The role of defining in pre-proving activity

Maria Pericleous

UCL Institute of Education

In this paper, I present and discuss findings from a research study the aim of which is to investigate the activity of proving as constituted in a Cypriot classroom for 12 year old students. By drawing on Cultural-Historical Activity Theory and collaborative task design, this study explores the way the teacher is working with the students to foreground mathematical argumentation. Analyses of video-recorded whole class discussions show how processes of explaining and exploring are key sub-systems within the central activity of proving as they provide a key pathway, which often includes defining. I refer to these developments as pre-proving. However, emerging contradictions within explaining and exploring make the constitution of pre-proving in the classroom inherently complex. In this paper, I illustrate how defining, an activity integrated in the activity of explanation, plays a crucial role in regards to pre-proving activity.

Keywords: proving; defining; explanation; justification; CHAT

Introduction

It is now acknowledged that proof and proving should become part of students' experiences throughout their schooling (Hanna, 2000, Yackel and Hanna, 2003, Stylianides, 2007). It is also argued that argumentation, explanation and justification provide a foundation for further work on developing deductive reasoning and the transition to a more formal mathematical study in which proof and proving are central (Yackel and Hanna, 2003). Research has responded to the need to conceptualize proof and proving in such a way that it can be applied not only to older students but also to those in elementary school (Stylianides, 2007). The challenge remains however to understand how proof and proving is shaped by the practices in the mathematics classroom. This is in accordance with Herbst and Balacheff (2009), who argue that the focus should not only be on proof as the culminating stage of mathematical activity, but also on the proving process and how this is shaped by the classroom environment. Thus, in understanding how proving is constituted in the classroom, a wider network of ideas is required as these ideas no doubt have an impact on how proof in the narrow sense is constituted.

To address this issue, I refer to pre-proving, that aspect of mathematical reasoning that might nurture proving. What are the roots of proving? The purpose of this study is to investigate proof and proving in the naturalistic setting of the classroom and the way the structuring resources of the classroom's setting shape this process. Instances of students proving statements have been identified in this classroom community but instances where the argument was not in the conceptual reach of the classroom have also been identified. However, this study also points to those aspects of reasoning that appear to have the qualities of proving, even though they may not be proving in themselves. That is, analyses of video-recorded whole class discussions show how processes of explaining and exploring are key sub-systems within the central activity of proving as they provide a key pathway, which often includes defining. Thus, pre-proving refers to those elements that direct mathematical reasoning towards the ultimate goal of formal proving, namely exploring, explaining, justifying and defining. There is insufficient scope in this short paper to consider in detail these various levels and so this specific study focuses on the role defining plays in regards to pre-proving activity.

Definitions and terminology in mathematical reasoning

Definitions are considered fundamental in mathematics and mathematics education. Among the main roles attributed to definitions is that they constitute fundamental components for concept formation as they introduce the objects of a theory and capture the essence of a concept by conveying its characterizing properties. Adding to the above, definitions form a generative basis for logical deduction, not only of known properties of the concept but of new properties. Definitions may also facilitate the generation and construction of different types of theorems, proofs and solution methods (Morgan, 2005). In discussing definitions in mathematics, mathematicians and mathematics educators address, initially, the distinction between ordinary or dictionary definitions (descriptive or extracted definitions) and mathematical definitions (stipulated or analytic definitions). In describing the way students use definitions in mathematics, Tall and Vinner (1981) introduced the terms concept image and concept definition. The concept image is a nonverbal representation of an individual's understanding of a concept. The term concept definition refers to a form of words used to specify a concept and can be personal or formal (Harel et al, 2006). The concept definition can be the stipulated definition assigned to a given concept.

In addition to distinguishing types of defining in actively engaging students in defining mathematical concepts, the requirements for mathematical definitions valued by mathematicians are also being described in the mathematics education research literature. That is, a mathematical definition should be unambiguous, non-contradictory, hierarchical, as well as invariant under changes of representation. Also, mathematical definitions should have precision in terminology and be easily comprehended by students (Morgan, 2005). In discussing these characteristics of mathematical definitions, Borasi (1992, pp.17-18) identifies two functions that definitions must fulfil; definitions should (i) allow us to discriminate between instances and non-instances of the concept with certainty, consistency, and efficiency and (ii) 'capture' and synthesize the mathematical essence of the concept.

