Difficulties in teaching calculus concepts in undergraduate courses: the notion of limit of a sequence

Marius Ghergu
School of Mathematics and Statistics, University College Dublin, Ireland

It is commonly agreed that undergraduate students encounter difficulties in understanding the notion of the limit in calculus courses. Both the formal and informal approach do not convey a setting that bridges the intuition and mathematical rigour. In this work we explore the formal and informal definition of the limit of a sequence existent in several calculus textbooks in light of the recent research in the literature. A case study was conducted on a large group of first year undergraduate students in University College Dublin, the largest university in Ireland. The findings reveal students' difficulties in formalising their mathematical observations along with a reluctance to employ formal or informal definition of a limit of a sequence in their arguments.

Keywords: Undergraduate courses; calculus; limit of a sequence; formal and informal definition

Introduction

Teaching calculus in undergraduate courses can be a delicate task for the lecturer. Some of the main concerns one faces in this regard are summarised below (Kotecha, 2012):

- Students are generally unhappy with the fast pace at which the lectures and tutorials are delivered. This is mainly due to the large variety in students' academic and cultural background.
- Students' expectations from the lecturer and their individual learning styles are different.
- Students need additional guidance and support as they find it hard to cope with the material covered in lectures.
- Students find that a significant part of the course is not relevant to their respective career.
- Some students seem to lack confidence due to low academic self-efficacy which acts as a barrier to learning.

In undergraduate courses the lecturer has to find a balance between the formal and informal approach of teaching the concepts. This task becomes even more sensitive when it comes to introducing topics which are fundamental to the understanding of the course material. One of such examples is the notion of limit of a sequence, an infinite process initiated by finite processes involving large numbers. It is widely acknowledged that the notion of limit is one of the fundamental ideas in understanding calculus; it is perhaps the first infinite process students face in their mathematics coursework and it is regarded as a major concept when aiming for mathematical rigour. For most students, the concept of limit is the first instance in which mathematics is not restricted to a finite computation which gives a definite answer (Cornu, 1981; Cottrill et al. 1996).

Background and literature review

By now there is a well-documented research literature (Cornu, 1981; Cotrill et al. 1996; Tall & Vinner, 1981; Williams, 1991) that testifies to students' difficulties with the understanding of the limit concept. We state here just a few misconceptions reported (Flores & Park, 2016; Oehrtman, Swinyard & Martin, 2014):

- The limit is the last term in an infinite sequence.
- Limit is a boundary that cannot be surpassed.
- A limit is a number which eventually, theoretically can be reached.
- As n increases, a_n gets closer and closer to its limit.

Research (Davis & Vinner, 1986; Tall & Vinner, 1981) brings evidence that such misconceptions are "persistent and rather than replacing earlier notions with notions learned during formal instruction, students often retain both sets of ideas" (Nagle, 2013, p.3). For instance, these misconceptions are carried over to other related topics in calculus such as continuous and differentiable functions or infinite series. Mammona-Downs (2001, p. 261) suggested three didactical steps in teaching the concept of limit as follows:

(i) Initiate and develop intuition through raising issues in a classroom discussion environment.

(ii) Introduce the formal definition and analyse it in conjunction with (i) above.

(iii) Endorse or revoke opinions made in step (i) by comparison with the formal definition, especially via the representation in (ii) above.

More recent studies (Flores & Park, 2016; Oehrtman, Swinyard & Martin, 2014; Swinyard & Larsen, 2012) focus on a guided reinvention of definition of limit of a sequence with or without interactive technology.

Difficulties with the notion of limit

Difficulties with the notion of limit are conceptual rather than computational. Current textbooks focus more on techniques for limit calculation and propose either a formal (rigorous) or an informal (intuitive) definition for the limit of a sequence. In fact, the dispute between intuitive and formal teaching of calculus concepts is "a problem that goes through the history of calculus" (Moreno-Armella, 2014, p. 621).

