How much cream for 6 people? Some of the complexities that emerged as three 13 year old students attempted to solve 'a quarter plus an eighth'

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In this paper I report on an interview on a fractions task (in the context of a recipe) with three 13 year old (Year 8, grade 7) students, undertaken by their maths teacher and myself. I also look at a follow-up interview with two of the students that took place two weeks later. In the first interview, the students eventually solved the task but difficulties they encountered along the way re-emerged in the second interview.

I will argue that the students' responses fit a socio-constructivist view that mathematical understanding is formed of a complex network of ideas, which are consolidated, developed, challenged and restructured in part through social interactions with others.

Keywords: fractions; formative assessment; constructivism

Introduction

The extract below is taken from the grade descriptors for 'outstanding' teaching that OFSTED published in January 2012. I have underlined aspects concerned with the assessment of students and want to argue that, with regard to assessment, these are criteria that no teacher can satisfy.

> Drawing on excellent subject knowledge, teachers plan astutely and set challenging tasks based on <u>systematic, accurate assessment of pupils' prior skills, knowledge and understanding</u>. They use well judged and often imaginative teaching strategies that, together with sharply focused and timely support and intervention, match individual needs accurately. (Ofsted, 2012, p.12)

The notion that we can have accurate information about the 'prior skills, knowledge and understanding' of students seems to posit a model of learning where students progress in a stable, consistent, step-by-step way. I don't think this provides a good fit with reality, at least not as far as mathematical understanding is concerned.

Interestingly, the above extract has disappeared from OFSTED's January 2015[1] guidance, though the one below remains (my underlining again):

> Teachers <u>systematically and effectively check pupils' understanding throughout lessons</u>, anticipating where they may need to intervene and doing so with notable impact on the quality of learning. (Ofsted, 2012, p.12; Ofsted, 2015, p.61)

Here the reference to assessment is more ambiguous. A relaxed interpretation might be 'elicit some ideas from some students about key notions in the lesson'. This might well be attainable - and could, it seems to me, provide a sound basis for highly effective lessons.

I want to argue in this paper, that it can be useful to view students' understanding from a socio-constructivist perspective (eg, Cobb, 1994), as formed of a

[1] A new version of this handbook came out in June 2015, with the extract loosened slightly to "Teachers check pupils' understanding systematically and effectively in lessons, offering clearly directed and timely support".

complex network of ideas, where these ideas are not always well-formed or well-connected or well-fitting, and where students will tend to select ideas haphazardly, and where the development of the network will tend to be slow and fluctuating. Further, ideas voiced by students are likely to be of value to other students, in terms of consolidating, extending, or challenging their own ideas.

To illustrate this viewpoint, I shall report on an interview with three Year 8 students on fractions, undertaken by their maths teacher and myself. I will also look briefly at a follow-up interview. The students, all boys, were from the second of six classes, ordered by maths attainment, in a small 11-16 comprehensive school, located in a fairly deprived area on the outskirts of a large town in southern England.

The first interview

I want to focus particularly on the students' responses to the very last item in this Recipe task (Fig 1), which is taken from the CSMS Ratio test developed by Kath Hart (1981). As can be seen, the item asks students to find the amount of cream needed to make Onion Soup for 6 people, given that a $\frac{1}{2}$ pint is needed for 8 people. The students saw that this could be found by solving $\frac{1}{4} + \frac{1}{8}$, which they attempted to do by using a part-whole area model or by manipulating the numbers (often with no heed given to the context and perhaps echoing previous encounters with fractions in the classroom). The students flipped from one approach to the other, often without seeming to connect the two. At one point one of the students tried to use decimals and later he introduced a fresh method involving scaling ($\frac{3}{4}$ of 8 people is 6 people) which he used successfully before again applying an inappropriate numerical rule.

