Sharing chocolate bars: Year 8 students' use of narrative, visual and symbolic representations of fractions

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In this paper we discuss incidents from video recordings of a lesson and of a follow-up interview arising from a story about 'sharing chocolate bars'. The students, from a relatively 'low attaining' Year 8 (Grade 7) class, made use of various representations of fractions. They sometimes made good sense of a representation by linking it to the story, but it often proved challenging to make fruitful connections between representations. And representations were sometimes used procedurally in ways that fitted what students remembered or (re-constructed) about fractions, rather than with how the procedures and outcomes might have related to the story.

Keywords: fractions, division model, part-whole model

Introduction

Rational numbers, and fractions, can be interpreted in several different ways (see, for example, Kieren, 1980): as a part-whole relationship, a ratio, division (or sharing), an operator, and a measure (or number). The dominant representation of fractions in UK schools is the part-whole model, though students might also meet the number line and division (sharing) models. Nunes (2006) argues that young children bring informal notions of division with them to school and that initially they tend to find this more accessible than the part-whole model. Streefland (1991), in a teaching experiment over several years with primary school children, used a division model involving 'table arrangements', ie of people seated around a table sharing equal amounts of pancake, say. A nice feature of his model is that it can lead to notation that morphs almost seamlessly into formal fraction notation.

We used this idea of 'sharing' in some fractions lessons developed for the ICCAMS project (Hodgen, Coe, Brown & Küchemann, 2014). In this paper we discuss video recordings of a lesson and interview involving the task shown in Figure 1. The students were from a 'low attaining' Year 8 class.

Sharing chocolate bars

Eight girls and two boys are sitting at a table.
They want to share six bars of chocolate.

The two boys run off with one of the bars.

So there are now **8 girls** with **5 bars**.

Does this leave the girls with
more, the same, or less chocolate each?

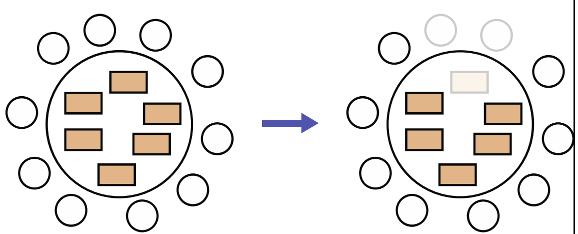

Figure 1

The video recording was undertaken by two professional film makers. It was a trial run, to help us decide how to we might produce edited lesson- and interview-extracts to support teachers using the ICCAMS materials. As such, the recording was not systematic and sometimes only captured parts of interesting incidents. In this paper, we focus primarily on one student, W, who appears several times in the lesson video, including a segment where I sat with his group during the lesson. He was also one of the three students in the group that the teacher and I interviewed after the lesson.

The lesson

At the start of the lesson, the teacher introduced the ‘sharing chocolate bars’ task and asked students to write on their mini whiteboards their initial response to the question, “Does this leave the girls with more, the same, or less chocolate each?”. The responses were fairly evenly spread, with W being one of several choosing ‘the same’.

The students were then asked to discuss the task further with their neighbours and this was followed by a whole class discussion. This pattern of working was repeated, with plenty of time given to the group discussion on each occasion.

During the lesson, the idea emerged that in both sharing chocolate scenarios (be it with the initial group of 2 boys and 8 girls, or the subsequent group of 8 girls only) each person could be given half a bar of chocolate, and that this would leave one whole bar still to be shared out. This idea is contained in the ICCAMS lesson notes but it seemed to emerge spontaneously in the class, including in W’s group.

Towards the end of the lesson, the teacher, using contributions from the class, had written $\frac{1}{2}+\frac{1}{10}$ and $\frac{1}{2}+\frac{1}{8}$ on the class whiteboard, for the amount each person would get in the two scenarios. There seemed to be general agreement that the second amount was larger, though students struggled to put this into words, ie to explain why in particular $\frac{1}{8}$ is larger than $\frac{1}{10}$ - even though the teacher had also drawn a bar cut into 10 pieces for the first scenario and a bar cut into 8 pieces for the second scenario.