An informal definition of limit

In many calculus textbooks an informal definition of the limit of a sequence is proposed. According to Stewart, (2015, p 696),

> A sequence $\{a_n\}$ has the limit L and we write that $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$ if we can make the terms of $\{a_n\}$ as close as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say that the sequence $\{a_n\}$ converges (or is convergent). Otherwise we say that the sequence diverges (or is divergent).

It was argued (Oehrtman, 2008) that there is a risk that the student may assign a simpler meaning to limit concept. In particular students interpret the modifiers *sufficiently* and *arbitrarily* as indicators of degree. Thus, "*sufficiently small* means *very small* and *arbitrarily small* means *very very small*" (Oehrtman, 2008, p. 68).

A formal definition of limit

Many reputable textbooks present the following formal definition of the limit of a sequence (Apostol, 1974, p.70):

> We call a sequence $\{a_n\}$ to be convergent to a real number L if for any positive number ε, there is a natural number N such that $|a_n - L| < \varepsilon$ for all $n > N$.

We note here the presence of the quantifying symbols and that the order in which they appear is very important. Besides, the above formal definition of the limit requires the use of other mathematical symbols such as inequalities, absolute value. Documented research (Tall, 1992; Williams, 1991) states that students who are given a formal limit definition may have great difficulty making sense of it.

Another view on the definition of limit

Another definition of the limit of a sequence requires the treatment first of monotone sequences and the notion of *least upper bound* and *greatest lower bound*. One may introduce (Dineen, 2012) the limit of an increasing sequence which is bounded from above as the least upper bound of the set that contains all of its terms. Similarly, the limit of a decreasing sequence which is bounded from below is defined as the least upper bound of the set that contains all of its terms. This has the advantage of allowing students to use a visual approach to countable sets and then determine the limit. However, it introduces another subtle concept namely least upper bound and greatest lower bound for a set. Now,

> A sequence of real numbers $\{a_n\}$ converges to the real number a if there exists an increasing sequence $\{b_n\}$ and a decreasing sequence $\{c_n\}$ both of which converge to a such that $b_n < a_n < c_n$ for all n (Dineen, 2012, p. 139)

In practice, the difficulty lies in finding the two monotone sequences $\{b_n\}$ and $\{c_n\}$ which have the same limit and squeeze the initial sequence $\{a_n\}$.

A Case Study

A formative assessment was conducted by the author in the middle of the first semester of the academic year 2016/2017 on a group of 120 first year undergraduate students at University College Dublin, the largest university in Ireland. The students were enrolled in the Business Analytics and Economics & Finance joint programme and were taking their first calculus course at undergraduate level. Among the entry requirements to this programme is a good mathematical ability demonstrated in the State Examinations held at the end of secondary school system in Ireland.

The central objective in running this experiment was to provide a fresh insight into students' understanding of the limit concept. Students were exposed to both formal and informal definitions of the limit of a sequence as discussed above. They were told that the test is anonymous and that this assessment would not contribute towards their formal evaluation imposed by the university. Three questions were asked over a period of 20 minutes as follows:

Q1: What do you understand by a divergent sequence?

Q2: Give example of two sequences $\{a_n\}$ and $\{b_n\}$ such that $\lim_{n\to\infty} a_n = \infty$, $\lim_{n\to\infty} b_n = -\infty$ and that $\lim_{n\to\infty} (2a_n + b_n) = 5$.

Q3: Let $x_n = cos\frac{n\pi}{4}$. What can be said about $\lim_{n\to\infty} x_n$?

