Enlarging a tangram: do the pieces still fit? An account of an interview and its use in the design of a lesson on multiplicative reasoning

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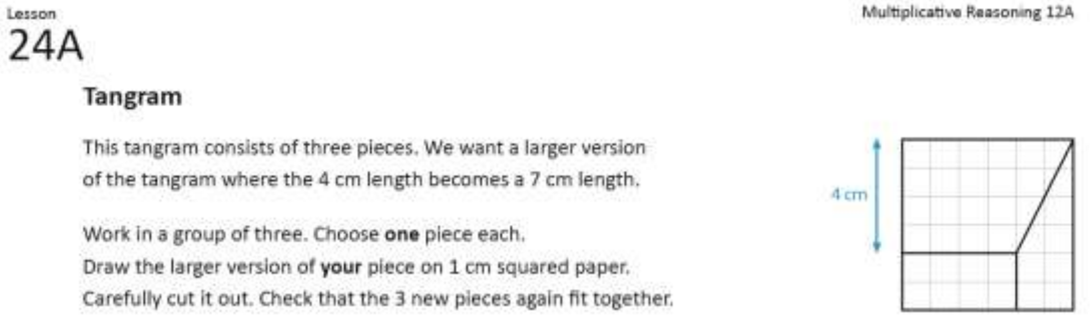
In this paper I report on an interview with three Year 8 (Grade 7) pupils in which they each attempt to enlarge one piece from a simple tangram drawn on squared paper. I go on to discuss how the interview informed the design of an ICCAMS lesson on multiplicative reasoning.

Keywords: enlargement; scaling; multiplicative reasoning

Introduction

In this paper I report on an interview with three Year 8 pupils (Grade 7, 12-13 year olds) on a Tangram task, and discuss how this interview helped inform the development of a lesson on multiplicative reasoning in the domain of geometric enlargement. The lesson belongs to a set of 24 pairs of lessons on Algebra and Multiplicative Reasoning developed for the *Increasing Competence and Confidence in Algebra and Multiplicative Structures* (ICCAMS) *Project*, aimed primarily at pupils in Key Stage 3. Figure 1 shows the task, in the form in which it currently appears in the ICCAMS lesson materials. The interview task was identical to this but presented orally.


Lesson
24A

Multiplicative Reasoning 12A

Tangram

This tangram consists of three pieces. We want a larger version of the tangram where the 4 cm length becomes a 7 cm length.

Work in a group of three. Choose **one** piece each.
Draw the larger version of **your** piece on 1 cm squared paper.
Carefully cut it out. Check that the 3 new pieces again fit together.

Figure 1

The tangram lesson is one of several ICCAMS lessons that explore the notion of enlargement, and although it is presented as the last of these lessons, it can be used independently of the others, ie as a lesson in its own right. The task arose from a suggestion by Jeremy Hodgen, who leads the ICCAMS project. It is a simplified version of a task developed by Guy Brousseau (Brousseau, Brousseau & Warfield, 2008) as part of a project in which lessons on rational number were developed over a period of more than 15 years with 5th grade classes (10 – 11 year olds) in a school in Bordeaux.

Stimulus task

The use of a ‘stimulus task’ (or mathematical Situation - Brousseau) at the start of a lesson is a key feature of the ICCAMS lessons. The design of the lessons derives from a socio-constructivist perspective on learning (eg, Cobb, 1994) and fits the design principles articulated by Swan (2008), in particular that one should “Confront difficulties rather than seek to avoid or pre-empt them” (p.8). Swan elaborates on this principle as follows:

> Effective teaching challenges learners and has high expectations of them (Bell, 1993; Wigley, 1994). It does not seek to 'smooth the path' but creates realistic obstacles to be overcome. Confidence, persistence and learning are not attained through repeating successes, but by struggling with difficulties. (p.8)

