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A MODEL OF STUDENTS' SIMPLIFICATION OF TRIGONOMETRIC EXPRESSIONS

Ali Delice
Centre for Studies in Science & Mathematics Education
University of Leeds

I present an operational model of how students' simplify trigonometric expressions. The model has three main components: recognising, recalling and doing. This paper describes the interaction between these components and links this model to other models of doing mathematics.

INTRODUCTION

This paper is self-contained but is one aspect of a wider study of 16-17 year old English and Turkish students' performance and understanding of aspects of trigonometry in the institutional context of each country. This paper concentrates on aspects of simplification of trigonometric expressions.

Trigonometry is an important area of senior secondary mathematics in every country in the world but surprisingly little research has been conducted in this area in terms of students' conceptions/performance or curricula design. Professional journals occasionally publish articles with ideas for teachers, e.g. Barraclough (1990) suggests unit circle activities that generate basic identities and Ellis (1990) stresses the importance of practical activities which link trigonometry to real world problem solving. Of the few research studies available Blackett and Tall (1991) found that students using computer graphing made a greater improvement in introductory trigonometry than those who did not. Kendal (1992) found the ratio method of introducing trigonometry more effective in terms of performance and retention of concepts than the unit circle approach. Pritchard and Simpson (1999) examine the role of pictorial images in students' solutions of trigonometry problems and found that propositional knowledge took precedence over imagery in students' thinking. I have found no research linked to students' understanding of higher trigonometry.

'Simplification' is an odd term to use with trigonometric expressions simply because it is far from obvious which of, say, **$\sin 2\theta$, $2\sin\theta\cos\theta$** is more 'simple'. Indeed this is an area of socio cultural mathematics and student initiation into the 'ways of mathematicians' is important as cognitive factors in analysing student performance. It is also important to note that simplification actions cannot be divorced from the tools employed to carry out these actions. Simplifying in this paper means simplifying with paper and pencil – simplifying with, say, a computer algebra system would change the actions significantly. This paper presents a model of what students do when they attempt to simplify a trigonometric expression. An apparently trivial but important aspect of this model is the act of rewriting the expression. The remainder of this paper sketches the methodological approach, presents the model and illustrates it with a protocol from one student and concludes with a discussion.

METHODOLOGY

My approach is 'naturalistic' in that I observed, as far as possible, 'what is' in both countries without manipulating the teaching and learning. The student sample consisted of 55 A-level mathematics students from one English college and 65 similar aged students in one Turkish school. The wider study involved various student tests, student and teacher interviews and classroom observations. To obtain data on students' manner of simplifying trigonometric expressions I used concurrent verbal protocols as students (4 English and 8 Turkish) solved simplification items to gain insight into their thinking. Students were selected for the protocol work to represent a range of attainments (in my tests and in school work) and for their ability to communicate well. In the spirit of naturalistic enquiry students used the resources they used for trigonometry in their school work: calculators and formula sheets for English students but only trigonometric tables for Turkish students.

A MODEL OF STUDENTS' SIMPLIFICATION

There were several differences in the overall performances of students from the two countries. Turkish students were 'better' at algebraic aspects of trigonometry whilst English students were 'better' at word problems. This, I believe, reflects what the curricula in the two countries privilege. However the concurrent protocols revealed a uniformity of approach with regard to simplification. The following informal summary is designed to give the reader an initial feel for the protocol which follows.

Students begin by reading the question. This may be explicit (aloud) or implicit (unspoken). NB I am aware that protocols should not, strictly speaking, include implicit actions but implicit actions are important for this paper. Students then focus on a subexpression or a form, e.g. $\mathbf{\sin^2x+\cos^2x}$ or $\mathbf{\sin^4x-\cos^4x}$. I call this 'recognising'. They then 'recall' (from memory or with the aid of formulae sheets) trigonometric or algebraic properties, e.g. that the first of the above expression equals **1** and the second one is $\mathbf{(\sin^2x+\cos^2x)(\sin^2x-\cos^2x)}$. Please note that I believe but have not demonstrated that recognition precedes the recall of properties. In the schematic of my model (Figure 1 below) read, recognise and recall are grouped together under the term 'recognition' because they all rely on sign association. Students then 'rewrite' the given expression with some form substituted for another, e.g. '**1**' substituted for $\mathbf{\sin^2x+\cos^2x}$. Students may (explicit manipulation, a continuous line in figure 1) or may not (implicit manipulation, a dotted line in figure 1) write ancillary 'jottings', e.g. $\mathbf{\sin^2x+\cos^2x}$=1, prior to rewriting the expression. What is important with regard to my model is that rewriting the expression always occurred. I will say more about this in the discussion section. The students then examine the rewritten form and recognise/recall another subexpression/property and enter a further manipulate/rewrite phase or accept their rewritten form as the result. Recall, manipulate and rewrite are grouped under the term 'doing' because they all rely on transforming signs. The model has 'recall' in both the 'recognition' and 'doing' groups. I do not see a contradiction here and would add that this duality appears to be a function of the dialectic between recognition and doing in this context.