Method

As this study is exploring the various forces that impact on the activity of proving, Cultural-Historical Activity Theory (CHAT) is being employed as a descriptive and analytical tool alongside collaborative task design (a means of gaining access to the teacher's objectives), to capture the interaction of different levels, such as the actions of teachers, students and the wider field as evidenced in curricula and research documentation. The study was conducted in a year 6 classroom in a primary school in Cyprus. Apart from the researcher, the participants were the teacher, a Deputy Principal at the school who endorses the integration of technology in teaching mathematics and 22 students (11-12 years old) of mixed abilities. The data collection process as relevant to this paper included video data from the classroom observations and field notes. The content of the curriculum covered during the classroom observations as relevant to this paper was the circumference and area of a circle. The overall process of analysis of the collected data was one of progressive focusing. According to Stake (1981, p.1), progressive focusing is "accomplished in multiple stages: first observation of the site, then further inquiry, beginning to focus on relevant issues, and then seeking to explain". The systematisation of the data led to the evolution of two broad activities: (i) the activity of exploration including the exploration of mathematical situations, exploration for supporting mathematical connections and exploration of Dynamic Geometry Environments and (ii) the activity of explanation which focuses on clarifying aspects of one's mathematical thinking to others, and sometimes justifying for them the validity of a statement. These activities were then interpreted through the lens of CHAT, by generating the activity systems of both exploration and explanation. Analysis of the classroom data revealed that the activity of explanation unfolds and expands around mathematical definitions and defining as activity. What is the connection between definitions and explanation? Definitions are conventions that require no explanation. However, the teacher wants reference to the attributes that involve properties. That is, the move from a definition involving only perception to a definition that involves properties needs explaining.

Findings

This section provides a detailed description of three classroom protocols. In the following dialogues, T represents the teacher and S1, S2…Ss the students that participated in the discussion.

Protocol A

The following discussion constitutes the first part of the lesson where students were introduced to the area of mathematics related with circle.

> The teacher began by asking the class for the definition of a circle, T: "What is circle?" S1 replies, "It is a shape that does not have sides or angles." The teacher probed for a definition that was precise. T: "S1 says that a circle is a shape without sides or angles. I draw a shape according to this definition". The teacher draws a non-regular shape with curved lines. T: "According to what S1 said this is a circle." The students reply "This is not a circle". S2 says, "A straight shape" while S3 says, "Without curves." The teacher still insisting for an accurate definition says, "I want an accurate definition. S4". Responding with a precise definition was not easy for the students, who looked instead for analogies. S4 says, "We call a circle the shape that … it has the shape of a sphere." The teacher asks, "What is the difference between a circle and a sphere?" S5 says, "The sphere has volume." The teacher says, "The sphere has volume, it is 3-dimensional, whereas a circle is …" The students say, "Flat." The teacher continues, "Flat. Thus, a circle is a flat shape whereas a sphere is a 3-dimensional shape. Which shape do we call circle S6?" S6 says, "The shape that does not have angles." The teacher agrees, "Yes." S6 continues, "And has a curve as a side." The student draws a circle in the air with his hand. T: "Like this?" The teacher draws an ellipse on the whiteboard. The students' response to the teacher's drawing is "No." S6 says, "No, I mean … it is like …" The student is using again his hands to show what he means. At this point, the teacher makes explicit this specific characteristic of mathematical definitions, T: "We said that in mathematics, our definitions must be accurate. Is there a detail that is missing?" S7 says, "A circle is a flat shape." T: "Correct." S7 continues, "That …" Having made the point that it is not so easy to be precise, the teacher turned the class's attention towards properties. T: "A circle has some characteristics." S8 says, "You take the compass …" T: "Yes." S9: "Oh I know." S8 says, "The center of the circle." S9 says, "When you fold it the two parts are equal." The teacher says,

> "This applies for this shape as well (the ellipse)." S9 says, "It's more circular." S8 agrees, "I know." S10 says, "Because the distance from the center to the …" T: "Circumference." S10 says, "It's the same." T: "Exactly." Finally, the teacher summarized what she felt had been important in the prior discussion. T: "We will talk about that later. Thus, S11 says that a circle is the shape that, according to S8 has a center, has a circular circumference and all the points of the circumference have the same distance from the center. Right?" The students say "Yes."

Protocol B

Following the classroom discussion on defining circle, the teacher asked students to look at several shapes illustrated on the interactive whiteboard and determine whether these shapes were circles by also justifying their answer.

> T: "Based on what we have said so far, look at these shapes on the screen. Are these circles?" The students reply "No." The teacher asks, T: "S1, is this a circle?" S1 replies, "No." At this point, the teacher made the following comment: "I don't accept your answer." S1 says, "No its not." At this point, the teacher guided students in making the definition operable: "Why?" S1 says, "They are not circles because …" The teacher says, T: "Because … you were not paying any attention earlier … S2." S2 says, "We have one circle here …" The teacher asks, T: "Which one is the circle?" S2 replies, "There … on the right …the other shapes are not circles because their center does not have the same distance from their circumference." The teacher in effect affirms the importance of a justification by accepting the response with an explanation and not accepting the previous responses: "Yes." S2 continues, "The others are not circles because their center is not in the middle … the center is not equidistant from the circumference."

Protocol C

After the class reached a conclusion regarding the mathematical formula for the circumference of circle and made hypotheses concerning the mathematical formula for the area of circle, the teacher gave each pair a circle divided in either 8 or 10 pizza pieces. The teacher commented that they could use the pizza slices to explore the area of circle.