Findings

None of 120 students used the definition of a convergent sequence when answering Q1. Their answers were formulated using the notion of limit. Here are some answers from students' tests:

> A divergent sequence is one which has limit or no limit at all.
>
> A sequence in which the distance between a term and the one previous is always increasing as n approaches infinity.
>
> It means that the sequence has no exact limit. Divergent sequence is a sequence that does not tend to a fixed point as the variable goes to infinity, e.g. 5^n.
>
> Any sequence whose limit is not a real number.
>
> Is a sequence that can continue infinitely. It is not limited in any way, i.e., it will never occur that the sequence will not pass or reach a certain value. It will never converge at any given point

In answering Q2 students struggled to find examples of sequences tending to infinity, so that their linear combination $2a_n + b_n$ has finite limit. However, a good number of students managed to match constants in such a way that $2a_n + b_n$ is constant (for instance $a_n = n$ and $b_n = -2n + 5$).
Below we state some answers to Q3 taken from students' scripts:

> As n tends to infinity the values of x_n will always shift from -1 to 1. Every fourth value of n will give the same result. No limit of x_n exist.
>
> The limit of x_n is nonexistent. When n is odd no limit exist as the values alternate between $\sqrt{2}/2$ and -$\sqrt{2}/2$. When n is even no limit exist as the values alternate between -1, 0, 1.
>
> Because cosine is a cyclical function going from -1 to 1 continuously, it will never converge to any number in between as n tends to infinity.
>
> The limit does not exist as x_n oscillates over and back between positive and negative values.
>
> The values repeat themselves every fourth integer. Thus, we can say there is no definite limit. The limit ranges from -1 to 1 depending on the value of n. This is easily seen by looking at a graphical representation of $cos\left(\frac{n\pi}{4}\right)$.

Two more answers are presented in the figures below.

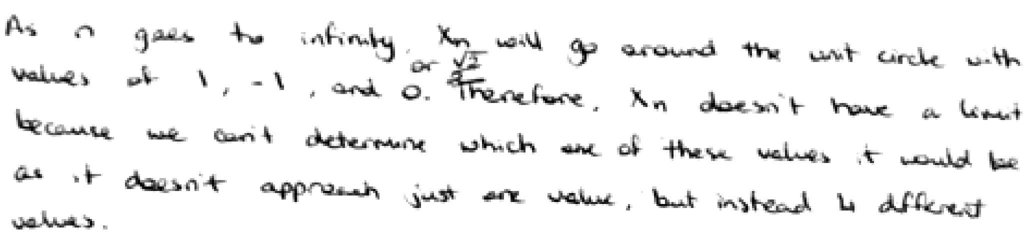

Figure 1: Student work

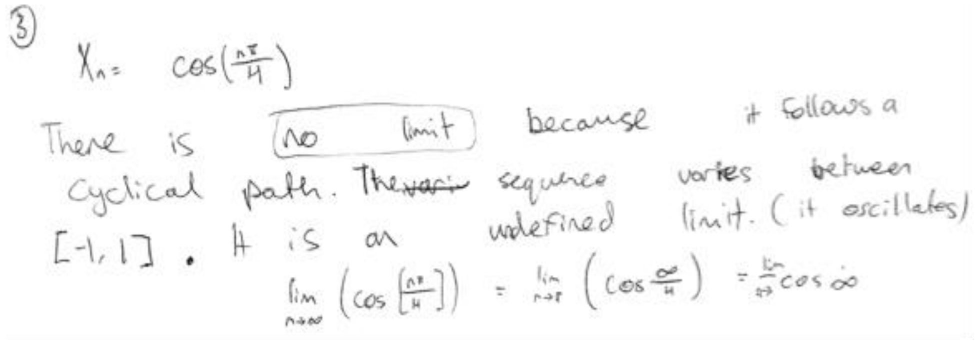

Figure 2: Student work

The test reveals that students have difficulties in formalising their mathematical observations related to the notion of limit of a sequence. Students rely heavily on their informal observations and routine calculations; they find it hard to build a systematic approach articulated in rigorous mathematical terms. Nonetheless, the findings of this test gave the instructor scope to intervene in the learning process and revise the notion of limit of a sequence.