One student suggested we could show this by cutting the left-over bar for the 8 girls into 10 pieces, as we had for the 10 boys and girls; if each girl was given one of these pieces ($\frac{1}{10}$, as in the first scenario) there would be two further pieces to share, so the girls in the second scenario would have (a bit) more. The teacher illustrated this on her whiteboard by cutting each of these two pieces into four (Figure 2). The teacher did not take this any further, but in the subsequent interview, we discussed how these pieces could be represented as vulgar fractions, individually and in combination. The teacher ended the lesson by asking students once more to explain why $\frac{1}{10}$ is less than $\frac{1}{8}$.

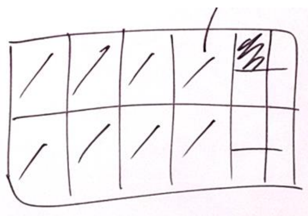

Figure 2

Figure 3 shows W’s mini-board early in the lesson, about 10 minutes in, and about 20 minutes in.

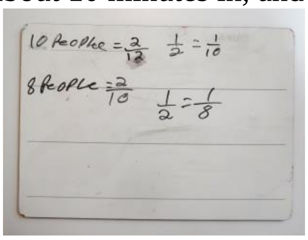

Figure 3a

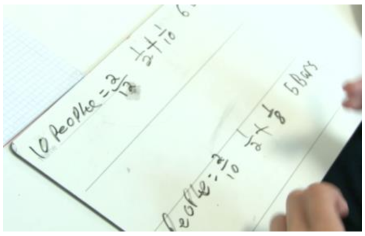

Figure 3b

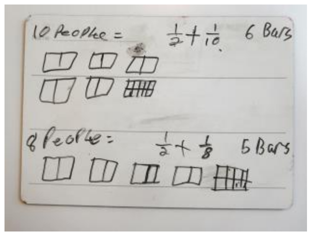

Figure 3c

Figure 3b shows the mini-board just after W, prompted by the teacher, has changed the = sign between the fractions $\frac{1}{2}$ and $\frac{1}{10}$ and the fractions $\frac{1}{2}$ and $\frac{1}{8}$, to a + sign. During this discussion with the teacher, we hear W say, “Miss, that's what we are trying to work out, how we can put them together”. This suggests W is trying to add the fractions, as the teacher had earlier (see below) asked him to do. In turn this suggests that he is not entirely happy with the ‘add tops and bottoms’ responses of $\frac{2}{12}$ and $\frac{2}{10}$ that he has written on his board. By the end of the lesson, we can see that W has wiped these off (Figure 3c), though we don't know when this happened or why.

Early in the lesson [3:43 on the video] the teacher asks W, “How much would the girls get when the boys are there?” (scenario 1). W replies,

> They get half and then a tenth.... because there's 10 people and then.... they all get half because there's 5 bars and then there's a bar remaining so that it has to be split between 10 people so they all get a tenth of a bar.

This verbal account suggests W has a good understanding of the task. It also suggest his insight into how the chocolate bars can be shared would enable him to conclude that each girl’s share would be greater in the second scenario. Unfortunately [from a research point of view!] he is not asked to make the comparison at this moment. Instead, the teacher poses another, equally interesting, question [5:00]: “So what’s that as one fraction, then?... Work it out.... Write it in your books”.

Seven minutes later, at the point in the video where we see the mini-board as in Figure 3b, W’s focus still seems to be very much on the fractions, in compliance with the teacher’s request. However, when she points to the mini-board and asks [12:15] “Where did you get the 8 from?”, he can relate this back to the story: “There was a spare bar so we split it between 8 people”.

The teacher now leads W and his group away from trying to add the fractions as a way of deciding which scenario provides the bigger shares, to simply considering the $\frac{1}{10}$ and $\frac{1}{8}$. This seems to produce an interesting shift away from the story:

> [14:04] Teacher: So what have I got to look for, in the bit that was different here? [pointing to the fractions 1/10 and 1/8]
> W: What bottom number's bigger and smaller.
> Teacher: Is it just the bottom number I look at?
> W: The fraction...
> Teacher: So what fraction's bigger, one tenth or one eighth?
> W: So they [the girls] get less!! In the first one they would get one tenth and a half but now the boys have gone they only get an eighth and a half, so it's less.