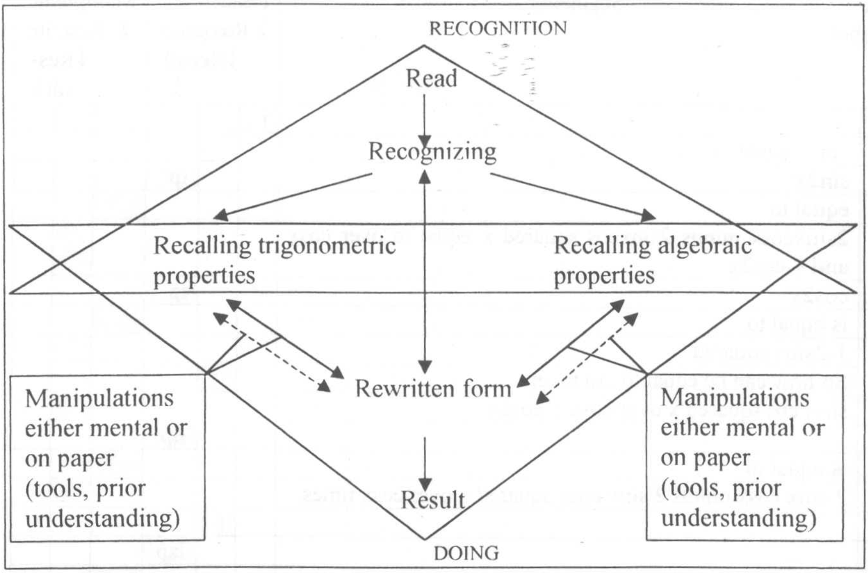

Figure 1 Schematic model of students' simplification of trigonometric expressions. (The upper triangle represents the RECOGNITION actions, the lower triangle the DOING actions and the intersection of two triangles the RECALLING actions)

A PROTOCOL

I restrict my attention to one student's protocol for the task 'simplify the expression $\dfrac{\sin 2x - 2\sin x\cos^2 x}{\cos x(1-\cos 2x)}$', an English student who produced a correct answer (in a longer paper I contrast this with a Turkish student who produced an incorrect answer). This student answered most of the protocol items correctly. The columns to the right of the protocols correspond to components in my model. I have used the abbreviations: 'i' for 'implicit' and 'ap'/'tp' for 'algebraic property'/'trigonometric property'.

$$= \frac{2\sin x\cos x - 2\sin x\cos^2 x}{\cos x\,(1-\cos 2x)}$$

$$\cos 2x = 1 - 2\sin^2 x$$

$$\frac{\sin 2x - 2\sin x\cos^2 x}{\cos x\,(1-\cos 2x)}$$

$$= \frac{2\sin x\cos x - 2\sin x\cos^2 x}{\cos x \times 2\sin^2 x}$$

$$= \frac{1-\cos x}{\sin x}$$

$$= \frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$$

$$= \tan\frac{x}{2}$$

Line number	Segments	Read ↓	Recognise ↓	Recall ↓	Manipulate ↓	Rewrite ↓	Result ↓
1		i					
2	..err...equal to						
3	sin2x			tp			
4	equal to						
5	2sinxcosx minus 2sinxcos squared x equal to over cosx and 1-cos2x						
6	cos2x			tp			
7	is equal to						
8	1-2sinx squared.						
9	So how can be equal to ..to.to..err..						
10	sinx cos squared x over cosx 1-cos2x						
11				itp			
12	is equal to						
13	2 sinx cosx minus 2 sinx cosx squared x over cosx times						
14			i				
15				iap			
16					i		
17	2 sin squared x						
18			i				
19	so cancelled out 2sinxcosx			ap			
20					i		
21	so it is equal to 1-cosx over sinx						
22	err... errr... no not yet						
23	1-cosx			tp			
24	is equal to 1 cos co						
25	1 sin squared x over 2						
26	and sinx			tp			
27	is equal to						
28	2 sin x over 2 cosx over 2						
29			i				
30				iap			
31					i		
32	so it is sinx over 2 cos x over 2						
33			i				
34				itp			
35	which is equal to tan x over 2						
36							i

