Before *Pythagoras*: A brief cultural history of the *Pythagorean relation*

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Pythagoras' theorem is standard fare for schoolchildren around the world. Often presented as a result to be memorised, rather than appreciated (it is a remarkable result), understood and proved. Rarely do learners have the opportunity to find out about the extensive history of this important mathematical result, or to question whether *Pythagoras* actually existed. In this paper, we highlight some sources that might allow teachers to offer a different experience.

Keywords: Pythagoras' Theorem; history of mathematics

Introduction

As a standard part of the school mathematics curriculum, *Pythagoras' theorem* is often presented to students 'algebraically' or 'geometrically', but rarely with any particular motivation or context. In this paper, we offer some insights into the long and varied history of the *Pythagorean relation*, exploring its origins, development, and the meaning of its current status as a 'theorem' in mathematics. We will discuss the opportunities for the mathematics curriculum and implications for the classroom.

The result is established in Euclid's Elements (c. 300 BCE), Book 1 Proposition 47 without any reference to Pythagoras (or theorem). Since Mediaeval times, the *Theorem of Pythagoras* has been a staple of school mathematics. One of the earliest English records of the result attributed to *Pythagoras* dates back to 1725 (Stone, 1725). Geometry was one of the seven liberal arts taught in mediaeval universities; however, the knowledge of the relation between the areas of squares (or any similar shapes) on the three sides of a right-angled triangle, has been known for at least 3500 years.

A simple, modern definition of a theorem (Oxford English Dictionary) is a general proposition, not self-evident, but proved by a chain of reasoning; a truth established by means of accepted truths. The 'accepted truths' are called axioms. Axioms are statements generally accepted to be true that form the basis of an argument. According to this definition, the result known as *Pythagoras' theorem* is not a theorem.

There is considerable controversy over whether *Pythagoras* actually existed, all accounts of his life come from writers who lived at least 600 years after he is claimed to have lived, although a quick trawl of the internet would reveal much to perpetuate the myth of this vegetarian Greek philosopher.

Having established the name is wholly incorrect, we look to see how humans have known and worked with this surprising relation concerning the areas of similar shapes (often squares) on the sides of right angled triangles – have you ever looked at what happens for similar shapes on the sides of non-right-angled triangles?

Ancient Sources

Considerable research over the last thirty years or so in archaeo-astronomy, archaeology and evolutionary linguistics has increased our knowledge of ancient peoples, their cultures and languages, so that we can now see them as living societies with seasonal customs and regulated daily lives.

In Europe, ancient artefacts surround us, from the islands of Scotland, through England to Brittany, we can see stone circles of various sizes that have ritual significance. Aligned to the rising and setting of the sun and the appearance of other celestial bodies, they have been investigated to show that Neolithic people (from about 10 000 BCE) already possessed a knowledge of the construction and area properties of right-angled triangles. Megalithic man discovered and recorded in stone, many (*Pythagorean)* triangles that were right-angled all with integer sides: 3, 4, 5; 5, 12, 13; 8, 15, 17; 7, 24, 25; 20, 21, 29; 12, 35, 37. Ten sites are known with peculiarly egg-shaped shaped rings (figure 1). They can be classified into two types both of which are based on a *Pythagorean* or 'near *Pythagorean*' triangle.

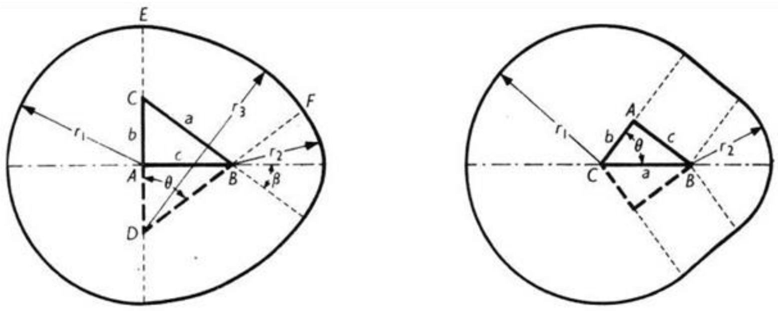


Figure 1: Egg shaped stone rings (Thom, 1971)

There are many locations around the world, where the conditions favoured early agriculture. The Nile in Egypt flooded regularly depositing fertile silt, in Iraq where the Tigris and Euphrates rivers run from the Taurus Mountains to the Persian Gulf formed the 'Fertile Crescent' (figure 2) that provided similar opportunities for land measurement after seasonal flooding.

Figure 2: Early agriculture c. 9000 BCE (BBC, n.d.)

The earliest measurement of the heights above the horizon of celestial bodies for regulating time seasons, and ritual purposes can be found in ancient astronomy (Plofker, 2009; Imhausen, 2016). In the Babylonian culture, the earliest writing was mathematics. Metrological systems developed to deal with social administration, land measurement and taxes (Robson, 2008). The Babylonian approach for completing the square to solve quadratic equations is discussed in another article (Rogers & Pope, 2015).

