The complexities of using multiple representations in the teaching of fractions

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Fractions are considered to be a complex concept as they are associated with part-whole, quotient, operator, ratio and magnitude interpretations, and have a range of representations. Evidence indicates that drawing attention to magnitude interpretations enriches learning opportunities, but the part-whole perspective dominates in England. One lesson on fractions from a larger video study in England is analysed here, where the teacher's focus was the role of multiple representations. Whilst the students described fractions using language associated with a range of interpretations, the part-whole perspective remained central to the teacher's management of classroom discourse; the implications for learning are considered.

Key words: multiple representations, fractions, part-whole model

Introduction

Mathematics classrooms are acknowledged as being difficult to conceptualise due to the multiple and interrelated aspects that need to be considered (Larsson & Ryve, 2012). For example, task design, sequencing of activities, management of classroom discourse, sociomathematical norms and teacher knowledge are just some aspects that can contribute to the mathematics made available to learners (e.g. Boston & Smith, 2009; Cobb, Stephan, McClain, & Gravemeijer, 2001; Rowland, Hodgen, & Solomon, 2015; Watson & Mason, 2007). Analysing classrooms is further complicated by the fact that the presence of particular features *per se* provides only limited information, as subtle difference in enactment can engender significant differences in learning opportunities, and events generate meaning through their relationship with other activities (Stigler & Hiebert, 2009).

Developing an understanding of 'typical' mathematics classrooms also encounters the difficulty that the structure, content and features of lessons can vary considerably. Moreover, the interactive and dynamic nature of lessons means that whilst teachers may plan a particular lesson trajectory, it is only as the lesson unfolds that the mathematical learning opportunities can begin to be discerned. The Orchestration of Mathematics Framework (OMF) (figure 1) has evolved out of a larger classroom study in England (Baldry, 2017), as an instrument to capture teachers' pedagogical practice and the learning opportunities associated with those moves. It is designed to allow different lenses to be brought into play as the lesson unfolds and thereby allow teachers' 'typical' lessons to be analysed, and without any preconceptions regarding content or style. In this paper, one lesson on the topic of fractions from the larger study is analysed, with the aim of examining the links between the teacher's actions and the learning opportunities for students. The teacher made an explicit choice to use multiple representations as a means of addressing perceived weaknesses in the students' understanding of fractions; the efficacy of the OMF as an analytical tool when the teacher focusses on this particular approach is also considered.

Theoretical framework

The OMF draws on a range of theoretical perspectives that are prominent within mathematics education research. To encapsulate the teachers' iterative planning, actions and reflections, Simon's (1995) hypothetical learning trajectory has been developed into the cycle of the lesson image, the teachers orchestration of mathematics (TOM) and the assessment of classroom activity. Within this cycle, notions such as professional noticing of student reasoning (Jacobs, Lamb, & Philipp, 2010) can be considered alongside other features, such as the teacher keeping an eye on the mathematical horizon (Ball, 1993), and the potential tensions there within. Classroom norms, which include social and sociomathematical norms (Cobb et al., 2001), allows for local interactions to be interpreted with less inference, whereas cognitive demand offers a way to categorise the learning potential of activities (Boston & Smith, 2009).

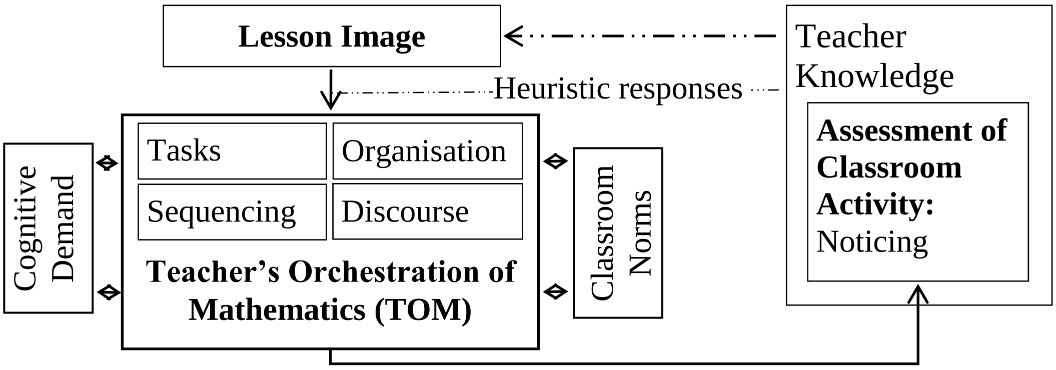

Figure 1: The Orchestration of Mathematics Framework (OMF)

The four dimensions of TOM offer a classification of the teacher's in-class activities that research indicates have pedagogical significance. Whilst these lenses have levels of interdependence, in this analysis the task and discourse dimensions came to the fore. The task dimension capture features of mathematical tasks as presented to students, such as the use of context or the availability of multiple solution strategies. Multiple representations are a constituent part of this dimension and have a key role in the understanding of complex concepts (Ainsworth, 2006), but developing this understanding is hard to achieve. Individual representations can rarely capture all the facets of complex concepts, so the flexibility to switch between representations is needed, but in order to do so the learners must already understand the critical features of each representation, and how they link to the underlying concept and to other representations (Duval, 2006). The discourse dimension captures the teacher's management of classroom discourse, such as the use of mathematical registers, patterns of interaction and the use of errors.

