Different levels for basic knowledge and skills assessment and mathematical rigorousness

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China's *Mathematical Curriculum Standard* is a programmatic document issued by the national education authorities. In the *Mathematical Curriculum Standard*, comprehensive recommendations have been proposed for teachers on how to carry out assessment. Therein, for assessing the results from students' learning of basic knowledge and skills, four different levels have been proposed, such as "understanding, comprehension, mastering and application". We'll use some mathematical examples to explain such four different levels and discuss the mathematical rigorousness embodied in such examples.

Keywords: assessment; mathematical curriculum standard; mathematical rigorousness; mastery

Assessment recommendations

As a teaching guidance document issued by the competent national education authorities, China's *Mathematical Curriculum Standard* has two books (Ministry of Education of the People's Republic of China, 2012a, 2012b). One is suitable for the phase of the nine-year compulsory education, including the phase of elementary education (usually at the age of 6 to 11) and the phase of junior high school education (usually at the age of 12-14); the other is applicable to high school students (usually at the age of 15-17).

The *Mathematical Curriculum Standard* mainly consists of such three parts as the course objectives, course contents and implementation recommendations. Therein, the implementation recommendations also cover such contents as the teaching recommendations, assessment recommendations and textbook compilation recommendations.

In the assessment recommendations, the main purposes of the assessment have been proposed as follows:

> Fully understand the process and results of students' mathematical learning for motivating students' learning and teachers' improvement on their teaching. The assessment should follow the specific objectives and requirements in each phase of studying as the standard to examine students' understanding and mastering of the basic knowledge. While assessing the results from students' basic knowledge and skills learning, we should accurately grasp the requirements for such different levels as the "understanding, comprehension, mastering and application".(Ministry of Education of the People's Republic of China, 2012a, p.52)

Four different levels for basic knowledge and skills assessment

We will explain such four different levels based on specific examples from the Senior High School Entrance Examination(SHSEE) Paper (Beijing) 2016. We have taken the Paper examples just because of its authority. As the final exam of the nine-year compulsory education stage, the SHSEE examines the students' ultimate mathematical achievements in this phase of studying. Therefore, it is an achievement examination to test whether junior high school students have achieved the corresponding academic level and also a recruitment examination for senior high school students.

While explaining the four different levels, we'll also talk about the rigorousness of mathematics.

Understanding (Synonyms: knowing, speaking out, recognition and identification): *Know about or illustrate the object relevant characteristics from specific examples; recognize or illustrate the objects from specific situations based on their characteristics*.

Let's look at the following examples from SHSEE(Beijing) 2016.

Example 1: The oracle bone inscriptions are a kind of ancient characters in China as the early form of Chinese characters. Which of the following inscriptions is not axisymmetric?

Figure 1: The oracle bone inscriptions

Here the knowledge point examined is the concept of the "axis of symmetry". The *Mathematical Curriculum Standard* requires "understanding" of the concept. For the concept, we should not necessarily let students give a strict mathematical definition. The intuitive understanding is that, if the two halves of a graph can be completely overlapped when folded along a straight line, then the graph is axisymmetric. The "folding in halves" is just a descriptive term rather than a mathematical one, but it can enable students to understand this concept in an intuitive manner. For this question, the choices A, B and C are axisymmetric, while D not. If in class, we can design such questions as follows to promote students' thinking:

> Question 1: Find out the axis of symmetry in the graphs A, B and C.

In fact, most students can easily find out the axis of symmetry. However, we really intend to put forward the question below.

> Question 2: Give your reason why the graph D is not axisymmetric?

Many students will immediately give an explanation:

> The choice A, B and C are axisymmetric and the answer is unique to the question as required, so we can only choose D and thus the graph D is not axisymmetric.

For students' answers, first we should give a positive response:

> Your logic is correct and flawless! But that's the logic of problem solving, but not a mathematical explanation.

We can further ask:

> If this is not a multi-choice question and no options A, B or C is provided, how can you directly determine whether the graph D is axisymmetric or not?

This question is rather difficult. To prove a graph is axisymmetric, we only need to find an axis of symmetry, but if we want to prove a graph is not axisymmetric, do we need to prove that each line is not the axis of symmetry? And unfortunately, there are innumerable straight lines on a plane and such lines can not be enumerated.

When putting forward this question, we don't intend to get students' solutions and in fact, the question has been beyond the requirement for "understanding" of the concept. However, we really intend to guide students to think (even if they can't solve it within the current scope of knowledge), and after all, the question is really in existence. Moreover, we also intend to let students experience the rigorousness of thinking. It is reasonable that we can't solve some questions, but we can't deny their existence. This is a good attitude towards learning.