Discussion

Students' answers and interpretations of the limit concept as shown above indeed lack mathematical rigour; the notion of limit appears to students as a difficult concept mainly because it involves infinite mathematical processes which are not discrete. The limit of a sequence paves the way to continuous processes which will appear more and more frequently in the calculus syllabus; their understanding requires not only intuition but also a good level of accuracy when working with abstract concepts. In order to have a better insight into how students understand infinite mathematical concepts we have perhaps to go deeper into the study of cognitive operations involved in the learning process.

Conclusion

The case study presented in this work confirms the previous research in the literature done on understanding the notion of limit: the limit concept has either a too formal or a too light meaning. Traditional instruction in calculus courses is not enough if one wants to provide a thorough understanding of calculus concepts to our students. This task becomes even more challenging in the case of calculus courses when time constraint and a fast pace of content delivery are experienced by the lecturer. The teaching approach has to be complemented by other methods that take into account the cognitive procedures involved in students' acquisition of mathematical concepts related to infinite processes and identify the nature of students' conceptual difficulties in learning. As Siegel (1988, p. 82) wrote many years ago, "our need to learn about cognitive science is acute". Iterative methods such as guided reinvention of the notion of limit of a sequence which aims at helping students reason critically is one alternative currently explored.

References

Apostol, T. (1974). *Mathematical analysis*. Reading, MA: Addison-Wesley.

Cornu, B. (1981). Some obstacles to learning the concept of limit. *Recherches en Didactique des Mathematiques, 4,* 236-268.

Cottrill, J., Dublinsky, E., Nichols, D., Schwingendorf, K., Tomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process schema. *Journal of Mathematical Behavior*, *15*, 167-192.

Davis, R.B. & Vinner, S. (1986). The notion of limit: some seemingly unavoidable misconceptions stages. *Journal of Mathematical Behavior*, *15,* 281-303.

Dineen, S. (2012). *Analysis. A gateway to understanding mathematics.* Singapore: World Scientific.

Flores, A., & Park, J. (2016). Students' guided reinvention of definition of limit of a sequence with interactive technology. *Contemporary Issues in Technology & Teacher Education, 116,* 110-126.

Kotecha, M. (2012). Teaching mathematics and statistics: Promoting students' engagement and interaction. University of Bristol, UK.

Mamona-Downs, J. (2001). Letting the intuitive bear on the formal; A didactical approach for the understanding of the limit of a sequence. *Educational Studies in Mathematics*, *48,* 259-288.

Moreno-Armella, L. (2014). An essential tension in mathematics education. *ZDM Mathematics Education*, *46*, 621-633.

Nagle, C. (2013). Transitioning from introductory calculus to formal limit conceptions. *For the Learning of Mathematics*, *33*, 2-10.

Oehrtman, M. (2008). Layers of abstraction: Theory and design for the instruction of limit concepts. In M.P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 65-80). Washington, DC: Mathematics Association of America.

Oerthman, M., Swinyard, C., & Martin, J. (2014). Problems and solutions in students' reinvention of a definition for sequence convergence. *Journal of Mathematical Behavior*, *33*, 131-148.

Siegel, M.J. (1988). Teaching mathematics as a service subject. In A.G. Howson, J.-P. Kahane, P. Lauginie & E. de Turckheim (Eds.), *Mathematics as a service course* (pp. 75-89). Cambridge: Cambridge University Press.

Stewart, J. (2015). *Calculus: Early transcendentals (8th ed).* Cengage Learning.

Swinyard, C., & Larsen, S. (2012). What does it mean to understand the formal definition of limit? Insights gained from engaging students in reinvention. *Journal for Research in Mathematics Education,* 4*3*, 465-493.

Tall, D. (1992). The transition to advanced mathematical thinking: functions, limits, infinity, and proof. In D. A. Grouws (Eds.), *Handbook of research on mathematics teaching and learning* (pp. 495-511). New York: Macmillan.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*, 151-169.

Williams, S. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education*, *22*, 219-236.