We set pupils the same numerical challenge as in Brousseau's task, namely to produce a larger version of the tangram such that a 4 cm length becomes a 7 cm length. This can be described mathematically as an enlargement with scale factor ×1.75. This is quite challenging - numerically and conceptually - deliberately so. We could have made the task simpler by choosing a multiplier greater than 2, and a great deal simpler by choosing a whole number multiplier, eg by changing the requirement '4→7' to '4→8'. However, starting with a mapping like 4→8 would in all likelihood have reduced the level of engagement in the task (pupils would have solved the task without much effort and moved on) and would have let pupils sidestep a key feature of enlargement, namely that it involves scaling (see eg Küchemann, Hodgen & Brown, 2014) and that this scaling is uniform. Of course, this does not mean that it might not be useful to introduce such a relatively simple mapping at a later stage.

There is considerable evidence to suggest that for a mapping like 4→7, many pupils will adopt an addition strategy (Hart, 1981), whereby they decide that the mapping involves the operation +3 rather than ×1.75. This seems to be particularly the case for geometric contexts (Küchemann, 1989). Brousseau et al (2008) make the following observation with regard to their tangram task:

> Almost all the students think that the thing to do is to add 3 cm to every dimension. Even if a few doubt this plan, they rarely succeed in explaining themselves to their partners and never succeed in convincing them at this point. (p.155)

The interview

The pupils in our interview responded in just this manner. They were from a relatively 'able' Year 8 mathematics class (set 2 out of 5) and consisted of two girls, P and A, and a boy, B. Pupil P took the most active role in the interview and was also the most confident mathematically. Pupil A was less confident initially but took an increasingly active part, while Pupil B tended to stay in the background. The pupils seemed engaged in the task throughout, and my announcement after about 22 minutes that we would have to stop because it was break time, drew a disappointed "Oh!" from P.

I introduced the task and we agreed on who would enlarge which piece of the Tangram (see Figure 2, where the Tangram is shown 'upside down', to match the stills shown below that were taken from the video recording of the interview). I then gave the pupils a fairly short period of thinking time after which I asked, "What are your thoughts?". Pupil A immediately responded with "I don't know" while P said "If we add 3 onto 4, do we need to add 3 onto A's bit as well?". The conversation continued with me trying to hedge my response:

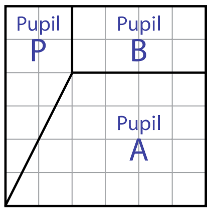


Figure 2

> DEK: Well, that's one strategy... that would make it... You need *a* strategy for making it bigger, everything is going to be bigger, ... but when we cut the three out they've got to fit together again, that's going to be the test. [8 seconds pause]
> Pupil P: So! If we're adding 3 to the 4, do we need to add 3 to all of the sides?
> DEK: Well, it's a very good question. That is a strategy, the question is, is it the right strategy? [6 seconds pause] We could try it, and see what happens.
> Pupil A: - being me, I'll probably just add it, so every side, what you do to one side you do to the other.
> DEK: OK, let's try that, and let's see what happens.

The pupils proceeded to apply the +3 rule. Pupil A, having started with a 4 cm by 2 cm rectangle, drew and cut out a 7 cm by 5 cm rectangle. Similarly, Pupil B produced a shape in which he had added 3 cm to the given vertical and horizontal edges of his original shape which he then closed by drawing a slanting line. Pupil P proceeded in the same way but paused as she was about to draw her slanting line. She had noticed that the original slanting line "goes through two squares" and that this was steeper than her new line would be. She now lengthened the 9 cm vertical line ("I think it has to be 11") and then carefully drew the slanting line in repeated segments going 1 square across, 2 up. In this way she succeeded in reproducing the original slope; however, she had to extend the 9 cm line further until it was about 17 cm long.

