STUDENTS' PROBLEM SOLVING PROCESSES: TWO DISTINCT CATEGORIES

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Abstract.- This study looks at the problem solving ideas and experiences of a group of students that took part in a problem-solving course. The aim is to identify the essential issues behind students' problem solving processes and to build a model of the situation. This paper discusses the two problem solving categories that emerged from the analysis of the data. These categories were labelled 'panning for gold' and 'building a solution' and it is argued that they represent two different approaches to solving mathematical problems. The data consisted of the 'scripts' that students created for their problem solving processes. These 'scripts' are written descriptions of the process students went through for solving a non-routine mathematical problem and were created by the students during the course.

INTRODUCTION

The study of non-routine mathematical problem solving allows the researcher to look at students in the processes of 'creating' mathematics (Ernest, 1991). As opposed to content-based mathematics, problem solving does not provide the student with ready-made alternatives as to how to tackle a problem. Thus, s/he will face the need of defining viable courses of action and to make decisions about their implementation. From a researcher's point of view, this 'freedom' to react in front of a problem provides data that can be used to learn about students' idiosyncrasies in relation to mathematics.

The idea of "what mathematics really is" and how mathematicians approach it has been discussed by a number of authors (Davis and Hersh, 1986; Ernest, 1998; Hersh, 1997; Lakatos, 1988). In relation to mathematics learners, these issues suggest the question of what views students hold towards mathematics and how these views affect the way in which they do mathematics. It may be said that problem solving—for the reason described above—may provide the researcher with useful information for gaining some insight into the way students approach mathematics.

The results that will be discussed in this paper correspond to a study on problem solving. The participants were undergraduate students that took part in a problem solving course and whose solving processes were considered useful for learning about their approaches towards mathematics. The analysis suggests that students hold different approaches towards problem solving and it may be speculated that their ideas about doing mathematics vary accordingly.

THE PROBLEM SOLVING COURSE

As mentioned above, the research focuses on a problem-solving course offered at the undergraduate level. The main objective of the course is to allow students to reflect on their own mathematical thinking and to identify and develop their problem-solving strategies. Furthermore, the course does not focus on any aspect of mathematics in particular but on mathematical processes in general. Mason, Burton and Stacey's (1985) Thinking Mathematically is used a basis for the course, but other sources are usually used as well.

The data that will be analysed corresponds to the course as offered during the autumn term 2001. In this case, students attended lectures and seminars every week, for 10 weeks. In lectures, they were introduced to Mason et al.'s (1985) framework for problem solving. This framework considers the processes of 'specialising/generalising' and the phases of 'entry/attack/review'. Moreover, the framework suggests a number of strategies for dealing with these phases and processes. As said above, ideas from other sources were discussed as well. For instance, students had the opportunity to discuss DeBellis and Goldin's (1997) views about how affective issues influence mathematical problem solving and to experiment with Schoenfeld's (1992) problem solving 'maps'.

During lectures, students were given the opportunity to solve one or two problems and to discuss the use of the different strategies with their classmates. At the end of each lecture, students were given one or two more problems. They were supposed to tackle these additional problems on their own and to discuss them during the seminars.

Developing a Rubric. During the course students were required to develop their own rubric for solving problems, i.e., their own guidelines for tackling problems successfully. For doing this, they could refer to the strategies suggested by the course or they could create their own. Thus, students were prompted to keep a written record of their solving processes and to discuss the strategies they used and how they used them. Furthermore, it was through these scripts that students shared their problem solving processes and rubrics with their classmates and with the lecturer.

It was observed that the scripts produced for examination purposes contained a clear description of the way students tackled a given problem and the way they solved it (or not). Since the course focused on the solving process, and since the scripts were going to be marked, students were particularly clear in describing how they coped with the problem [1]. The findings presented here correspond to an analysis of the scripts developed for examination purposes during the autumn-2001 course.

THE 'LIOUVILLE' PROBLEM

As mentioned above, the data that will be analysed in this study corresponds to the scripts that were submitted for examination purposes during the autumn term 2001. These scripts contained students' processes for tackling a non-routine mathematical problem, namely, the 'Liouville' problem:

> Take any number and find all of its positive divisors. Find the number of divisors of each of these divisors. Add the resulting numbers and square the answer. Compare it with the sum of the cubes of the numbers of divisors of the original divisors.

The comments that will be quoted in the following section correspond to students' processes for tackling this problem. It may be observed that students began by experimenting with "random" numbers and calculating the quantities specified. They soon noticed that the quantities were true in every case. In general, students set themselves to show that equality was true for every natural number "n" or to try to find a reason why this was the case.

TWO CATEGORIES

This section looks at students' written accounts of their own process for solving a non-routine mathematical problem. An analysis of the data suggested that students' approaches towards the problem varied and could be classified according to two categories: 'panning for gold' (PG) and 'building a solution' (BS). In the case of the scripts classified as 'panning for gold' it may be said that students were following a process in which the overall strategy was to try to find an answer that was 'out there' and ready to be discovered. The scripts classified as 'building a solution', on the other hand, suggested that students tried to make use of the available information in order to build an answer. The main characteristics of these two approaches are explained and a number of examples are discussed.

Panning for Gold (Looking for Key Ideas)

For students classified as following a PG approach, one of the main activities was to look for key ideas. These students' processes and comments suggest that they assumed that the problem contained some 'hidden' characteristics that could be unveiled (and that, after this, the problem would be solved or at least a few steps away from being solved). For instance, Heather described her objective while solving the problem in the following way:

> Right, I know that the number of divisors of divisors seems to make the two calculations in the problem equal. I want to prove this claim. In order to do this, I need to see *what is so special about* the number of divisors of divisors that makes these two calculations equal. In doing so I shall be able to determine whether the number of divisors of divisors are the only positive sets of numbers that satisfy the claim. (Heather, emphasis added)

As shown in this quote, Heather set her strategy as that of finding "what is so special about" the problem. According to her, this would not only help her to prove the equality under discussion but would also allow her to investigate further and extend the problem. In practice, this was not the case since Heather never managed to discover what was "so special" about the quantities to be compared and was not able to conclude the problem satisfactorily.

