Goodchild, S. (Ed.) Proceedings of the British Society for Research into Learning Mathematics 22(1&2) February/May 2002

STUDENTS, BICYCLES AND THE QUIRKS OF SYMBOLIC LANGUAGE IN MATHEMATICAL LOGIC

Elena Nardi and Paola Iannone

School of Education and School of Mathematics, University of East Anglia

For many learners of mathematics the symbiosis of the language of mathematical logic with the ordinary language of 'real-life' problems is an uncomfortable one. Here we draw on the written responses of sixteen Year 1 mathematics undergraduates to a question that asked them to use quantifiers in order to rewrite the sentence **"In each year group in the university there are at least ten students with bicycles"** *and discuss: the question setter's intentions and proposed preferred solution; typical and untypical student responses; and, alternatives on how the question setter's intentions could perhaps be more efficiently realised. The discussion aims to demonstrate that students need to be exposed to the* necessity *and* effectiveness *of mathematical language, not simply to its* authority.

The study we report here on is funded by the Nuffield Foundation, is a small, exploratory data-grounded theory study (Glaser and Strauss 1967) of the mathematical writing of the students (60 in total) in the Years 1 and 2 Pure Mathematics courses of the 3-year degree in Mathematics at the UEA and is an on-going collaboration between the School of Mathematics and the School of Education (see for example Nardi and Iannone 2000). The aims of the study are: identifying the major problematic aspects of the students' mathematical writing in their drafts submitted to tutors on a fortnightly basis; increasing awareness of the students' difficulties for the undergraduates' tutors; providing a set of foci of caution, action and possibly immediate reform of practice; and setting foundations for a further larger-scale research project. The first two phases focused on Year 1 Calculus, Linear Algebra and Probability courses. The current phase focuses on the Year 2 Abstract Algebra course.

The Year 1 course on Analysis we draw on here ran for 12 weeks during the Autumn Semester, with 3 hours of lectures per week. There were 6 tutorial sessions for each of ten groups of about 5-6 students. The lectures were traditional front-teaching sessions. Three of the tutorial groups were run by the second author. At the end of each tutorial the students were given a selection of exercises to be submitted for marking. These written responses form the bulk of data gathered for this project.

The first level of data analysis, Data Analysis Version 1, involved the production of a student–by–question table that focused on the written responses of 16 out of 60 students registered for the course. This table summarised observations and comments of the second author on the students' responses to the set homework. After this table was completed, informal conversations were held with the other tutors and lecturers to the course to record their observations and comments after having marked part of the homework of the remaining student cohort. Following a detailed discussion of Data Analysis Version 1, the first author produced Data Analysis Version 2, a question by question table where the major issues were summarised, characteristic examples of the students' work were referred to and links with current literature were made. In these analyses it became clear that probing further into the students' thinking would be greatly helped if they could be asked to provide justification for parts of their writing, for example, via interviewing them (a step we have been taking in more recent phases of the project). Six such cycles of data collection took place and the following question, which we wish to focus on in this paper, comes from Cycle 4:

> Write down using quantifiers the statement:
>
> " In each year group in the university there are at least ten students with bicycles."
>
> Using your quantified statement, write the negation of the statement down, both in words and using quantifiers.

What were the question setter's intentions? In the Notes for Tutors, a short commentary on each problem sheet, compiled by the lecturers and distributed to the tutors for the course, the two lecturers offer no specific comment on their intentions for setting this question. We were particularly intrigued by the question as it was a rather untypical one: since the beginning of semester no question had been intended as an exercise in quantified logic and negation. In particular none linked the language of mathematical logic with the ordinary language of 'real-life' problems. Our first impression was that it was odd and rather out of place. Still it seemed to address a worthy issue. As widely reported in the literature (e.g. Laborde 1990) for most learners of mathematics the symbiosis of these two forms of language is a very uncomfortable one. Therefore any pedagogical technique - here: an exercise in a problem sheet - addressing this issue is welcome. However our puzzlement increased at the sight of the solution suggested by the lecturers in their Notes on Solutions, a document distributed to students and tutors once the students have submitted their work:

> The statement: "In each year group in the university there are at least ten students with bicycles" may be written using quantifiers as follows: $\forall j \in \{1,2,3\}$ $\exists$ ten students in year j that have bicycles. The negation is then $\exists j \in \{1,2,3\}$ such that no more than nine students in year j have bicycles. In words " there is a year group in the university with no more than nine students in that years group having bicycles" [There are many different ways to write these answers.]

