Developing a concrete-pictorial-abstract model for negative number arithmetic

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Research findings and assessment results persistently identify negative number arithmetic as a topic which poses challenges to learners. This study aims to build on existing research which identifies four conceptualisations involved in negative number arithmetic: unary, binary, and symmetric operations, and magnitude, to develop a concrete-pictorial-abstract (CPA) model which uses the vertical number line and ‘number bar’ manipulatives as the key representations. A trial group (n=7) who were taught using the CPA model are found to have made significantly greater increases in post-assessment scores (p=0.025<0.05) compared to a control group (n=5) who were taught using a non-CPA approach. Analysis of results reveals that the trial group made significant increases in post-assessment scores in addition (p=0.034<0.05) and subtraction (p=0.040<0.05) with negative numbers.

Keywords: negative number arithmetic; concrete-pictorial-abstract; vertical number line; instructional manipulatives; representations.

Introduction

Existing research finds that that the multiple roles of the negative sign, as a *unary*, *binary*, and *symmetric* operator, present a fundamental challenge for learners (Vlassis, 2008; Bofferding, 2010). The *unary* conceptualisation is the idea of a negative number as a position on the number line below zero; the *binary* conceptualisation corresponds to the subtraction operation; and the *symmetric* conceptualisation is the idea of a directed magnitude: the *opposite* of positive. Research has shown that students’ application of these different, and often ambiguous, conceptualisations can lead to incorrect reasoning in calculations. A fourth conceptual challenge arises from the fact that while positive numbers can be visualised in terms of the cardinality of sets, negative numbers cannot. Tatsuoka (1983) identified 89 different rules that students used to erroneously carry out integer addition and subtraction calculations, so a major challenge for teachers lies in effectively addressing these conceptual elements in our instructional models.

In UK secondary schools, negative number arithmetic is conventionally taught as a set of rules such as ‘adding a negative number is equivalent to subtracting a positive number’ and often coupled with analogies like ‘adding cold water to a bath makes it colder’. Our models and analogies often fail to represent the underlying concepts properly and this leads to the development of misconceptions. For example, the commonly heard phrase ‘two minuses make a plus’ does not specify the type of calculation to which this applies. When I asked a year 10 student in my class to explain their reasoning behind their statement $-3 - 7 = 10$, they responded by saying “a minus and a minus makes it a plus”. Askew and Wiliam provide a key observation here when they state that “all pupils constantly ‘invent’ rules to explain the patterns they see around them." (1995, p.12). This may explain why negative number arithmetic is a topic so fraught with misconceptions; if the instructional model does not address conceptual understanding, students may actively seek out rules and justifications which may not necessarily be conceptually sound.

Teaching for conceptual understanding has long been a focus for mathematics education, and in recent years the concrete-pictorial-abstract (CPA) approach has been identified as a central feature of successful mathematics curricula that place an emphasis on conceptual understanding (DfE, 2012). Broadly speaking, the CPA approach involves first exploring a concept through the use of concrete manipulatives; next representing that concept pictorially; and finally representing the concept symbolically. CPA models have been shown to be effective in teaching negative number concepts to sixth-grade students in Turkey (Altiparmak and Ozdogan, 2008) and algebra to middle-school students in the USA (Witzel, 2005).

The use of the number line to teach negative number concepts is a subject of debate (Heeffer, 2011); research highlights the fact that no single representation seems to adequately represent all negative number operations (Kilhamn, 2008) and that manipulatives and representations can be contrived (Bofferding, 2010, p.1). One issue with representations in general is their inconsistent use; the number line, for example, is a ubiquitous feature of the mathematics classroom and it is taken for granted that it will be used by teachers and students in a consistent and conceptually coherent manner. But the number line forms a part of a *semiotic representation* (Duval, 1999); an overall model which consists of the number line itself, coupled with an understanding of what concepts it represents and how it represents these concepts, and "confusion is very likely if … the nature of the number line model is not understood and the constituting aspects of its nature are not clearly recognised." (Heuvel-Panhuizen, 2008, pp.25-26). Kilhamn (2008) notes that some teachers use several different models simultaneously during a lesson and this variety can confuse students. So, even if the representation is an effective one, it is possible that its inconsistent use renders it ineffective.