The Sulba-Sutras of India

In the Indus Valley of NW India, the Vedic People developed a culture from about 3000 BCE that involved building altars of various geometrical shapes and sizes according to specific rules, including representations of animals. These procedures are described in the Sulba-Sutras, part of the Vedic literary corpus, now part of modern Hinduism.

Determining the exact East-West line (figure 3) at a given place was a pre-requisite for all constructions, be it a residence, a temple, a sacrificial altar or a domestic fireplace. The only tools used were 'peg and cord'. Many of the instructions implicitly contain transformation rules for preserving areas, such as changing a rectangle into a square of the same size, doubling a square, or 'squaring' a circle, etc.

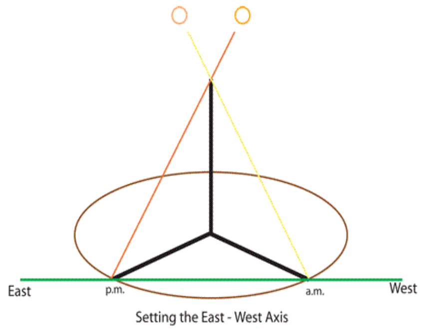


Figure 3: Establishing the exact East-West line (Plofker, 2009)

Once this line is established, other objects can be described using just a peg and a cord (figure 4).

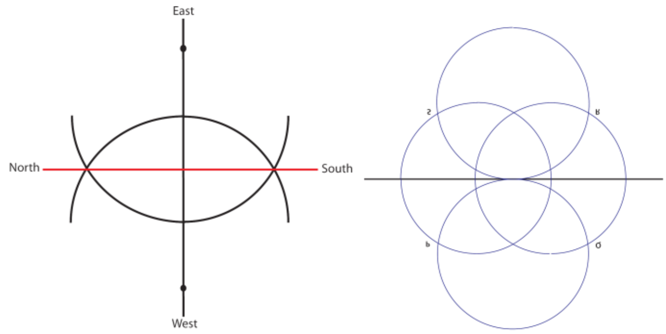


Figure 4: Constructing other objects on the East-West line (Plofker 2009)

Many of the instructions for creating altars contained rules for preserving areas, such as changing a rectangle into a square of the same area, doubling a square, or combining two squares that are not of equal area (figure 5, Rogers, n.d.a).

> Katayana Sulba-Sutra: The rope which is stretched along the length of the diagonal of a rectangle produces an area which the vertical and horizontal sides make together.
>
> Baudhayana Sulba-Sutra: If it is desired to combine two squares of different measures, a rectangle is formed with the side of the smaller [square] [as breadth] and that of the larger [as length]; the diagonal of the rectangle [thus formed] is the side of the combined square.

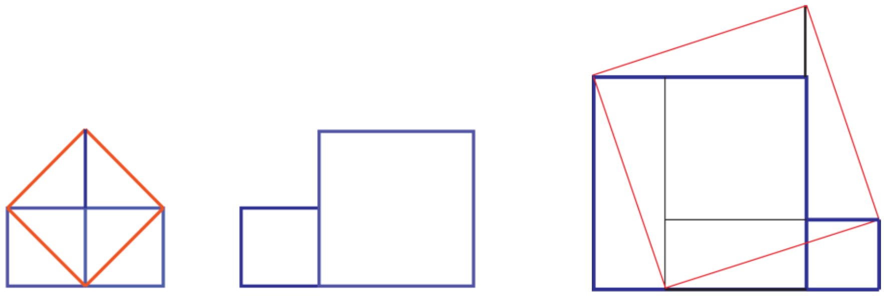


Figure 5: Combining squares of equal area and unequal area

In early Indian mathematics, the transformation of areas was effected by constructing objects with drawing instruments, the 'proof' of the correctness of the drawing was intrinsic to the act of construction. By the 12th century, the *Pythagorean* relation in geometry and arithmetical form was so well-known that Bhaskara II (1114-1185) described the diagram (figure 6) in some detail stating:

> the product of the arm (short side) and upright (long side), multiplied by two, increased by the square of the difference of the arm and the upright should be equal to the sum of their squares. (Plofker in Katz 2007, p.477)

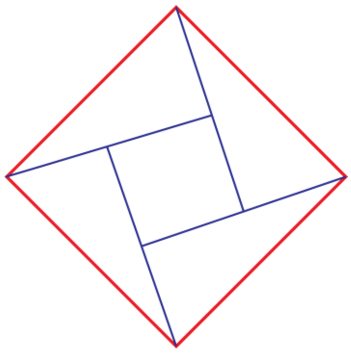


Figure 6: Diagram referred to by Bhaskara II as 'making the truth obvious'

Mathematical traditions from China

As in other cultures, Chinese mathematics grew from a long history of accumulated practice. Peg and cord were probably used for the original construction of geometrical figures. Evidence from settlements along the Yangtze River thousands of years ago, are currently being investigated by archaeologists that may provide more insight into early mathematical practices (Hoyrup, 2015). Characteristic of Ancient Chinese mathematics were general principles (both arithmetical and geometrical) for solving problems that could be applied in many different situations.