In regard to this question, we should also give students necessary explanations and especially encourage those interested further to think about it. It is necessary to explain to our students as follows:

> We can't solve this question now just because the concept of "axial symmetry" is defined by means of geometric intuition while the "folding in halves" is a descriptive statement. First of all, we should use a mathematical language to depict such kind of concepts before further study. Moreover, the study of this question may involve some higher concepts, such as "transformation".

Comprehension (synonyms: cognition, capability): *Describe the object characteristics and origins and simultaneously explain the difference and relationship between one object and relevant others*.

Example 2: If the real numbers a and b are located on the number axis as shown in the figure, then the correct conclusion should be

A. $a > -2$ B. $a < -3$ C. $a > -b$ D. $a < -b$

Figure 2: The real numbers a, b located on the axis

Here the knowledge point examined is the concept of the "real number and real axis", which should be "comprehended" as required in the *Mathematical Curriculum Standard*. In fact, the real number is not a very simple concept, although it has been introduced to the students since the beginning of their junior high school study. To completely understand the concept of real numbers, we should get help from the tool of the "limit", which, in China, won't be taught until college study.

For junior high school students, we should let them have an intuitive understanding of the real numbers:

> All real numbers feature one-to-one correspondence to the points on a directed line, namely, a number axis. The order of the real numbers is defined on the basis of the direction of the axis. Geometrically, the distance can be defined between two points on a straight line, so that a sense of the distance has come into being between two real numbers, namely the concept of the absolute value. Moreover, the concept of opposite numbers can be viewed as two numbers symmetric about the origin.

If a student can know about the above-mentioned points, it can be concluded that he/she has "comprehended" this concept. For this question, first look at the number axis, then work out the value ranges of a and -b correctly and then we can compare with the choices and achieve the answer.

If students can well comprehend the concept of the number axis, it will help them learn irrational numbers. In terms of the concept of irrational numbers, we can introduce it as follows. First of all, we can ask our students:

> If a right triangle has its two right-angle sides both equal to 1, then what is the length of its hypotenuse? Please mark this length on a number axis.

We can let students use compasses and rulers to complete the following graphs. As shown in the figure, the hypotenuse of the right triangle is the line segment OC. Take the point O as the origin and take the OC length as the radius to draw a circle intersecting the positive axis at the point A. Consequently, the coordinates of the point A should be the length of the hypotenuse.

By virtue of construction, we can let the students know that the length of the hypotenuse is a number really in existence and the number corresponds to the point A on the number axis. Moreover, this "number" is objectively real. However, this "number" is different from rational numbers. A rational number can be expressed as the fraction of two co-prime integers. We can ask students:

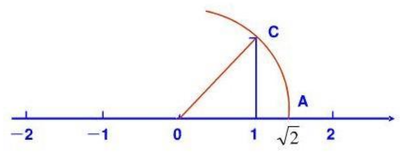

Figure 3: demonstration of irrational number using the number axis

> Can we express the number corresponding to Point A as the fraction of two co-prime integers? (or, equivalently, Can we express the root of the equation $x^2-2=0$ as the fraction of two co-prime integers?)

We can lead students to prove that the above-mentioned question is negative by means of the proof by contradiction. In other words, the root of this equation is not rational. Then we can introduce the concept of irrational numbers.

We take advantage of the number axis to make students understand irrational numbers, because irrational numbers are not simple or direct response to empirical facts. At the stage of elementary school, students obtained the concepts of natural numbers and fractions respectively by means of counting and measurement. These concepts are direct responses to empirical facts in life. However, the irrational numbers are revealed through logical reasoning based on such empirical facts. Preparations are needed when we ask students to accept new "things" beyond those facts.

Mastery (Synonym: competence): *Apply an object to a new situation on the basis of comprehension.*

Let's look at such a question as follows. The Mathematical Curriculum Standard requires that the students should be "able" to use the discriminant to judge the root status of a quadratic equation and "able" to solve the quadratic equation.

Example 3: The quadratic equation $x^2+(2m+1)x+m^2-1=0$ with one unknown *x* has two unequal real roots.

(1) Work out the value range of *m*;

(2) Write out an *m* value satisfying the condition and work out the roots of the equation accordingly.

The quadratic functions and equations are very important topics in junior high school mathematics. To solve the quadratic equation $ax^2+bx+c=0 \quad (a \neq 0)$, the method used is called as the "method of completing the square", i.e., the equation can be equivalently transformed into such an equation as follows:

$$(x+\frac{b}{2a})^2=\frac{b^2-4ac}{4a^2}$$

It is not recommended to ask students to directly recite the quadratic formula when they start to learn the contents in this part, but let them become skilled in algebraic operation by using the "method of completing the square". Then, use specific examples to let the students find out that the root extraction operation is really meaningful only when $b^2 - 4ac \geq 0$. Thus, let them understand that $b^2 - 4ac$ has determined whether the equation has real roots or not, so it is called as the “discriminant”. In the example above, it requires that the equation should have two unequal real roots, so the discriminant needs to satisfy

$$b^2 - 4ac = (2m+1)^2 - 4(m^2 - 1) > 0$$

Thus, the question has been transformed into solving a linear inequality of one unknown. The *Mathematics Curriculum Standard* also requires that the students should be “able” to solve any linear inequality of one unknown.