When the pupils put their three new shapes together there was genuine surprise that they didn't fit (Figure 3). However, the pupils struggled to determine how things had gone wrong. They all felt that A's shape was correct because (unsurprisingly!) the two edges that touched the adjacent shapes exactly matched the corresponding edges. Pupil P decided that "it's B's fault" - his slanting line was wrong - although she ignored the fact that she had broken their +3 rule and that her shape was far taller than the combined height of A's and B's shapes.

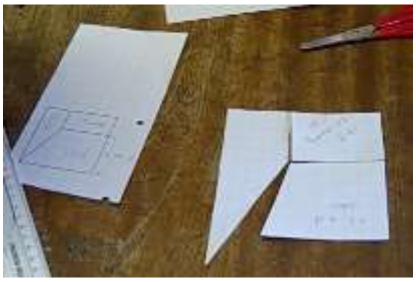


Figure 3

At one point Pupil A said that the left hand edge of the combined new shape should be 9 cm long, not 12 cm, since it was 6 cm long to start with. Here she was on the brink of a telling insight, namely that the increase in a line's length must be independent of whether (or how many times) the line has been partitioned. However, influenced by P perhaps, she quickly changed her mind and decided that this edge (and the base) should be increased by 6, in conformity to what had actually been done.

I initiated a discussion about the slope of Pupil P's (and the original tangram's) slanting line. This led Pupil B to produce a revised shape with the correct slope, but still with a base and height of (4+3) cm, and this time with an upper horizontal edge that turned out to be about 10.5 cm long to accommodate the 1-by-2 slanting line, instead of the previous (6+3) cm.

This new shape produced a better fit with P's shape, though P's shape still protruded above the rest of the tangram. P tried to rectify this by splitting her shape using a vertical cut. This produced a shape with a base of 2 cm that was too short and too thin. But it also produced a shape with a base of 3 cm whose slanting side seemed to match B's quite well, in length and slope, though its lower part was too tall compared to A's shape. (Figure 4). This prompted P to ask A whether she could make her shape bigger. In reply, A suggested they could do this by drawing "the whole big square by adding 6 to each side and then split it up as it goes". [Here she was using her earlier mistaken idea of adding 3+3 cm to the base and the height of the whole tangram.]

Pupil P started to draw a 12 cm by 12 cm square and while this was going on, B replaced A's 7 by 5 rectangle with one that more closely fitted the space between the other two shapes (right hand shape, Figure 5). Pupil A now pointed to P's shape and said, "But that's not in proportion to those [other two]... Because that's so much skinnier". Here A is displaying an intuitive understanding of 'proportion' which would clearly be worth developing into something more precise and analytic.

At this point, P had drawn and cut out a 12 cm square and proceeded to mark on it the edges of A's and B's shapes, using their original +3 rule. She then drew a slanting line from their common vertex to the top-right corner of the 12 cm square (see left hand shape, Figure 5). The pupils all notice that the 'line has changed', ie that it had a different slope from before. P expressed her dissatisfaction with this, but A tried to make sense of the change by suggesting they should get a protractor, "Because you have to add 3 to that angle". Overall, she felt that the tangram "sort of looks all in proportion". At this point we had to stop the interview unfortunately, as it was break time and people were entering the room.

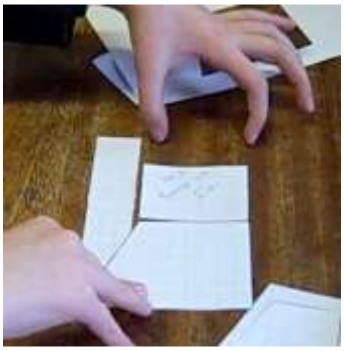


Figure 4

Figure 5

Developing the lesson

It was clear from this interview (and an earlier one), that our simplified version of Brousseau's Tangram task has considerable potential for generating conflict in pupils' thinking about enlargement which, if resolved, could enrich their understanding. However the interview also supports our experience that making productive use of cognitive conflict is far from straightforward. Pupils can easily ignore conflicting ideas, as Pupil A did when she rapidly discarded the idea that the new tangram should (in effect) form a 9 cm rather than a 12 cm square. Or pupils' thinking can be too localised so that they don't coordinate all the features involved in the conflict. This suggests that the use of the Tangram task in a lesson is more likely to be fruitful if the initial exploratory work is followed by a carefully thought-out intervention from the teacher. [Note that we are not advocating that the lesson be given over to the kind of 'direct instruction' favoured by many in the swelling ranks of cognitive load theory devotees!]

