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THE QUALITY OF STUDENTS' REASONS FOR THE STEPS IN A GEOMETRIC CALCULATION

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We report on the variety of reasons that students give for each step in a three-step geometric calculation. Where these reasons are non-standard this may be partly due to a lack of familiarity with the appropriate conventions, but it may also indicate that students need to articulate such reasons as scaffolding for their explanations.

INTRODUCTION

In this paper we report on students' responses to a geometry question, G4, which formed part of a written proof test developed for the Longitudinal Proof Project (see, for example, Hoyles and Küchemann, 2000). The test was given to high attaining Year 8 students from randomly selected English secondary schools and a similar test was given to the same students in Year 9. Altogether, 1984 students from 59 schools took both tests. The Year 8 version of G4 contained a multiple-choice component, which was changed to an open-response format in Year 9.

Anderson et al (1997), in a study of naturally occurring arguments in 4th grade classrooms, found that the students' utterances were often vague and with no explicit conclusion, and that they were usually missing, or seemingly missing, explicit warrants to authorise conclusions. They suggest that this is because students take the shared knowledge of the participants as given and not needing to be spelt out, and they go on to suggest that the underlying arguments are usually perfectly sound. Reid (1999), on the basis of observing grade 10 mathematics classes, suggests there are several modes of explaining, including non-explanations (where, for example, students refer to their own or the teacher's authority), explaining how, explaining why, explaining to someone else (spontaneously, or in response to a question) and explaining to oneself (in an attempt to come to a personal understanding).

Completing a written test, for researchers that the students do not know, is clearly different from the classroom activities considered by Anderson and by Reid, but it is nonetheless possible that some of the factors that they identified are operating with the written test. For examples, some students might think they are being asked to explain how they arrived at their answer, rather than why; or they might assume, out of habit or from a lack of familiarity with the conventions of mathematical argument, that some of the knowledge that they share with the researchers (whom they don't know, but who presumably know 'everything') does not need to be made explicit. For our sample, many of the 'inadequate' response to G4 may be largely due to a lack of familiarity with what is required for a mathematical reason in this particular context, although the sometimes thorny issue of how far 'back' one needs to go to justify a mathematical explanation might also play a part. [This is not to suggest that students' difficulties with mathematical explanations are always primarily a matter of familiarity with convention. As we hope to show in an expanded version of this paper, there are questions on the proof test where other factors seem to apply, in particular the need to use 'transformational reasoning' (see for example Simon, 1996) to transmute the givens in the question into something more explicit.]

Students' responses to G4

The Year 9 version of question G4 is shown in Figure 1, together with one of our students' largely successful and well set out responses. In the first part of the question, students are asked to calculate the size of an angle. The standard calculation involves three steps and knowledge of three geometric relationships (in this case, 'angle at a point', 'angle sum of a triangle', and 'base angles of an isosceles triangle'). [It is of course possible to find the angle in other ways, though we only spotted 12 cases where students seemed to be using a viable alternative method.] The Year 8 version is similar, and again involves three steps and the same geometric relationships, except that 'angle on a straight line' is required rather than 'angle at a point'. There is a marked improvement in performance from Year 8 to Year 9, with 54 percent finding the correct value (apart from possible arithmetic errors) in Year 8 and 73 percent in Year 9. Improvements on most of the other geometry items on the proof test were noticeably smaller.

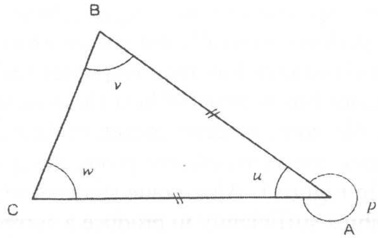
In the Year 8 version, students were then presented with three 'randomly' ordered calculations that represent the standard steps for finding the required angle, which they were asked to list in the order in which they should be carried out. They were then given three geometrical reasons, which they were asked to match to the three calculations. This multiple-choice format was used because we felt that Year 8 students (even relatively high performing students as in our sample) would not necessarily have had much experience of this kind of activity, especially the last part, of giving geometrical reasons for their steps. In the event, these parts were answered successfully by 70 percent and 45 percent of students respectively.