Fractions are a complex concept due to their multifaceted interpretations where, for instance, Pantziara and Philippou (2012) identify five subconstructs: part-whole, ratio, quotient, measure and operator. The part-whole representation retains a dominant position in England (Küchemann, 2017), whereas research indicates that learning is improved if the measure interpretation is also emphasised, (e.g. Torbeyns, Schneider, Xin, & Siegler, 2015). Moreover, a comprehensive understanding requires engagement with multiple representations, as a range of different language and visual representations can be used to highlight different facets. For example, four fifths could be represented as a circle, equipartitioned into five parts with four shaded, or as four white dots and one black; the former is more likely to draw attention to a part-whole interpretation, with the latter to ratio (Küchemann, 2017). However, the interpretation of the visual representation would depend on the features being attended to: for example, the dot representation could be interpreted as part-whole if the group is 'seen' as a unit whole. A range of formal and informal language was also heard in the classroom, including "four out of five" and "four over five"; the former being more attuned to part-whole and the latter to the written form. Rau and Matthews (2017) argue that learning is enhanced if students articulate how different visual features compare, and if the students themselves establish mappings between the visual features of the representations and between visual features and the concept.

Method

As part of a larger classroom-based study undertaken in England, eighteen secondary mathematics lessons have been observed by this author and video recorded. Three teachers have participated, with each teacher being observed with two different classes. Two static video cameras were used and audio data was transcribed. Lesson artefacts, such as resources used and students' work, was collected, along with pre- and post-lesson information from the teacher. One lesson with a class of year 9 students (13-14 year olds) on the topic of fractions is analysed here. As anticipated the task and discourse dimensions were the most pronounced features of TOM; lesson extracts are presented below to exemplify how the finer grained analysis was undertaken.

Initial coding categorised activities as either mathematically relevant or classroom management, with the former then being mapped against the OMF. The wider study did inform this analysis through the understanding that had been developed about classroom norms. This allowed individual interactions to be interpreted with lower levels of inference: for example, it became possible to interpret the manner of the teacher's re-voicing of student contributions as evaluative feedback. Identifying which interpretation of fractions may have been attended to was inferred through how language, diagrams and gestures were used. For example, "four parts out of five" was taken as a clear indication of a part-whole focus, whereas reference to decimal equivalence was more likely to be a measure interpretation.

Classroom episodes

Before the lesson, the teacher stated that the students would be familiar with part-whole representations, but probably not with others. At the start of the lesson the following prompts were displayed and the students were given four minutes to discuss with their peers and write down ideas:

What does fraction mean? How many ways can you represent fractions?

This was followed by a whole class discussion in which "one number divided by another" and "a part of a whole" were initially offered; these were taken to indicate quotient and part-whole interpretations. The teacher refocussed the discussion by asking "how could two thirds be represented?", which was followed by:

13 Ross: it's two thirds of a whole
14 T: two thirds of a whole (.) how are you going to represent that?
15 Ross: by making three then shade in two
16 T: make three of what
17 Ross: whole
18 T: what are you calling the whole?
19 Ross: circle

20 T: a circle (.) OK (.) any other ways?
21 Tom: you could really do it with any shape can't you

Student contributions continued and included "two to one" and "zero point six recurring"; these were taken to indicate ratio and measure interpretations. So by the end of this initial phase four out of five subconstructs offered by Pantziara and Philippou (2012) had been offered. Whilst the teacher has indicated acceptance of all the student contributions, either by repeating or writing on the board, the exchange with Ross and Tom was the one example where the teacher interrogated meaning.The teacher then displayed the following PowerPoint (figure 2).