This study aims to develop a semiotic representation using the vertical number line and 'number bar' manipulatives as the key representations, for addition, subtraction, multiplication, and division with negative numbers. The 'number bars' we have designed are card rectangles, with a positive number printed on one side and its negative on the other, upside down, so that 'flipping' the bar negates the number.

Method and research questions

The participating students (n=12) were aged between 14 and 15 years, forming two existing classes in a UK secondary school which were both ranked fifth out of five in the year group, based on prior attainment. The trial group (n=7) was taught the topic of negative number arithmetic using the trial CPA model (8 hours in total), and the control group (n=5) was taught using a conventional non-CPA approach (10 hours in total). Both trial and control groups completed a pre-assessment prior to studying the topic and a post-assessment afterwards. This research was guided by the two following questions:

1. Is there a difference in change in score from pre-assessment to post-assessment between trial and control group? We assume the null hypothesis that the mean change in scores is equal for both trial and control groups.
2. Is there qualitative evidence that trial or control group demonstrate conceptual understanding? Lessons were observed in-class; filmed and observed by

researcher and trial group teacher; and post-lesson discussions were held with

Table 1: Design of the concrete-pictorial-abstract model for negative number arithmetic

		Conceptualisation	Representation	Rationale
Concrete		Unary, symmetric, binary, magnitude	Thermometer	Position, direction, and movement may be represented in concrete contexts.
			Sea level	
			Building section with basement levels.	
Pictorial		Unary, symmetric, binary, magnitude	Vertical number line.	Direct abstraction from the concrete representations. Red: positive, blue: negative.
		Symmetric, binary, magnitude	Number bar manipulatives.	Sign as direction and magnitude as distance from zero. Red: positive, blue: negative.
Abstract	**Vocabulary**	Unary, magnitude	Higher Lower Zero	To avoid possible confusion over the terms *smaller*; *larger*; *greater*; *less*. Zero as a position on the number line as opposed to a zero-quantity. Negative numbers ‘count downwards’, away from zero. Misconceptions about magnitude, e.g.: a student might think that (–7) > 4 because 7 > 4.
		Symmetric, magnitude	Negative Opposite Direction Up Down	The terms *negative* and *opposite* are equivalent. The sign of a number denotes the direction in which it is pointing. Avoiding use of the term *minus*, which could be confused with the subtraction operation.
		Addition (binary)	Addition Forward	Addition as forward movement.
		Subtraction (binary)	Subtraction Backward	Subtraction as backward movement.
		Multiplication (binary)	Multiplication Add on … times. Lots of	Multiplication as repeated addition.
		Division (binary)	Division Take away ... times. How many times does … go into …?	Division as repeated subtraction.
	Notation	Unary, symmetric, binary	Use of brackets around negative numbers.	To distinguish between the negative sign of a number and the subtraction operation.
	Formal rules	Addition (unary, symmetric, binary)	Adding a negative number is equivalent to subtracting a positive number.	Procedural fluency based on conceptual understanding.
		Subtraction (unary, symmetric, binary)	Subtracting a negative number is equivalent to adding a negative number.	
		Multiplication (unary, symmetric, binary)	The product of an even/odd number of negative numbers is positive/negative.	

both trial and control group teachers.

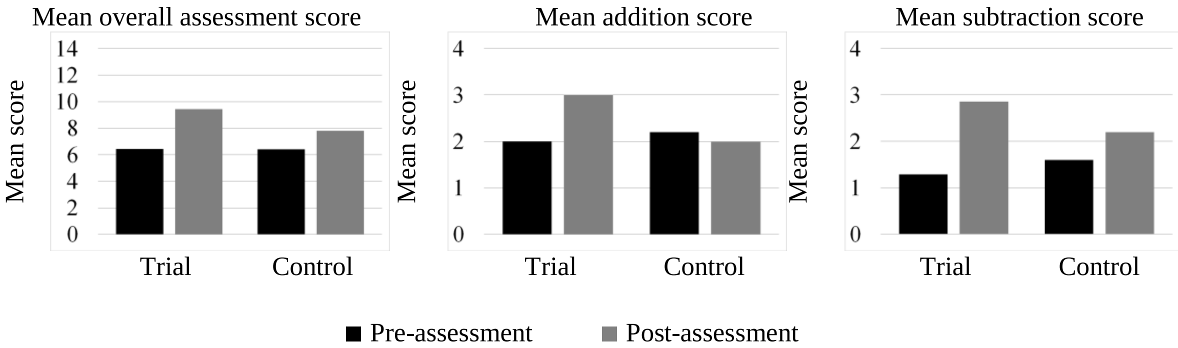
		Division (unary, symmetric, binary)	The quotient of an even/odd number of negative numbers is positive/negative.	