Commentary of the protocol

The student clearly read the statement and recognised the terms. Note that it is irrelevant to my model whether this is implicit or not. She then focused on *sin2x* and rewrote the expression with *2sinxcosx* in place of *sin2x* (the manipulation being a mental substitution). She then focused on the *cos2x* of the rewritten expression and recognised that it is $1\text{-}2sin^2x$ and writes this down. Further rewritten forms accompany the rewritten expression. I will not continue this commentary for the complete protocols.

Note the general diagonal pattern in the right-hand column. These diagonals reflect the iterative 'recognise, recall, manipulate, rewrite' components of my model. 'Rewrite' is imperative and central for this model. In this component, terms of the initial expression and the initial expression itself are written in their equivalent form. Then these forms become either another expression to simplify or the final result.

Note that a trigonometric expression may be simplified by using either trigonometric or algebraic properties. This distinction is important for the wider study alluded to in the introduction but it is possible to employ an undivided 'symbolic properties' component in the model. Analysis of other students' protocols, not presented here, reveals that the model 'holds' for incorrect and for partially simplified expressions.

DISCUSSION

I will focus on the import of the rewritten form and on links between this model and other models of mathematical activity. I commented above that students always rewrote the expression to be simplified at least once before claiming that they had reached a simplified result and I claimed that this was important, but is it central to students' simplification? At one level, students may simply rewrite the expression because they are told to simplify it – so there is an implicit didactic command to rewrite the expression. For a student who has had little or no initiation into the community of practice of mathematicians a rewrite (right or wrong) is not important. This was not the case with any of the students who provided protocols in this sample, they all had many hours practice in simplifying and knew the 'rules of the game'.

My model has similarities with other models of mathematical activity. The links to other models also highlight the importance of the rewritten form and the dialectic between recognition and doing. Rewritten forms in my model has links with Saxe's (1991, p.17) model where 4 parameters – activity structures, prior understanding, artifacts/conventions and social interactions – interact with emergent goals. Although social interactions and activity structures are somewhat limited in these protocol items his inclusion of 'conventions' seems particularly apt in the case of trigonometry, where symbolic manipulation conventions abound. What is particularly interesting, however, is that Saxe's emergent goals appear to coincide with my rewritten forms – both appear and fade away in the course of the activity. My model, however, appears more relevant to an analysis of school mathematics symbolic manipulation.

My model also has direct links to Dreyfus *et al.*'s (2001) model for abstraction. Their model has three epistemic actions: *constructing, recognising* and *building-with*. *Constructing* new conceptual knowledge has no place in my protocols because the tasks rely on students working with existing knowledge. However, *recognising* and *building-with* are, I believe, parallel to my recognising and doing. There is clearly scope for further work into the relationship between their work and mine but what my model emphasises, and theirs does not, is the central place of rewritten forms.

The issue of mental manipulations and their link with written manipulations is important. A possible criticism of my hypothesis that rewritten forms are central to simplification is that many people can simplify expressions 'mentally' without rewriting. I agree and there was some evidence of this in this sample. However, the ability to mentally simplify without rewriting appears to be dependent on the simplifier's experience and the simplicity of the task. I expect that most people reading this paper can mentally substitute **$\sin 2\theta$** for **$2\sin\theta\cos\theta$** but I believe that there was a point in our development when this was not the case, when we had to perform a written substitution. Regarding the simplicity of the task there was a student who, it appeared, did not need to rewrite an expression (though he did so because he was asked to). This was an easier (for him, **$\tan^2x\cos^2x+\cot^2x\sin^2x$**) expression to the one in the protocols above. However, although he 'saw' that this was 1 I claim there was a point in his development when he would not have seen this.

What my explanation of students' trigonometric actions, rather than my model, lacks is a consideration of student use of memory. This requires further work.

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