Commognitive analysis of a teacher's mathematical discourse on the derivative

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Research on the teaching of the derivative (and limit) is still not as extensive as the research on students' learning of calculus. This paper introduces the commognitive theory and reports on a commognitive analysis of a teacher's discourse on tangents, gradient and differentiation with a Year 12 class in England. Discourse is the core unit of analysis and mathematics is seen as a form of discourse. Mathematical discourse is characterised by four commognitive constructs: *word use, visual mediators, endorsed narratives* and *routines*. These constructs provide discursive foci for analysing mathematical discourse. However, this paper reports on the analysis for *word use* and *narratives* in the teacher's discourse. Data sets include two face-to-face interviews with the teacher, which were both audio-recorded and an observation of an introductory lesson on differential calculus, which was video-recorded.

Commognition; Discourse; Word use; Endorsed narratives

Background to the study

I set out to investigate how secondary school teachers of mathematics teach calculus, with a focus on differentiation. Many countries teach calculus to post-16 students. In Britain, calculus is part of the school mathematics programme for 16-18 age range, and this has been the case for over six decades. My prime motivation stems from my personal experiences, first, as a school teacher of mathematics, teaching calculus to Year 12 and Year 13 classes, i.e., 16 to 18-year-old students, and secondly, as a university mathematics teacher educator teaching calculus to postgraduate trainee teachers of mathematics. Teaching postgraduate students training to become teachers of mathematics, I quickly realised that, for many of these students, understanding of elementary calculus was often limited to some algebraic and algorithmic rules of differentiating functions, often disjointed, which they could hardly explain. A review of research literature on calculus education reinforced my quest for answers or explanations for the problem.

A review of literature

Other mathematics educators reported similar observations of students' difficulties with calculus. Berry & Nyman (2003), who, for a number of years worked with mathematics undergraduate students and postgraduate students training to be teachers, both in the United Kingdom (UK) and United States of America (USA), described their students' understanding of calculus as "a set of loosely connected actions based on a set of algebraic rules that can be applied in restricted, often artificial, algebraic situations" (2003, p.482). In a plenary presentation at the International Congress on Mathematical Education (ICME) conference in 1992 in Québec, Tall (1992) gave an extensive summary of the difficulties students encounter in learning calculus. Some of the challenges in calculus that Tall talked about had earlier been highlighted by Orton's

(1983a, 1983b) study on students' understanding of elementary calculus involving 16–22 year olds. Orton (1983a, 1983b) found evidence of 'instrumental understanding' from students' routine performance, knowing the rules for differentiation without reasons. The students lacked adequate intuitive 'relational understanding' of differentiation, i.e., "knowing *both* what to do and why" (Skemp, 1976, p.20, italics mine). Many more studies in the 1980s (Dreyfus and Eisenberg, 1983; Tall and Blackett, 1986; and Vinner, 1983) suggested that the students' understanding of fundamental concepts in elementary calculus such as the limit, functions, the derivative, and integrals was inadequate. There was an apparent general atmosphere of dissatisfaction in teaching and learning calculus in the 1980s. In the USA, the Calculus Reform Movement instigated changes in the teaching of calculus, which resulted in an extensive use of technology such as graphing calculators and dynamic computer software (Tall, 1992). Over the past decade research on teaching and learning mathematics and calculus has grown (Artigue, Batanero, & Kent, 2007; Oehrtman, Carlson, & Thompson, 2008; Park, 2013), which suggests that the derivative, among other calculus topics, remains difficult for many students. Although there is evidence from the literature to suggest procedural competence of elementary calculus by students, there is even stronger evidence to suggest that students lack conceptual understanding of the fundamental concepts of calculus.

Theoretical Framework

This study makes use of Sfard's (2008) commognitive framework, which is a discursive framework for analysing and interpreting human activity, to understand the 'intricacies of mathematical learning' (2008, p.566). It is built on the premise that 'thinking is a form of communication' (p.565). Commognition – 'both thinking and interpersonal communication' (Sfard, 2007, p.570) follow rules rooted in historically established customs. Thinking is considered as individualisation of (interpersonal) communication. Thinking is conceptualised as a form of activity of communication with oneself, thus cognition + communication (interpersonal exchanges) = commognition (p.570). The commognitivists reject any split between thinking and speech or thinking and communication; thinking processes and interpersonal communication are sides of the same coin.

In a commognitive framework, discourse is the core unit of analysis. Sfard (2007) likens commognition to games, a metaphor that originates in Wittgenstein (1953), and his famous notion of 'language games'. Just like with a vast number of different games, each is played according to certain rules, and with various tools, there exist many types of communicational activity each characterised and distinguishable by their own rules, mediational means and objects of communication. These distinctive types of communication are what Sfard calls 'discourses' (Sfard, 2007, p.570). However, Sfard (2008) distinguishes between two broad categories of discourses: non-specialised discourses – '*colloquial discourse'* (p.299), and '*literate discourses*' (p. 299) which are artefact-mediated mainly by symbolic tools designed specifically for communication. Another fundamental tenet of the commognitive framework is that mathematics is a form of discourse. According to Sfard (2008), a mathematical discourse is characterised by four features: *word use*; *visual mediators*; *endorsed narratives* and *routines*.