At one level this suggested solution clarified, partly and tentatively, the question setters' intentions: this may have been intended as a smooth, unthreatening introduction to exercises in logic, with a light use of notation - quantifiers $\forall$ and $\exists$, the index *j* and the rest being expressed in words. As such it was a noble approach demonstrating sensitivity to the students' well-known apprehension about understanding mathematical texts written in purely symbolic language (e.g. Furinghetti and Paola 1991). For instance Dee Lucas and Larkin (1991) found that proofs written in a verbal, ordinary language produced better performance than equation-based proofs on problems related to both equation and non-equational proof content. They also explain that equations cause students to shift attention away from non-equational content and learners have more difficulty processing equations than verbal statements of the same content. Similarly MacGregor (1990) noticed that writing sentences helps students write correct equations and contrary to expectations the most successful students were those who used common idiomatic forms of English that could not be directly translated into mathematical notation. At another level however the suggested solution struck us as bound to cause confusion to the students: it is a quasi-mathematical expression, it provides no answer to the question 'is this a useful way of expressing this idea?' despite making some use of quantifiers and the rules of logical language. In sum: it is an unsatisfying exercise in logic. An example of what we perceived as a satisfying response is the following:

Let s_j be the number of students in Year j that own a bicycle, where $j \in \{1,2,3\}$.

Then: $\forall j \in \{1,2,3\},\ s_j \geq 10$.

Negation: $\exists j \in \{1,2,3\}: s_j < 10$.

Of course the suitability of this, or other, 'satisfying' solutions relies heavily on whether these are within the range of tools available to the students at this stage of their mathematical studies. Not to our surprise, the students' performance was alarmingly disappointing.

How did the students respond to the question? The lecturers may have intended it as a smooth introduction to exercises in logic - and the link to a realistic situation succeeded in supporting the students nearly unanimously correct construction of the negated statement - but, as the failure of all students to provide a satisfactory response shows, their good intentions have not been realised at all. We hope to show here that, at this stage, the students do not possess the tools for completing the task. Placing them at the already blurred interface of ordinary with formal language is unwise or perhaps ought to be done differently (see final section for a comment). Before we move on to presenting our case, let us observe a general attitude in the students' work, established in the work of Alan Schoenfeld (e.g. 1987): the tension between what the students could (and were expected to) do and what they thought they had to do. Students believe mathematics has to be complicated and if the likely answer to a problem seems to be too easy they will treat this answer with disbelief and will move towards a more complicated one.

In general all students, while not necessarily in control of the relevant logic notation, attempted to do away with words and write the required statement fully in symbolic language - a known attitude amongst novice mathematicians (e.g. Nardi 1996) and an unsurprising one given the hitherto (see recent slight modifications to the A'level syllabus in this direction, e.g. EdExcel Syllabus 2000) limited relevant experience with such formalism within school mathematics. Here we present our case on the students' disappointing performance in three examples that were typical of the 16 pieces of written work we examined (and our informal knowledge of the remaining 44).

Example 1 - Nicolas. Most students' responses demonstrated limitations in the ability to write out formally a set and its contents (e.g. omission of curly brackets in defining a set when a set contains non-numerical quantities). E.g. Nicolas:

> (5) Let y = year group of UEA
> s = student with bicycles
> In each year group of UEA there are at least ten students with bicycles. $\Rightarrow \forall y \in \{1,2,3\} \; \exists s \quad s \geq 10$
>
> Negation: $\exists y \in \{1,2,3\} \; \forall s \quad s < 10$
> i.e There is a year group at UEA with less than 10 students who have bicycles.

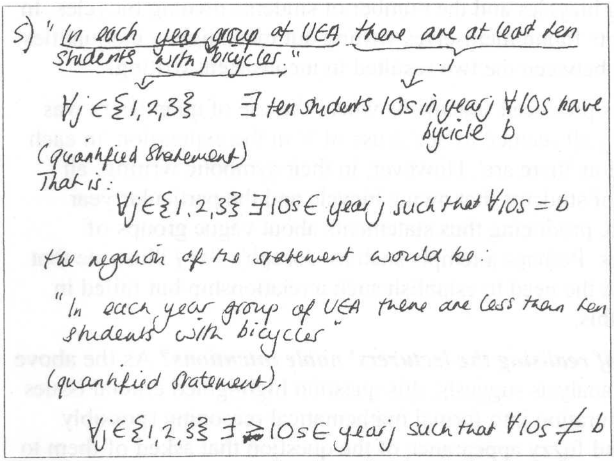
Nicolas' response is characteristically non-verbal but, unlike most of his peers, he felt he had to define y and s before using them. The way these two are defined makes no relationship between them possible or apparent - how can any choice of y have any implication for the choice of s, for example? Moreover, while s is defined as 'student (s?) with bicycles', it seems to denote both this AND the 'number of students with bicycles'. Implicit here seems to be the notion of cardinality of a set but the use is ambiguous. Also = here seems to mean 'define as' but what follows on the right side of = is not a conventional definition of a set.