The intervention used by Brousseau et al (2008) was to list all the key horizontal and vertical lengths in their tangram (namely 4→7, 5→, 6→, 2→, 9→ and 7→) with a view to tabulating the image-lengths. Brousseau et al report that this invariably led to someone adding 8→14 to the list (presumably derived by doubling 4→7) and to someone suggesting that "We need the image of 1". And once this value was found, it seems that the pupils used multiplication or rated addition rather than the simple +3 strategy to find the various images:

> Either they multiply the image of 1 successively by 5, 6, 7 and 9 or they add the image of 1 to the image of 5 to get that of 6, the image of 4 and that of 2 to get that of 6, and so on. (p.157)

It seems surprising that these pupils switched to a multiplicative approach so readily, although it is worth noting that they had already engaged in a great deal of work on rational number. Such a switch might not work so readily in a UK school. And even if it did, would pupils *appreciate* the multiplicative nature of the relation between object and image? For example, in the case of A's 4 by 2 rectangle, would they now be aware that the two sides are *not* increased by the same amount, but, rather, that the longer side is increased by more than the shorter side, and would they have some sense of why this must be so?

Ainley, Pratt and Hansen (2006) make an interesting case for designing tasks that involve information technology. They suggest that with such technology it is possible to design *purposeful* tasks in which pupils create virtual objects, and that this can be done with tools whose mathematical *utility* can emerge through their use. Experience suggests that the Tangram task is already purposeful - at least once pupils reach the stage of seeing that their pieces don't quite fit. However, it is interesting to consider how the technological tool of *dragging* could be used to tease out the multiplicative nature of enlargement. Pupils could be given a computer image of the tangram, or of their one piece, which they could make bigger by dragging. A constrained dragging tool (in which a figure's aspect ratio doesn't change) would enable them to produce a correct image, whose properties could then be analysed. More demanding, and more illuminating perhaps, would be to use a free dragging tool, which might force pupils to think more deeply and in advance about aspect ratio.

For pragmatic reasons, we decided not to require the use of information technology in the next stages of our Tangram lesson (though we provided optional Word files that allowed for constrained and free dragging). However, the issue of how much one leads pupils towards a correct solution to be analysed post hoc or asks them to discover a correct method, was still to the fore. The interview indicated that there were lots of potential, and potentially fruitful, conflicts arising from the Tangram task that one could focus on, though of course one couldn't be sure which ones would arise in an actual lesson or how widely they would occur.

In the end, we decided not to home in on specific conflicts. Also, rather than rushing to resolve the task (eg by somehow eliciting the multiplier ×1.75) we would focus attention on qualitative aspects of enlargement, in particular the grounded notion of 'a square remains a square'. We thus decided that the lesson would continue with a brief discussion of pupils' responses to the Tangram task, followed by a third stage involving a simpler version of the task (see Figure 6). Notice that here the Tangram is the same as before, except that one of the shapes has been partitioned into two squares. (Note too that there are many other squares implicit in the tangram that the teacher can point to if desired: the 2×2 'base' of the right-hand shape; the 4×4 left hand portion of the top shape; the 6×6 square of the whole tangram; the unit squares that form the background grid.) We also involved a somewhat simpler mapping, of 6→9 instead of 4→7. This is likely to elicit a rated addition approach ('add half as much again') from some pupils which could fruitfully compete with a +3 addition strategy.