Word use refers to the types of words used in the discourse (Sfard, 2008), including 'the use of *mathematical terminology* (such as 'topology')' (Nardi, Ryve, Stadler, & Viirman, 2014, p.184, italics mine). It also refers to ordinary words used in everyday communication, but with unique and specific meanings in mathematics, such as differentiation, limit, point.

Narratives are utterances within the discourse. A narrative is any form of spoken or written text that is 'framed as a description of objects, or of relations between objects or activities with or by objects, and that is subject to *endorsement* or rejection, that is, to being labelled *true* or *false'* (Sfard, 2007, p.572). Examples of *endorsed narratives* include mathematical theories, definitions, proofs and theorems. Mathematical narratives can be categorised into either '*object level'* or *meta level* narratives. Mathematical objects such as ' $-3 \times -7 = 21$' and 'the sum of the exterior angles of a polygon is 360^{o}' are examples of object-level narratives. The '*meta level'* narratives are propositions about the discourse itself and the activities of the participants in the discourse, rather than about its objects, which says how mathematics is done, for example, when calculating use the order of operations (BIDMAS).

Methodology

The data was collected through interviews with a teacher of mathematics and lesson observation of an introductory lesson about the derivative to a year 12 class in a post-16 college in England. Data sets include two audio-recorded interviews with the class teacher and one video-recorded classroom observation in which the teacher discussed tangents, gradient and differentiation. The first interview was done before the observed lesson and the second one after the observed lesson. The video data from the lesson observation and the audio data from the interviews with the teacher were transcribed with attention given to the participants' utterances and actions. The discourses of the teacher (and the class) were then analysed with respect to *word use, visual mediators, routines, and endorsed narratives*.

For this paper, the analysis focuses primarily on *word use* and *narratives*. The study seeks to investigate the kind of words that are used in the discourse on the derivative and to analyse the extent to which specialised mathematical language (Morgan & Sfard, 2016) is used. *Narratives* are made up of words. For the analysis of narratives, the focus was on both written and spoken text about definitions, proofs, theorems, and facts related to the derivative within the mathematical discourse. In the following, I report on selected results that focus on the words gradient and tangent.

Selected results

Gradient and *tangent* are the most frequently used specialised mathematical words in the classroom discourse. Consider these two object-level narratives regarding gradient from the teacher (T) during the classroom discourse:

[99]. T:	The gradient of a curve is not constant, it would depend on x, and it's called the gradient function.
[119]. T:	What I want you to try to understand is that the gradient of a curve is the gradient of a tangent, that I do want you to appreciate, that's important.

These two narratives are contradictory. The teacher's narrative in [99] that 'gradient of a curve is not constant' is inconsistent with his definition in [119] that 'the gradient of a curve is the gradient of a tangent'. A tangent here is a straight line, and the gradient of a straight line is, indeed, constant. It follows, therefore, from the narratives above that the *gradient of a curve* [99] cannot be the *gradient of a tangent* [119].

In *literate* mathematical discourse, the gradient function describes the gradient of the tangent *at any point* on the curve. As it would 'depend on x' [99], it is a function of x. So, the word use 'gradient' in [99] is for the *gradient (or derivative) function*. However, the word use 'gradient' in [119] is for the *gradient (or derivative) at a given point*. Unlike the 'gradient' in [99], this gradient is not a *function*, but a *scalar*, i.e., a number. At any given point on a curve, the gradient (derivative) of a curve is, in fact, equal to the gradient (derivative) of the tangent to the curve at that given point. What should rather be important about this narrative [119] is in fact what is missing from the narrative – *'at a given point'*. It could have been *substantiated* by specifying and including the words *'at a given point'* for it to be *endorsed*. The teacher's narrative in [119] is, therefore, inconsistent with *literate* mathematical discourse; should this narrative be *endorsed* or rejected?

In the entire 60 minute-lesson, there were eight utterances of the narrative 'gradient of a curve' by the teacher. The utterance 'gradient of a curve' appears four times within the first eight minutes, and again four times in the last eight minutes of the 60 minutes' lesson. Analysing each of the narratives within its context reveals that by the 'gradient of a curve', the teacher was referring to the gradient (or derivative) function. Consider the following two episodes. The first one is an extract of the classroom discourse just before the teacher's narrative in [99] above. The second one is an extract of the classroom discourse immediately after the teacher's narrative in [119] above.