These percentages seem high enough to suggest that this kind of activity could well be fruitfully developed in the classroom with students such as ours. However, the percentages don't tell us how successful students would be at writing out and justifying the steps for themselves. We therefore decided to ask this in open-response format in Year 9.

The change of format raises interesting research issues. The multiple-choice responses were easy to code and in an unambiguous way; also, the resulting frequencies have a seemingly clear meaning (that, for example, 70 percent of the sample could put three particular given calculations in the right order). However, the task is contrived and does not mirror normal mathematical activity, so it is difficult to decide what the frequencies really convey.

In contrast, the open-response form produced a rich variety of responses which gave far more insight into how students tackle this kind of activity but which made the

The diagram shows a triangle ABC.
Side AB is the same length as side AC.



a) Find the size of angle v, when angle p is 320°. 70

Write down each step of your calculation.

360 − 320 = 40 u = 40
180 − 40 = 140
140 ÷ 2 = 70 u = 70 v = 70

b) Write down your first step again and give a reason for the step.

360 − 320 = 40
because I needed to find out the angle of u.

c) Write down your next steps again and give a reason for each one.

180 − 40 = 140
Because a triangle has a total angle of 180° & I wanted to find out what angles u & v added up to.

140 ÷ 2 = 70
Because it ~~each~~ AB & AC are the same length the angles @ the end must be the same.

Figure 1: The Year 9 version of Question G4 and a student's responses

task of coding, and of producing a reliable coding scheme, much more difficult. Though students generally did not have too much difficulty in writing down each step of their calculation, their responses varied not only in terms of the nature of their reasons but in terms of how these were laid out and the vocabulary used. Thus, for example, some students presented their calculations and reasons in the classic format used for a two-column proof, though more often students used a narrative style as in Figure 1. Also, some students who knew the underlying geometric relationships sufficiently to produce a correct series of calculations, were quite casual in their use of terminology, for example writing that the *area* of a triangle is 180° or that two angles of the triangle are the same because the sides are *parallel*. Of particular interest was the finding that students' often gave reasons which were procedural (Reid's 'Explaining how'), rather than, or as well as, giving a geometrical justification for their step. An example can be seen in Figure 1 where, for the calculation 360 – 320 = 40, the student has written "because I needed to find out the angle of *u*" rather than a geometric reason like "The angle at a point is 360°".

We had to make difficult decisions about what to code and what to leave out. In the end, for each step of the standard three steps required to calculate *v*, we coded students' reasons in the following way (Table 1):

code 4	Correct mathematical reason (with or without a procedural reason), written directly next to the appropriate step in the calculation
code 3	Correct mathematical reason (with or without a procedural reason), somewhere on the page but not next to the appropriate step in the calculation
code 2	Incorrect mathematical reason
code 1	Procedural reason only
code 0	No reason for the specific step
code 9	No reason for any of the steps (or 'unclassifiable', eg because step incorrect)

Table 1: Coding scheme used for reasons in G4 (Year 9)

The standard way to find v is to find u (step 1: $360 - p$), then $v + w$ (step 2: $180 - v$) and finally v (step 3: $[v + w] \div 2$). Reasons were deemed to be mathematically correct in the following way. For step 1, reasons had to include the fact that the angle at a point, or in a circle, is 360°. However, we were liberal about terminology, and accepted statements like "The angle at a corner is 360°". For step 2, reference had to be made to the fact that the angle sum of a triangle is 180°. In step 3, we noticed that many students who gave $v = w$ (or the equivalent) as their reason, did not take the extra step of justifying this (by referring to the fact that the sides AB and AC are the same length, or by stating that the triangle is isosceles). We decided to class $v = w$ as an adequate mathematical reason, but also to note whether these students gave a further justification (in the event just over half of them did). [Perhaps not surprisingly, we found no evidence of students giving further justifications for the first two steps.] The frequency of the basic codes, for each step of the calculation, is shown in Table 2, below.