> Onion Soup Recipe for 8 persons
>
> 8 onions
> 2 pints water
> 4 chicken soup cubes
> 2 dessertspoons butter
> $\frac{1}{2}$ pint cream
>
> a) I am cooking onion soup for 4 people.
>
> How much water do I need?
>
> How many chicken soup cubes do I need?
>
> b) I am cooking onion soup for 6 people.
>
> How much water do I need?
>
> How many chicken soup cubes do I need?
>
> How much cream do I need?

Fig 1

Overall, we spent nearly a quarter of an hour on the Recipe task. The students quickly solved part a), by halving. They also had little difficulty with the first two items of b), where they used a 'rated addition' approach, by finding the required amount for 2 people and adding this to the amount they had found in a) for 4 people.

The item involving cream is less structured, but they approached it in a similar way, by finding the amount of cream needed for 4 people (half of $\frac{1}{2}$ pint = $\frac{1}{4}$ pint) and for 2 people (half of $\frac{1}{4}$ pint = $\frac{1}{8}$ pint). Two of the students (possibly all three) called out

> ...so that will be a quarter plus an eighth.

Student R, who was the most dominant of the three boys, immediately said 'two twelfths' and then 'one sixth'. The group wasn't sure whether this was right, but when the teacher asked the students once more to consider 'a quarter plus an eighth', R stuck to his method:

> I just added them and got them down to the simplest form.

Thus R was using a simple addition rule (commonly described as 'add tops and bottoms') which doesn't work with fractions or make sense in this context. The teacher then asked, 'Is $\frac{1}{6}$ bigger than $\frac{1}{4}$?'. The students agreed it wasn't and, interestingly, R now attempted to explain this geometrically. He did so by running his finger three times across the A4 sheet on which the task was printed, in effect halving the sheet vertically, then horizontally and then once diagonally. As a result, the paper was cut into 6 (virtual) pieces, as shown schematically here (Fig 2).

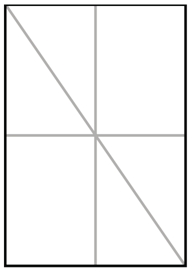

Fig 2

This move towards using a geometric model looked promising, the more so that it was spontaneous, and especially after having used the ill-fitting 'add tops and bottoms' rule. However, the fact that the pieces were not all identical suggests that R might not have been entirely clear about how this representation models fractions, especially as this statement by W was ignored:

> If you cross it again [draw the second diagonal], it will be an eighth.

Nonetheless, R came out with this nice statement, which might have been based on his geometric model: "It's got to be higher than ¼ but less than ½".

To gain some clarity, we asked the students to make an actual drawing. R proceeded to draw this (Fig 3a, right), which exactly mirrored his earlier gestures. We then asked where ⅙ is in the drawing, and R replied, "six pieces". This suggests that he was not at that moment concerned about the size of the pieces and that he would probably be prepared to call each 'a sixth' (whatever this word might mean). This is supported by the fact that he then spontaneously drew a circle cut into six, again quite different, pieces (Fig 3b, right), and then drew a rectangle cut into three different size pieces, each of which he called "a third". By contrast, while R was talking of "6 pieces" in the first shape, D pointed to one of its triangular regions and said "that's half a quarter", presumably as a way of saying it is *not* ⅙.