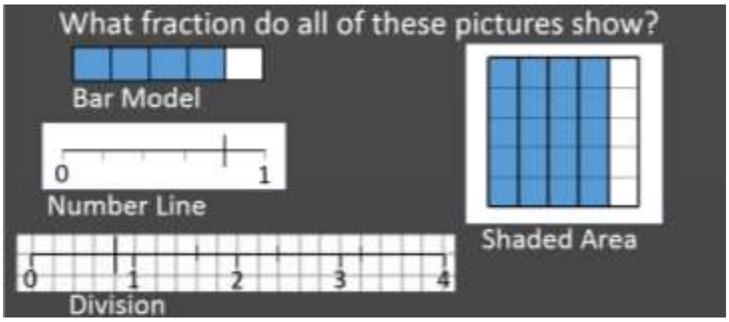

Figure 2: PowerPoint slide of four different visual representations of fractions.

Several students called out "four fifths" and about half the class raised their hand, indicating that they wanted to comment; a whole-class discussion commenced:

27 T: which one of those did you get your four fifths from?
28 Daya: the bar one (.) because it's four parts out of five
29 T: Alex which one are you going to take on?
30 Alex: shaded area
31 T: you're going to tell me why the shaded area is four fifths
32 Alex: cause it's twenty-five and (.) the number is twenty shaded
33 T: OK (.) what about those thick black lines drawn down like that?

By moving on, the teacher indicates acceptance of Daya's part-whole contribution (line 28). Whilst Alex's contribution is mathematically correct (line 32), the teacher treats this explanation as unsatisfactory (line 33), as indicated by the OK and pause. The teacher continues to ask about vertical blocks until Alex says "four out of five", when he indicated acceptance by moving on and asking about the other two diagrams; no students' hands were raised.

51 T: mmm bit tough aren't they (…) Sam
52 Sam: four over five as a decimal is zero point eight and the number line is on zero point eight
53 T: OK (.) but why does the number line when we were just talking about fractions (.) now why does that number line represent four out of five?
54 Sam: been split into five
55 T: been split into five (.) and where's that large mark been made?
56 Sam: on the fourth one
57 T: on the fourth one (.) OK what about then that bottom one? (…)

In line 52 Sam appeared to be attending to magnitude through her reference to decimals and the number line. In line 53 the teacher treats this contribution as unsatisfactory and through questioning redirects to part-whole language (line 54). When students did not volunteer for the last diagram (line 57) the teacher moved away from multiple representations to multiplication, which is analysed in the larger study.

Discussion and conclusion

In the initial phase of the lesson, the students' comments included language associated with four of the five subconstructs, those of part-whole, quotient, ratio and magnitude. However, the early examples chosen by students were part-whole (lines 15, 28 & 30), and fewer students volunteered to answer when diagrams were not closely attuned to part-whole (line 51); this offers some evidence that students accessed the part-whole interpretation more readily. Whilst the teacher accepted all the student contributions, the interaction with Ross and Tom was the one occasion where he highlighted links between features of the visual representation with particular facets of the fraction concept (lines 13 to 21). In line 18, he asked Ross what was representing the unit whole, and in line 20 his question prompted Tom's response that the shape chosen was not a critical feature; this more detailed interrogation appeared to focus on a part-whole interpretation.

After the PowerPoint was presented (figure 2) the teacher showed a preference for part-whole explanations. In particular, when a part-whole description was offered (line 28) this was immediately accepted, and subsequently he steered Alex towards "four out of five" (line 33). Moreover, when the number line was discussed (line 52), the teacher's response indicated that decimal language was not appropriate and again steered the student towards part-whole language. It appeared, therefore, that the teacher was directing travel towards his mathematical horizon, that of part-whole being the anchor for explanations, rather than exploring the students' reasoning. This could be seen as the dilemma that teachers face when using multiple representations; can reference to more familiar representations help students to understand alternative interpretations or does this approach serve to reinforce existing and possibly more restricted compartmentalised understanding?

In the initial phase of the lesson, the teacher highlighted how the diagram might show a general facet of fractions (lines 18 & 20), that of the role of a unit whole, whilst also providing an opportunity to discern critical from arbitrary features, in this case that the shape of the unit whole can vary. Rau and Matthews (2017) argue that articulating these types of structural links has the potential to enhance learning. In contrast, in the second phase, the language was related to the specific example of four fifths, and discussions about links to the concept of fractions were less in evidence. When these features and the teacher's focus on part-whole was mapped to the OMF a mixed picture of cognitive demand emerged; across this range of activities it appeared that there was substantial variation in the potential of these activities to generate higher levels of mathematical thinking.

The OMF provided a framework that allowed the teacher's lesson plan and prior knowledge about the classroom norms to inform the analysis of more local interactions. Moreover, as the lesson progressed, the dimensions of TOM allowed different lenses to come to the fore, in particular the task dimension in relation to multiple representations and the teacher's management of classroom discourse. However, further research will be needed in order to establish whether the OMF provides an analytical tool that links teachers' pedagogical moves to the learning opportunities made available to learners more effectively than consideration of separate elements of a teacher's classroom activities.

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