Results

Assessment scores

The changes in scores from pre-assessment to post-assessment were compared in relation to six skills: ordering numbers, addition, subtraction, multiplication, and division. A two-sided *t*-test suggests that the mean change in score was higher for the trial group in the areas of overall score ($p=0.025<0.05$), addition ($p=0.034<0.05$), and subtraction ($p=0.040<0.05$). In the areas of ordering numbers, multiplication, and division, there was no significant difference between groups.

Figure 1: Comparison of pre-assessment and post-assessment scores



Lesson observations and post-lesson discussions

During the addition and subtraction lessons, students were able to work for extended periods of time without asking for the teacher to check that their work was correct, assessing their own work against the visual representation. This is something that the teacher noted as being unusual for this class. The worksheets from the lessons also provide evidence that students were using the number lines correctly, and then continuing on to answer more challenging questions without the use of number lines. This suggests that some internalisation of concept may have taken place.

The consistent use of vocabulary and notation became an unexpected motivating factor in lessons, with students enjoying the opportunity to correct each others' use of the word 'minus' instead of 'negative', and to suggest where brackets should be written in order to distinguish between the roles of the negative sign.

The trial group teacher noted that the overall level of student engagement was improved, although this may have been due to the observer effect.

Discussion

While the model was effective in representing addition and subtraction with negative numbers, it becomes complicated as a representation for multiplication and division.

The concept of 'negative as opposite' was effective in developing students' understanding of the double-negative; the double-negative was not explicitly included in the pre- and post-assessments, however, so it is unclear what the students' exact starting points were with this concept. It was discussed that this may be a useful conceptualisation for multiplication and division. This, however, would rely on secure understanding of the associative property of multiplication, and that $(-a) = (-1) \times a$; so, for example, $(-4) \times (-3) = -(-(4 \times 3)) = -(-12) = 12$, that is, 'the opposite of 4 multiplied by negative 3 is equal to the opposite of negative 12'. It would also be important to consider the use of brackets here in order to maintain consistency but also to avoid over-complicating the notation.

A continuation of this research might also investigate the finer aspects of the model, for example, the effect of using a vertical number line as opposed to a horizontal number line; the efficacy of the 'number bar' manipulatives in representing negative number concepts; and how the 'number bar' manipulatives might fit into students' understanding of well-established models such as Cuisenaire rods and the bar model.

Reflecting on the design and trial process also raises the question: have we just created another analogy? Does the model represent the underlying mathematical structure of negative number arithmetic, or have we conceptualised negative number arithmetic so that it fits our model? It may be possible to answer this question by conducting further trials with students who have no prior experience with negative number arithmetic.

Conclusion

The results of this study suggest that students who were taught negative number arithmetic using a CPA model, based on the vertical number line and 'number bar' manipulatives as key representations, increased their scores in post-assessments by a significantly greater number of marks than students who were taught using a non-CPA approach, in the areas of addition and subtraction with negative numbers. There is also qualitative evidence to suggest that the CPA model had a positive effect on student confidence, independence, and engagement. The model would require further development in order to effectively represent multiplication and division, possibly by utilising the conceptualisation of 'negative as opposite'.

Whether or not the model does turn out to be successful, the systematic design process which draws extensively on existing research has, at least, raised some valuable ideas regarding the ways in which we design our instructional models.

Acknowledgements

This research was funded by Nottingham Trent University Small Scale Research Projects 2016. The authors would like to thank the participating teachers Joshua Cutts and John Hammond.

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