3. Simplify the task.

- Display this shape. →
  "This is a 6 cm square tangram again, but now with 4 pieces (2 of which are squares). Draw a larger version that forms a 9 cm square. Use 1 cm squared paper but *don't* cut out the pieces."
- Discuss students' drawings. "What clues can we use to decide whether the result is right or wrong?"

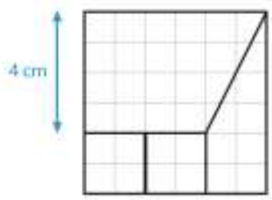


Figure 6: Stage 3 of the lesson

In the fourth and final stage of the lesson (Figure 7) we return to the original Tangram task, but this time we ask pupils to enlarge the whole tangram, not just individual pieces. This is a more 'efficient' task than enlarging the pieces individually, and more manageable than cutting them out.

4. Revisit the original task.

- Ask students to repeat the original 4 → 7 enlargement. "Draw a larger version but *don't* cut out the pieces." "What size square does the enlarged tangram form?"
- Evaluate the drawings. "What clues can we use...?"
- Why does the +3 addition strategy not work?
- Discuss the scale factor. "If I know a length on the tangram, how can I work out the enlarged length?"

Figure 7: Stage 4 of the lesson

However, it doesn't provide such telling feedback, though it does again provide opportunities to consider the result qualitatively. At the end we touch on the notion of *scale factor*, in anticipation of future, perhaps more analytic, explorations of enlargement and of the whole notion of multiplication as scaling. Note that our approach here is rather different from many text books where the notion of scale factor is often used to *introduce* pupils to enlargement. Our concern is that if scale factor is simply given to pupils as a (powerful) rule, it might shut down their thinking and thus hinder them from developing a richer understanding of enlargement.

Summary

We found that the Tangram task engaged pupils by exposing them to some of the demands of multiplicative reasoning in the domain of enlargement. And we found a lesson design that seemed to have a good chance of addressing some of these demands and of thus helping pupils make some sense of them.

Acknowledgements

This study formed part of the work of the ICCAMS project which is led by Professor Jeremy Hodgen. I am grateful to him as well as to ESRC who funded the original project and to EEF who currently fund the project.

References

Ainley, J., Pratt, D., & Hansen, A. (2006). Connecting engagement and focus in pedagogic task design. *British Educational Research Journal*, 32(1), 23–38.

Bell, A., W. (1993). Principles for the design of teaching. *Educational Studies in Mathematics,* 24(1), 5-34.

Brousseau, G., Brousseau, N., & Warfield, V. (2008). Rationals and decimals as required in the school curriculum: Part 3. Rationals and decimals as linear functions. *The Journal of Mathematical Behavior*, 27(3), 153-176.

Cobb, P. (1994). What is mind? Constructivist and sociocultural perspectives on mathematical development. *Educational Researcher*, 23(7), 13-20.

Hart, K. (1981). Ratio and proportion. In K. Hart (Ed.), *Children's understanding of mathematics: 11-16,* (pp.88-101). London: John Murray.

Küchemann, D. E. (1989). The effect of setting and numerical content on the difficulty of ratio tasks. In G. Vergnaud, J. Rogalski & M. Artigue (Eds.), *Proceedings of PME 13*, Volume 2, (pp.180-186). Paris: IGPME.

Küchemann, D. E., Hodgen, J., & Brown, M. (2014). The use of alternative double number lines as models of ratio tasks and as models for ratio relations and scaling. In S. Pope (Ed.), *Proceedings of the 8th British Congress of Mathematics Education*, (pp.231-238). Nottingham: BSRLM/BCME.

Swan, M. (2008). A Designer speaks: designing a multiple representation learning experience in secondary algebra. *Educational Designer*, *1*(1),1-17.

Wigley, A. (1994). Models for mathematics teaching. In A. Bloomfield, & T. Harries (Eds.), *Teaching and Learning Mathematics,* (pp.22-25). Derby: Association of Teachers of Mathematics.