Following on the teacher's explanation of the derivative of the function $f(x) = x^3$, which was *visually mediated* through the dynamic imagery of Autograph on the board, the following discourse with a student(S) resulted:

[88]. S: What does the derivative mean?
[89]. T: It means the gradient function, the gradient of the curve, is $2x$, of x^2. It's not a constant, is it?
[90]. S: No
[91]. T: The gradient, a constant?
[92]. S: No
[93]. T: It's a function of x.
[94]. S: Yeah
[95]. T: We call it a gradient function. We call it the derivative. There are other names as well, is that ok?

The mathematical object of the discourse here is the *derivative function*, certainly, not the *derivative at a point*. The teacher's frequent use of the specialised mathematical words in his utterances above [89, 91 & 93 & 95] explains what the teacher was referring to by 'gradient of a curve' or derivative. Notice that the teacher followed up this episode with the narrative in [119] above, which implied that the 'gradient of a curve' was constant. This prompted another episode of questions from a student to the teacher.

[121]. S: If that curve like you found out from the red curve, do you find out that the gradient is in that blue curve?
[122]. T: Yes/
[123]. S: But that's a curve, so the gradient changes a lot, doesn't it?
[124]. T: Yes, that's the whole point the gradient just change...that's exactly the point for a curve the gradient is changing all the time.

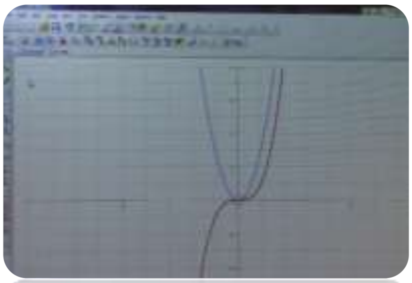

Fig I: Gradient function of $f(x) = x^3$

The student utterance, on the one hand, substantiates the narrative that is visually mediated graphically (Figure I), that of the function $f(x) = x^3$ and its derived function $f'(x) = 3x^2$. On another hand, it is a rejection of the teacher's narrative that 'the gradient of a curve is the gradient of a tangent' [119]. Note the teacher's utterance "the gradient just change" [124], the 'gradient' here is (incorrectly) used as an irregular plural. This is in fact consistent with his inconsistent use of the word 'gradient', for example, in his narrative 'the gradient of a curve'[99; 119]. The inconsistent use of the word 'gradient' for both "the gradient function" and "the gradient at a point" prompted the student, effectively, to challenge or dispute or reject the teacher's narrative [119]. Nardi et al. (2014) reports on a study by Park's (2013) which revealed that:

> Students' use of the word 'derivative' for both "the derivative function" and "the derivative at a point" as closely associated with evidence that the students often did not appreciate the derivative at a point as a number and the derivative as a function, and that *they often described the derivative as a tangent line – rather than seeing the value of the derivative at a point as determining the slope of the tangent at this point* (p.186; italics for emphasis).

It is therefore vital that teachers should carefully consider their use of the word *gradient* in their introductory lessons on calculus.

Conclusion

Analysing mathematical discourse of the teacher through their *word use* and *narratives* (plus visual mediators and routines or meta-rules), allowed for an investigation of how teachers of mathematics teach elementary differential calculus. In analysing the teacher's discourse for constructing the mathematical object – the derivative – the object-level narrative '*the gradient of a curve'*, draws special attention. The word *gradient* is a specialised mathematical term. The endorsed narrative in literate mathematical discourse describes the derivative of a function at a given point as the gradient of the tangent to the graph of the function at that point. Thus, the object-level narrative *'the gradient of a curve'*, is therefore inconsistent with literate mathematical discourse. However, since the teacher's narrative - *the gradient of a curve*, uses words characteristic of mathematical discourse, should it be endorsed or rejected?

Inconsistent *word use* of *gradient* could make it very difficult for the students to understand when the word derivative is used to mean the gradient at a point which is a constant, as opposed to the derivative as a function (Park, 2013). It is therefore essential that calculus teaching pays close attention to *word use* in the discourse, ensuring that a clear distinction is made between gradient at a point and gradient function.

The commognitive framework provides a powerful conceptual lens through which we can examine, on a micro-level, how teachers teach mathematics. Although the theory of commognition was initially developed for the study of thinking and learning, Cobb (2009, p.207 cited in Nardi, Ryve, Stadler, & Viirman, 2014) notes:

> The commognitive framework has the capacity to attend to "the macro-level of historically established mathematical discourse, the meso-level of local discourse practices jointly established by the teacher and students […] and the micro-level of individual students' developing mathematical discourses."

I would refine Cobb's comment, and further argue that the commognitive framework has the capacity to attend to the micro level, *not only* of the individual students' developing mathematical discourse *but also of the individual teachers' mathematical discourse*.

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