In George's case, he set his objective as follows:

> I need to work out why the two parts are equal. I know that the prime numbers can be explained. I need to investigate what is so special about the numbers generated by the given algorithm. (George).

Again, he was looking for "something special" about the quantities compared and, like Heather, he failed to discover the key idea behind the situation.

Another student who suggested giving importance to key ideas was Daniel. Commenting about his process, he mentioned how, had he continued trying to solve the problem, he would have focused on "understanding" two key ideas:

> I reflected on progress [before concluding the solving process] and wrote down two questions which I thought key to understanding why Liouville does what it does: (1) Why is it that the patterns I find for $\#D_2$ have this special property that $((\#D_2)^2 = ((\#D_2{}^3)$? (2) Why is it that a number of divisors, $\#D_1$, seems to have a unique corresponding pattern for $\#D_2$? (Daniel)

Daniel seemed to believe that the clue to solving the problem lied on "understanding" the ideas that he mentioned. This was something he did not do as he spent his processes exploring the situation and setting a clear notation and eventually ran out of time. However, these questions and his comments do suggest that the idea of trying to discover key ideas was part of his approach.

Building a Solution (Taking an Active Approach)

An important characteristic of the processes classified as BS was that students seemed to be actively constructing a solution. Their approach was to carry out purposeful operations on the data and the possibly useful facts they deduced. Instead of providing long and detailed explanations to why facts were true, they concentrated on trying to incorporate the new information to their ongoing process. For example, during his process, John devised a method of calculating the number of divisors of each of the divisors (which he later used in order to provide an inductive proof of the equality):

> AHA! I can do this [find the number of divisors of each of the divisors] by using summations over all the possible values which i_k can take. I.e., each possible combination can be covered by summing over all possible powers to which the prime factors of n can be raised, to produce a divisor of n, hence including every possible divisor of n. (John)

In fact, John's reflection at the end of the script made clear his BS approach. In this reflection, among other things, he summarised the activities carried out throughout his process. Overall, his comments suggest that various activities were carried in order to finally build a solution.

> I have successfully managed to investigate the two figures in the question and compare them in different numerical examples. I then looked at the case of a single prime number and proved equality. I then wrote any number n, in the form of a product of primes, this enabled me to determine its divisors easily and produce a formula for the number of divisors of a general divisor of n. I then needed to look at all possible positive divisors of

> n, I calculated X and Y by summations over the possible ranges of indices for each prime factor of n, this enabled me to simultaneously consider every possible divisor of n. I then used algebraic manipulation and two standard results to prove that X=Y, i.e., that equality holds between the two figures described in the question for any value of n. (John, p.12)

As expected, students that follow an active approach also fall in dead ends and frequently find themselves 'stuck'. A common situation is that students may find themselves working with overcomplicated calculations that force them to reconsider their strategy. Looking at Frank's process, for instance, he tried different routes before being able to calculate the number of divisors of the original divisors. He actively tried to construct an 'algorithm' for calculating the required quantity and it was only after several attempts that he was able to construct one. The following quote gives an overview of Frank's work:

> So I know that I can obtain the number of divisors that have a particular form and I know how many divisors that a divisor of a particular form will have. So I can multiply these two numbers.
>
> [...]
>
> Then the number of numbers of this form is... STUCK! This time in a big way. I keep trying to come up with a general formula for the number of divisors of divisors and failing.
>
> I know that a number when split in its prime factorisation is of the form $x=p^r$ then it will have r+1 divisors. These divisors are 1, p, p^2, ..., p^n and the number of divisors of these divisors is 1, 2, 3, ..., r+1. AHA! I've now seen how these summed and squared might give the cubes summed. If I consider summing the series 1, 2, 3, ..., r+1 I know from A-level maths I can find the formula. I can then square this formula. I should then be able to find the formula for $1^3+2^3+...+(r+1)^3$ and it should match. (Frank)

It may be said that Frank was putting different pieces of information together in order to construct the piece that was missing. He tried several avenues before he could actually achieve the result he wanted. In each of these, attempts he actively gathered or deduced possibly useful facts and then tried to relate them in a convenient way. In this sense, like John, he was taking an active approach.

CONCLUSION

The two categories discussed in this paper suggest that students approach mathematical problem solving in different ways. On one hand, students may look for key ideas and in this way expect to come across a piece of information that will directly lead to the solution. On the other hand, students may approach the problem as a process of constructing a solution. In this case, students adopt an active approach in the sense that they focus on transforming (as Lesh, Post and Behr [1987] used the term) the information that they are able to gather.

This categorisation allows the researcher to classify students' processes as 'panning for gold' or 'building a solution'. Moreover, it raises the question of whether students who follow a, say, 'panning for gold' process tend to work in the same way whenever they approach a problem, and vice versa.

It may also be speculated that both approaches can be useful for problem solving. In fact, it seems that 'panning for gold' may be an efficient approach for tackling the "types of problems that students [are] asked to solve at school"—to put in Lester's (1994, pp. 663 -664) words. However, when it comes to less well-defined problems, students may need a 'building a solution' approach in order to derive a solution from the problem.

NOTES

1. The advantages and disadvantages of using scripts produced for examination purposes were considered in De Hoyos (2002).

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