice, 12*(4), 2941-2952.

Cobb, P., Jackson, K., Dunlap, C., English, L., & Kirshner, D. (2014). Design research: An analysis and critique. In *Handbook of International Research in Mathematics Education* (3rd ed., pp. 1–56). New York: Routledge.

Fujita, T., Kondo, Y., Kumamura, H., & Kunimune, S. (2017). Students' geometric thinking with cube representations: Assessment framework and empirical evidence. *Journal of Mathematical Behavior, 46,* 96-111.

Gökkurt, B., & Soylu, Y. (2016). Examination of middle school mathematics teachers' mathematical content knowledge: The sample of prism. *Elementary Education Online, 15*(3), 946-973.

Gökkurt, B., Şahin, Ö., Erdem, E., Başıbüyük, K., & Soylu, Y. (2015). Investigation of pedagogical content knowledge of middle school prospective mathematics teachers on the cone topic in terms of some components. *Journal of Cognitive and Education Research 1*(1), 18-40.

Knuth, E. J. (2000). Student understanding of the Cartesian connection: An exploratory study. *Journal for Research in Mathematics Education, 31*(4), 500–508.

Kouba, V. L., Brown, C. A., Carpenter, T. P., Lindquist, M. M., Silver, E. A., & Swafford, J. O. (1988). Results of the fourth NAEP assessment of mathematics: Measurement, geometry, data interpretation, attitudes, and other topics. *Arithmetic Teacher*, *33*(1), 10–16.

OECD (2014). *PISA 2012 results in focus: What 15-year-olds know and what they can do with what they know*. OECD Publishing, Paris.

Pittalis, M., & Christou, C. (2013). Coding and decoding representations of 3D shapes. *The Journal of Mathematical Behavior, 32*(3), 673-689.

Saralar, İ., & Ainsworth, S. (2018). Turkish middle school students' understanding of polycubical shapes. In *European Conference on Educational Research 2018: Inclusion and Exclusion, Resources for Educational Research*. Free University of Bozen, Bolzano, Italy: European Educational Research Association.

Ural, A. (2011). Prospective teachers' criteria of dimension. *Pamukkale Üniversitesi Eğitim Fakültesi Dergisi, 30*(2), 13-25.

Usiskin, Z. (1987). Resolving the continuing dilemmas in school geometry. In M. M. Lindquist & A. P. Shulte (Eds.), *Learning and Teaching Geometry* (pp. 17–31). Reston, VA: National Council of Teachers of Mathematics.

Widder, M., & Gorsky, P. (2013). How students solve problems in spatial geometry while using a software application for visualizing 3D geometric objects. *Journal of Computers in Mathematics and Science Teaching, 32*(1), 89–120.