Example 2 - Joseph. Unlike Nicolas, Joseph defines s as the 'number of students owning bikes', not the 'students owning bikes'. This is perhaps an attempt to bypass the absence of a known-to-him symbol for cardinality but other problems ensue such as his attempt at relating s and j to each other via a puzzling 'belonging' relationship:

> S $\forall j \in \{1,2,3,4\}, \exists s \in j$ such that $s \geq 10$, where s = no. of students owning bikes.
>
> Negation $\exists j \in \{1,2,3,4\}, \forall s \in j$ such that $\exists s < 10$, where s = no. of students owning bikes
>
> In words There exists a year group at UEA for which there are less than 10 students with bicycles

Unlike s, a definition for what 1, 2 and 3 stand for - numbers or Year Groups? - is absent. So, while, because of the definition of s, inequalities such as $s \geq 10$ and $s < 10$ are meaningful, $\exists s \in j$ is puzzling as it seems to suggest that j is a set, and indeed one that s, a number, can belong to. As a result Joseph's problems seem to be different to those of Nicolas (s as both students and number of students; unrelated s and j): while he has conceptualised but not fully materialised the need to relate s and j.

Example 3 - Louise. The above are exacerbated in Louise's alarming response:


5) "In each year group at UEA there are at least ten students with bicycles."

$\forall j \in \{1,2,3\}$ $\exists$ ten students 10s in year j $\forall 10s$ have bicycle b

(quantified statement)

That is:

$\forall j \in \{1,2,3\} \exists 10s \in$ year j such that $\forall 10s = b$

the negation of the statement would be:

"In each year group of UEA there are less than ten students with bicycles"

(quantified statement).

$\forall j \in \{1,2,3\} \exists > 10s \in$ year j such that $\forall 10s \neq b$

Here *j*, *s* and *b* are not defined at all. The definition of *s* seems to be that of a single student. The repercussions of this are disturbing. 10*s* then translates into '10 students'. $\in$ means 'in', that is 'students' belong to *j*, itself a member of {1,2,3} (again are 1, 2 and 3 numbers or Year Groups? - implicit in Louise's writing 'year *j*' seems to be the former). $\forall$10*s* means 'all of these 10 students'. = means 'have' and *b* presumably stands for 'bicycle'. $\forall 10s=b$ then means 'all of these 10 students have a bicycle' and $\forall 10s \neq b$ 'all of these 10 students do not have a bicycle'. Also in the negation, $\exists>10s$ seems to mean 'there exist more than 10 students'. This suggests that, unlike quantifiers and set notation that are relatively novel to the students, a clear mathematical meaning for > and = also escapes a number of students. The difficulty in writing out a sentence in acceptable symbolic language by a student with such an intensely personal (solipsistic?) interpretation and application of mathematical notation must be a surprise to no-one!

In sum, the main difficulties identified in the students' responses were:

- Students used notation for entities such as the (number of) students owning bicycles and Year Groups, without always realising the necessity to define this use.
- When defining these entities as sets, parts of conventional set notation were ignored or inappropriate: for example, {} were missing therefore rendering the meaning of = unclear. $\in$ was often interpreted not as 'is a member of the set' but as 'belongs to' where this 'belonging' interfered with the ordinary notion of belonging located within the specific 'real life' problem (bicycle belonging to student). Also $\in$ was often interpreted as 'in'. An implication of this was that entities of one nature (e.g. students) appeared as being 'in' sets of entities of another nature (e.g. numbers or Year Groups). In extreme, but not negligible in number, cases certain idiosyncratic interpretations of <, = and $\neq$ resulted in unacceptable uses of formal notation.

- A set and its number of elements were often denoted by the same symbol, e.g. s as the set of students owning bicycles and the number of students owning bicycles. In the former case the students found themselves writing out meaningless inequalities such as $s<10$. Fluctuating between the two resulted in inconsistent writing.
- In most cases, the students produced statements where the use of quantifiers was implicitly acceptable - they all seemed to 'see' a use of $\forall$ in the expression 'in each year group' and a use of $\exists$ in 'there are'. However, in their symbolic writing, all failed to link the number of students that own a bicycle and the particular year these students belonged to, producing thus statements about vague groups of students who own bicycles. Perhaps attempts such as Joseph's $\exists s \in j$ illustrate that certain students did realise the need to establish such a relationship but failed in finding a way to express this.

Is there an alternative way of realising the lecturers' noble intentions? As the above cameo incident/microscopic analysis suggests, this question highlighted crucial issues regarding the students' enculturation into formal mathematical reasoning (possibly exacerbated by the sudden and fuzzy appearance of the question that asked of them to improvise non-standard usage of standard forms with which they were only slightly familiar anyway). In fact the appearance of this question in that Cycle's problem sheet provided an opportunity for the tutors to discuss set theory notation and the meaning of mathematical symbols in a concrete context. It seems to us though that this type of question can perhaps be dealt with far more efficiently in a more discursive mode: as it signifies a transition - and an interplay between - ordinary and mathematical language, a transformation of the students' suggestions (via gradually introducing the various relevant logical/symbolic elements, cardinality etc.) could be a suitable approach (given of course that the students are first appropriately 'de-conditioned' from being passive note-takers during lecturers and 're-conditioned' to honouring their part of the discursive contract...). Underlying our proposition is our increasingly stronger belief that students need to be exposed to the necessity and effectiveness of mathematical language, not simply to its authority.

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