G4 (Year 9)	Step 1 $u = 360 - p$		Step 2 $v + w = 180 - u$		Step 3 $(v + w) \div 2$	
	Number	%	Number	%	Number	%
Code 4 (Correct mathematical reason, linked)	957	**48**	961	**48**	1067	**54**
Code 3 (Correct mathematical reason, not linked)	7	**0**	44	**2**	172	**9**
Code 2 (Incorrect mathematical reason)	23	**1**	29	**1**	41	**2**
Code 1 (Non- mathematical reason only)	637	**32**	406	**20**	210	**11**
Code 0 (No reason)	33	**2**	217	**11**	167	**8**
Code 9 (No response or miscellaneous)	327	**16**	327	**16**	327	**16**

Table 2: Code frequencies for reasons in G4, Year 9 (N = 1984)

It can be seen from the table that for each step, about half the students gave a correct mathematical justification (codes 4 and 3), usually written next to the step in question (code 4). On the other hand, though it is not shown in the table, only 33 percent gave a correct mathematical reason for all three steps. Also, a sizeable minority gave only non-mathematical reasons for one or more steps (code 1).

Step 3 stands out from the other two: students more often gave a correct mathematical reason for this step (albeit often just the modest statement, $v = w$), and less often gave just a non-mathematical reason; however, there was a greater tendency for a correct mathematical reason not to be linked to this step. A possible explanation for all this is that many students regard the angle at a point and angle sum of a triangle properties as facts that can be read directly from the diagram and which are so obvious that they don't need to be stated. By contrast, the property that $v = w$ has to be derived from the particular information that AB = AC. Interestingly, this is one of the first things that many students wrote down in the course of calculating v and it is possible that many students saw it as their first real step in the calculation, and as the key to finding v.

Altogether, 40 percent of the students gave one or more procedural-only reason, but only 5 percent of the total sample gave just procedural reasons for all three steps. This lack of consistency might partly be explained by the special nature of the reason for step 3, as discussed above, but it also suggests that for many students in the sample, the notion of a mathematical reason that justifies a geometric calculation is not well articulated - probably because this has not yet been dealt with in any formal way in school.

It is also interesting to note that a sizeable minority of students (14 percent of the sample) gave another kind of unconventional response, which might be described as 'reflective' or 'strategic'. This happened particularly in step 1, with statements like "I did this to find u so that I could find v plus w", or "I started with 320° because that is the only angle I know". Such statements are more than just procedural and though they are similarly unconventional, they demonstrate a self-awareness that is worth encouraging and which might help students articulate their explanations.

CONCLUSION

The multiple-choice section of the Year 8 version of G4 suggest that many of the students in our sample can make good sense of the task of breaking a geometric calculation into steps and attaching geometric reasons to these steps. On the other hand, the Year 9 results suggest that only a minority of our students have a clear and consistent grasp of what is meant by a geometric reason. It may be that this is largely a matter of learning a convention with which most of our students are not yet familiar; at the same time, it is possible that the very articulation of these non-mathematical reasons provides students with important scaffolding in the construction of more mathematical explanations.

Our students made clear progress on the first part of G4, so that by the end of Year 9, nearly three quarters of the sample could successfully undertake a three-step calculation to find the size of an angle. This is a notable achievement in terms of developing some of the skills needed to produce a deductive proof, as each step can be said to involve a deduction. However, the task is only concerned with finding a specific value of an angle, rather than a general relationship between angles, which is more characteristic of a proof. Our Year 10 version of question G4 involves both kinds of task, and our preliminary findings suggest that there is a substantial difference between them.

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