I had just sat down with the group, and asked W, "Why is that less?" [pointing to $\frac{1}{2} + \frac{1}{8}$]. He replied [14:50], "Because when the boys went, they took a chocolate bar, so we only have 5 bars... between 8 people". Here W seems to be making the story fit his wrong interpretation of fractions, rather than letting the story inform his interpretation as he seemed to have done earlier. The conversation continued as follows [15:12]:

> DEK: What does it mean, 1 over 10?
> W: There's one piece divided by 10... There's one out of ten pieces.
> DEK: OK. And that one is one out of 8 pieces?
> W: Yes.
> DEK: So how big are the pieces? Which pieces are bigger?
> W: [pause] These pieces! [meaning 1/8] Oh yes, these pieces because there's less pieces. So they would get... oh... they would get more chocolate than they would have up here [points] because even though this fraction's smaller, the pieces are bigger because it's divided by less people. So the pieces are larger.

There are several points of interest here, of which the primary one perhaps is that he refers back to the actions in the story to correctly conclude once more that $\frac{1}{8}$ results in a larger share than $\frac{1}{10}$ even though he still says that the fraction itself is smaller. W’s phrase ‘less pieces’ is interesting too, for its ambiguity - it could be interpreted as meaning ‘smaller pieces’ rather than ‘fewer pieces’. Would it help his understanding (or indicate a better understanding) if he learnt to use (or had spontaneously used) 'fewer' here, rather than 'less'? Should we actively promote students' use of appropriate vocabulary, or wait for them to pick it up when they appreciate its significance?

W also expresses an interesting ambiguity in the meaning of $\frac{1}{10}$ when he says "There's one piece divided by 10.... There's one out of ten pieces". Both meanings are correct, but they are not quite the same, and we perhaps forget what a challenge it can be to reconcile the two. Nunes (2006) mentions exactly this phenomenon when she describes a division (as opposed to a part-whole) interpretation of fractions:

> In division situations, the numerator refers to the number of items being shared and the denominator refers to the number of recipients: if 1 chocolate is shared among 4 children, the number 1 refers to the number of chocolates being shared and the number 4 refers to the number of recipients: the fraction ¼ indicates both the division - 1 divided by 4 - and the portion that each one receives.

By the end of the lesson, W seems to go along with the teacher's perceived consensus in the class, that the fraction $\frac{1}{8}$ (and not just the result of the action it describes) is indeed larger than $\frac{1}{10}$. This is shown quite nicely by the diagrams that he has drawn on his mini-board 20 minutes into the lesson (Fig 3c). However, when the teacher asks him to give an explanation to the class, he refers instead to the diagram she had drawn on the class board (Figure 2). Unfortunately he gets into a muddle in trying to make use of this quite complex diagram, and one that does not even show the fraction $\frac{1}{8}$ directly.

This raises the issue of the role diagrams might play in representing fractions and in developing students' understanding. It is easy to assume that diagrams play a 'neutral' role of providing data about fractions similar to data that might be generated about, say, functions from feeding numbers into a function machine. Thus it might be thought that it is unproblematic to use diagrams to analyse the 'structure' of fractions, or to use them empirically to draw inferences about how fractions behave.

However, evidence suggests (eg, Kerslake, 1986; Hodgen, Küchemann, Brown & Coe, 2010) that to read a diagram effectively, and, in particular, to draw a diagram fit for purpose, students already need to know quite a lot about what they are looking for. Nunes (ibid) makes the point that students' drawings are often misleading in terms of what they portray and that they can distract students from using their knowledge about division.

This is not an argument against using diagrams, but, rather, for investing sufficient effort into constructing and interpreting them, since they can be a powerful tool for thinking about the structure of fractions.

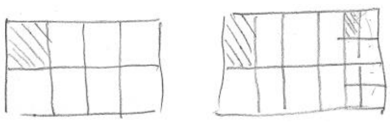

Figure 4

In the interview that followed the lesson I had made two drawings of the 'left over' chocolate bar for the 8-girl scenario (Figure 4). In one case (left) I had shown the bar cut into 8 (equal) pieces, in the other (right) I had repeated the teacher's drawing of the bar cut into 10 pieces, with the 2 left-over pieces cut into 4. Though these were mere sketches, I had gone to some length to explain that these represented the same bar, but cut differently. When I then asked whether the shaded piece on the left could represent the same amount of chocolate as the shaded pieces on the right, W initially said No, because if you covered up the 8 smaller pieces on the right, we'd be left with identical shaded amounts. Thus, though I had taken care with my sketch, and tried to make plain what it represented, W chose to interpret the drawing differently and to be persuaded by what he perceived.

On the other hand, after I had further clarified what the drawings were meant to represent, W, together with one of his colleagues, was able to make use of the right hand drawing to correctly resolve my question [4:55], "What fraction of the whole chocolate bar is that [smaller shaded region]?".