> S1: "Mrs, we made a rectangle. Radius times circumference … wait …" The teacher asks, T: "How do I find the area of a rectangle?" S1 replies, "Length times width." T: "Nice. Do I know the length and the width here? Write it in your notebooks." S1 responds, "Radius times diameter." The teacher asks, "Is this the diameter?" S1: "Aaa… its half the circumference." The teacher says, "Write it down." S2 says, "So… what are we going to write? We will write … area equals …" Soon, another group called the researcher. S3: "Mrs M we finished. Can we tell you? Radius times half the circumference. It's a rectangle thus the length is the radius and the width is half the circumference because its half." R: "Thus …" S4 says, "We wrote it down. Radius x circumference/2." R: "Nice. Now replace the circumference with the formula." S3 says, "Radius times radius times π …"

Discussion

In protocol A, the classroom discussion was initiated with a question. The teacher did not provide the definition of circle. On the contrary, the students were expected to explain what circle is. Circle is a geometric object for which the students have rich concept images and were therefore able to engage in constructing a definition. The teacher was drawing on the whiteboard following the students' responses. In this classroom discussion, the students, by focusing on the perceptual aspects of the teacher's drawing, were making alterations to the definition they were giving in order for the drawing to be a circle. The classroom discussion was guided by the students' responses to the question. The activity continued in this mode until an acceptable definition was given. The formulated definition captured and synthesised the mathematical essence of the concept. This is in accordance with what Borasi (1992) identifies as one of the functions of mathematical definitions.

Protocol B, differs from protocol A. In this protocol, the students had to say which of the shapes shown on the interactive whiteboard were circles and say why. The students could not rely only on perception but had to distance themselves from the 'geometrical drawing', and use the properties of circle in order to say why the shapes were or were not circles. At first glance, the question 'which of these shapes are circles?' seemed quite simple for the students. However, the students had to put effort in explaining why this was the case. Perception was not enough as the teacher would not accept their answers otherwise. The students had to draw on the definition of circle and its properties so as to justify why the presented shapes were not circles. This is in accordance with the function mathematical definitions should fulfil (Borasi, 1992). The definition students formulated in protocol A allowed them to discriminate between instances and non-instances of the specific concept. This can also be considered the first instance where it is attempted to make the definition of circle operable for the students.

In protocol C, the students had the opportunity to use the pizza demonstration to discover the mathematical formula for the area of circle. The students were encouraged to use the 'pizza slices' so as to construct a shape whose area they know how to find. Through this exploration the students were initially able to construct a rectangle. Following this, the students can use the formula for the area of a rectangle, replace its components with those corresponding in the circle and find the area of circle. With this activity, the students have the opportunity to relate the practical work and their observations with formulas and numbers as well as explain and justify their answer. What is more, the students are provided with the opportunity to find on their own the formal mathematical formula used for the area of the circle. This exploration opportunity consists of an illustrative example of exploration that reveals information necessary to prove, facilitates the understanding of proof, encourages the generation of conjectures as well as supports justification for the process of proving. This can be also characterised as constructive defining. The fact that differing numbers of pizza slices were given to different pairs of students strengthens the generality of the formula of the area of circle.

By considering these illustrative examples I now turn to the way defining is connected with pre-proving activity. Exploration can lead to explaining and justifying necessary for defining (protocol C). Defining can also be initiated by a question of the format 'what is' (protocol A). When formulating a definition and negotiating what one wants a definition to be, defining is explaining (protocol A). Definition construction entails justifying when one explains why. When discriminating between instances and non-instances of a concept and checking whether a potential candidate satisfies all the properties stated in the definition, the defining activity can entail both explaining and justifying (protocol B). Evidence from the data also point to instances where defining might not be explaining but can be considered pre-proving. That is, once a definition is formulated, the need for this definition to be enriched and developed so as to become part of students' explaining emerges. When the students are explicitly asked to repeat the formulated definition, it can be argued that the definition remains active. By revising or revisiting a formulated definition, the students are provided with the opportunity to revise their concept image and thus, broaden their concept definition. Consequently, the students can take control of the definition in their explaining.

Concluding remarks

The aim of this paper was to shed some light on the area related with the activity of proving as constituted in the naturalistic setting of the mathematics primary school classroom. The elements that drive pre-proving activity and influence the way proving may be established in the classroom have been identified. That is, in mathematical argumentation, pre-proving is coming out of reasoning through exploring, explaining, justifying and defining and can lead to proving. This paper reports on the role of defining in regards to pre-proving activity. Defining as a mathematical activity entails formulating a definition, negotiating what one wants a definition to be and why and refining or revisiting a definition. This activity can occur as the students are generating conjectures, creating examples, and trying out or 'proving' a definition or a statement. It entails explaining and justifying and thus, might give rise to proving. However, this does not tell the whole story. Defining can be exploited so as to negotiate and establish socio-mathematical norms in the classroom. As the socio-mathematical norms established in the classroom are related with the very nature, functions and characteristics of proof, proving and defining, their establishment strengthens the activity of explanation. Thus, the socio-mathematical norms related with explaining and justifying might give rise to explaining and justifying and defining and thus, proving.

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