By the Han Dynasty (200 BCE – 220 CE) there were two clearly stated mathematical principles concerning conservation:

a) The area of a plane figure or the volume of a solid remains the same under any rigid transformation

b) In the process of construction or reconstruction – of dissection and rearrangement – the sum of the areas or volumes of the parts, is equal to the area or volume of the original figure.

A good example of a Chinese basic principle is the 'In-Out' procedure. In the rectangle (figure 7) the areas of the two yellow shapes do not, at first sight, appear to be the same. However, the diagonal divides the rectangle in two, and there are two pairs of equal triangles (white areas), this means that the yellow areas must also be equal, balanced, as it were, across the diagonal.

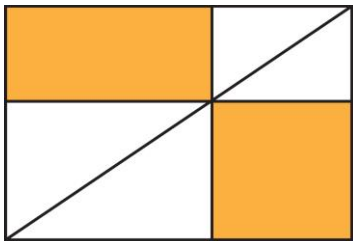
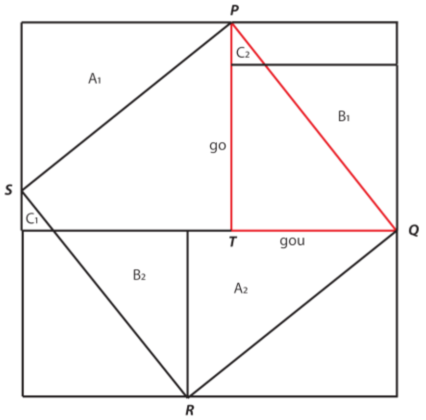


Figure 7: What is the relationship between the yellow areas?

The same principle is used in this explanation of the *Pythagorean* relation:



In the diagram (figure 8), the base is ‘gou’ and the height ‘go’

One square PQRS is tilted inside another square.

PQRS is the square on PQ

A1 is the square on ‘go’ and B1 is part of the square on ‘gou’

Areas of the triangles as part of the squares A1 = A2, B1 = B2, C1 = C2

To fill the square PQRS, A1 goes to A2, B1 goes to B2 and C1 goes to C2

Figure 8: Using the In-Out principle for the *Pythagorean relation* (Chemla & Shuchun, 2004)

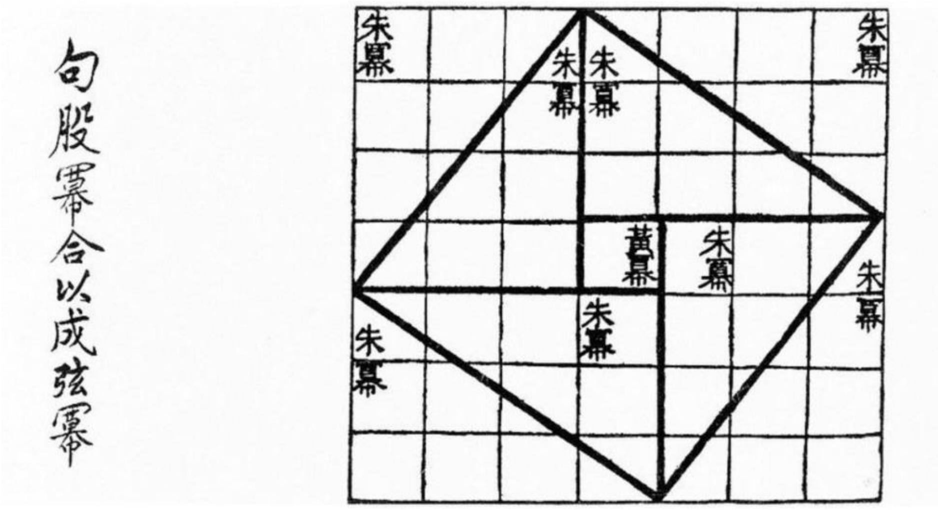


Figure 9: The *Pythagorean relation* from Zhou Bi Suan Jing: *The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven,* one of the oldest Chinese Mathematical texts, Zhou Dynasty (1046 – 256 BCE) (Dauben, 2007)

Conclusion

We have shared just a few rich historical sources that might provide a context for motivating learners to find out more about the *Pythagorean relation.* Attributing the result to the ancient Greeks is at best disingenuous, given its long history. Inviting learners to investigate the mathematics of ancient cultures can be intriguing and inspirational.

There are numerous proofs of the *Pythagorean relation,* those from India and China are probably more accessible than the Euclidean proof. For an accessible compendium with a variety of classroom opportunities see Knott (n.d.).

As well as looking at alternative proofs, generalisations of the *Pythagorean relation* offer possible explorations for the classroom using the 'what-if-not' approach (Walter & Brown, 1969). This is a way of exploring generalisations of mathematical situations, e.g. asking, *what if the angle is not a right angle?* can lead to discovery of the cosine rule for triangles, among other properties (Rogers, n.d.b).

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