W's first response was to say, "Maybe we have to quarter all of the squares, and then add all the small squares up and then we'd find out what the denominator would be". I then asked whether we could imagine doing this, ie find the answer without drawing. After quite a lengthy pause, one of the other two students, PR, suddenly exclaimed "40" [5:48]. The teacher asked how she got that, to which PR replied, "In my head I did 4, 8 and then 12, 16, 20, 24, 28, 32, 36, then 40" - all the while pointing to the $\frac{1}{10}$ regions in turn. In my view this is a small but significant step away from drawing the full 40 partitions and counting. At the same time, it is perhaps some way off seeing that there are 10×4 partitions.

I then asked [6:24] "Is one eighth [pointing to shaded piece] the same as one tenth and one fortieth [pointing to shaded pieces]?". PR replied, "Yes, it's still the same size, you've just split it up in different parts". Here she is using the story to justify her conclusion. The teacher now asked [6.44], "How could we add these two fractions?" [pointing to the right hand rectangle]. PR looked dumbfounded. "What do you mean?!" This move towards considering the fraction symbols in their own right left her bemused, even though she would probably have met 'adding fractions' numerous times in her school life.

At the teacher's suggestion, W wrote $\frac{1}{10} + \frac{1}{40}$ underneath the right hand rectangle, and I explained that "....we are wondering whether we can write that as a single fraction". PR replied with, "Maybe two over fifty". So PR is now working at a purely formal level, by using a mis-remembered or invented rule. None of the students considered whether this answer might be plausible in terms of the story or the diagram. W, though, rejected it at a similar, formal level [7:38]:

> I don't think so because you would have to simplify it to get that, but then if you simplify these ones [points to $\frac{1}{10}$ and $\frac{1}{40}$] would go to a decimal number which you can't really do with a fraction. The '1' will go to a decimal....

Perhaps W is thinking along the lines of $\frac{1}{10} + \frac{1}{40} = \frac{1}{10} + \frac{0.25}{10}$, but it is not entirely clear. He says he would like to be able to simplify "Because it's easier to add it together", so he seems at least partly to be remembering something he'd been taught about adding fractions.

W then has a sudden insight [9:15]: "Maybe we could make this one tenth [points to right hand diagram] into four fortieths, and then add the one fortieth too [points to right part of right hand diagram]". He writes, $\frac{4}{40} + \frac{1}{40}$ and partitions the shaded $\frac{1}{10}$ region on the diagram into 4 parts. Returning to his fraction sentence, he writes $\frac{5}{40}$ and concludes "...and so we get five fortieths".

I recap what W has just done. I then point to W's $\frac{5}{40}$, and to the fraction $\frac{1}{8}$ written above the shaded region of the left hand diagram.

> DEK [10:35]: Our hunch is they must be the same, because we've just shared the whole chocolate [bar] out amongst the same number of girls.
> W: Maybe if we simplified the five fortieths? Because this [1/8] hasn't got a double figure denominator. So if we simplify it, maybe we will get one eighth.
> PR [10.57]: You do! [face lights up] Divide by 5!
> W: Yes. Divide by 5. You have to divide both of them.
> Teacher: Why both of them?
> W: Because otherwise the relationship would be different.
> PR: What you divide by top you have to divide by bottom.

PR Takes the pencil and confidently draws an arrow diagram showing how $\frac{5}{40}$ becomes $\frac{1}{8}$. "It is the same!"

It is interesting how the discussion, and terms like 'simplify', triggered formal procedures that the students had met before. The students were excited and pleased by the outcome, though they gave no indication that they saw why 'divide by 5' made sense in terms of the diagrams or story.

We finished the interview by considering whether the shaded region representing an eighth in the left hand rectangle in Figure 4 could be changed into fortieths. After much thought, the students managed to do this, but again in a faltering way, ie without seeming to have gained insight from the previous task or from having earlier solved the similar task of changing a tenth into fortieths.

Conclusion

These observations suggest that even though students might make good sense of a story involving fractions, they can find it challenging to make fruitful links between the story and their verbal, visual and symbolic representations. At the same time, we would argue that engaging with rich and accessible tasks like 'sharing chocolate bars' may, over time, enable students to develop such links, which they need to do if they are to form a coherent and robust understanding of fractions.

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