Students can “master” the solution procedures of quadratic equations, which is the premise of learning complex numbers. When $b^2 - 4ac < 0$, the equation has no real solution. For example, let’s look at such an equation as follows: $x^2 = -1$

Obviously, it has no real solution. Then we introduced the concept of imaginary numbers commonly reasoning as follows: In order to let $x^2 = -1$ have solutions, we introduced a new number “i” (called as an imaginary number) and regarded it as the root of this equation.

Such a way of introduction is simple and direct. But still there are some puzzles about it. This reason is not easy for some students to accept. They may think it not reasonable. It might make them feel uncomfortable, or even unsafe. Some students may ask:

Why do we have to make this equation solvable?

Yes, why do we have to make it solvable? Of course, we have to admit that this way of introduction can make some students feel anxious and this kind of anxiety exists objectively. This is the feeling from the pursuit of rigorousness.

The author also encountered this problem at school. Moreover, such a sense of anxiety has not disappeared before I have learned abstract algebra and used the structural view to understand math. Later, when studying in the department of mathematics, I found that many classmates had had this feeling of anxiety. I felt relief that I was not alone in my anxiety - I was not an alien! We should confront such a sense of anxiety.

We should be observant in teaching. We should encourage students and tell them there is a better way to explain imaginary and complex numbers, but they should learn more about math. For example, we can also suggest that they study “symmetry and group theory”, “field extension” and other special subjects, which are optional ones in senior high schools. Guide them to study this problem in the view of algebra.

Application (Synonym: proof): The “application” is the highest level and top requirement. It means that *students should comprehensively use the obtained objects to choose or create appropriate methods to solve problems*.

Let’s take the following proof question as an example.

Example 4: As shown in the picture, the quadrilateral ABCD is a parallelogram and Line AE halves $\angle BAD$ and intersects with the extension of Line DC at Point E.

Prove: DA = DE

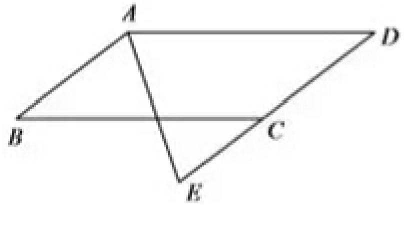

Figure 4: polygon ABCDE

It requires that students should be able to “apply” the parallelogram-related theorems to prove this proposition. Many math professionals regard the plane geometry as their first love, because it is their first experience in rigorous thinking. In the world of the plane geometry, everything is clear, neat and well-organized, able to give a sense of security to those incomers and simultaneously let them experience a sense of beauty. The significance of having students complete the proof questions in plane geometry is also to make most people have a simple idea of “proof”. At the same time, students experience the power of the axiomatization methods and accordingly improve their mathematical literacy in the process of question proving. It is not difficult to prove Example 4, but certain training should be necessary if the proof procedures can be written out normally based on sufficient reasons and proofs.

Students' weird questions

In fact, no question is bad. As long as a question exists, it is reasonable. Some are called “weird” questions just because they tend to challenge teachers. Let's look at the two functions below:

$$f(x) = \sin^2 x + \cos^2 x; \qquad g(x) = 1$$

Some students may put forward such a question as follows:

> Are these two functions the same?

There's no doubt that by using the identity $\sin^2 x + \cos^2 x = 1$, the two functions are equal. However, the students who presented this question had an opposite opinion. They don't think the two functions are the same, because the function is defined as follows in senior high school textbooks:

> *Let A, B be two non-empty sets. A function f is a RULE that assigns to each x in A a unique element y in B.*

We can note that there is a word “rule” in this definition and they think that the RULE of the two functions are different, so they are different from each other!

Actually, what they said is reasonable. In fact, the definition above is just a descriptive (not formalized) one and the word “rule” is not a mathematical term. That's why the ambiguity arises. However, such an unformalized definition adopted in textbooks is to give students a kind of intuitive understanding while their limited knowledge is taken into consideration.

Moreover, the thinking is fairly rigorous for those students proposing such questions, which we should pay more attention to while assessing students’ performance; simultaneously, we should encourage them to put forward such questions, but not say to them, “The textbooks are always right”. In fact, the development of mathematics is to discover the ‘unobvious’ hidden in the ‘obvious’.

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