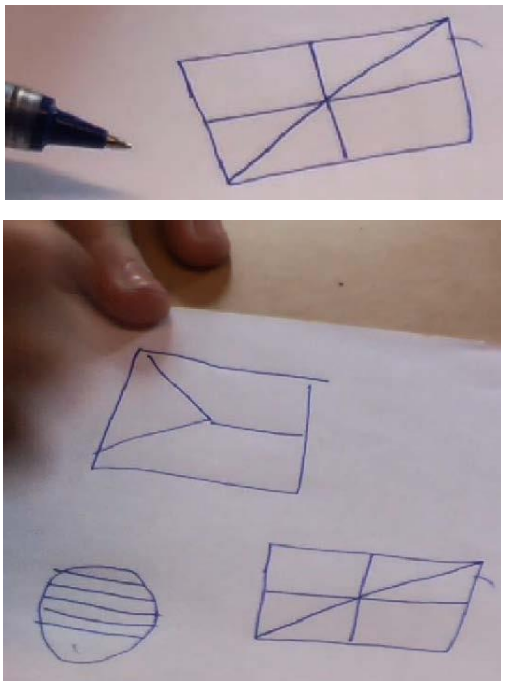

Fig 3a and 3b

We didn't discover why R made the last of his drawing, but when I then asked, "Do they (the three regions) all look the same to you?", the students all said No. My question inadvertently reminded the students that the pieces 'should' be 'equal' when we're dealing with 'fractions'. Thus, when I returned to the first drawing and again asked "where are there sixths here?", they all said, "They have to be equal". R then went on to draw the second diagonal and said, "8 pieces here".

We had now reached the stage where, aided by a cue from me, we had a diagram that could be used to model the fractions in the Recipe item effectively. However, it did not happen, at least not immediately. I asked the students, "Can you use the diagram to help you now, as you've got eighths there?". One of the students replied, "We are dealing with squares, but soup's normally in a bowl". We were all amused by this, though it also suggests that for him the visual model was still rather abstract! R's response, for some reason, was to switch to decimals. He started by writing "0.25 × 4 = 1", and then "1.3 recurring" (using the dot notation), which he seemed to think represented one third. We didn't probe why he wanted to work with a third, but he declared that the answer was 0.55 (from 0.25 + 0.3, presumably).

To get the students back on track, the teacher asked them to explain their original approach again. They did this quite effectively, though it took them nearly two minutes to re-establish that this involved adding a quarter and an eighth.

One student started by saying, “Take a quarter; what’s half of a quarter?”, to which R replied, “Half of a quarter is an eighth”. W then said, “Why don’t we do half of half a pint of cream which is a quarter, and add what half a quarter was to the quarter” and finally D said, “(Add) an eighth to a quarter, is it?”

However, having re-established the original, rated addition method, R adopted his inappropriate numerical rule again: “Two tenths..? A fifth! Is it a fifth?” (presumably from ½ + ⅛); W ‘corrected’ this by saying “two twelfths” (presumably from the correct expression, ¼ + ⅛).

During this discussion R had suddenly came out with this:

> It’s three quarters of a half! So split a half into quarters, that would be 8th, 8th, 8th, 8th and then using three of them, that would be the answer.

Here we have a completely fresh method, which not only works but is more powerful and abstract than rated addition, or the (potential) use of the geometric model. I asked R to explain this, which he did by writing ½ and then writing four ⅛s radiating out from the ½ (Fig 4, below). [For the first ⅛, he had originally written ¼ for “one quarter of a half”.] R then explained,

> Use three of them [draws a loop around three of the eighths, and crosses out the remaining one] and add them together...

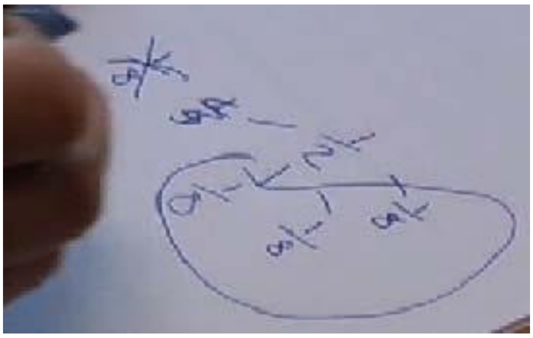

Fig 4

It might be thought that we had reached our goal. A powerful and quite abstract method of scaling, explained in a clear and quite concrete way. However, having derived three one eighths and having stated the need to ‘add them together’, R went on to give the answer “three twentysevenths (sic)”! W ‘corrected’ this to “three twentyfourths”.

Meanwhile, D had rather quietly said “three eighths” and when I asked him about this he replied:

> Oh yes, they’re all the same [denominator] ... so you just add [numerators] ... comes to three eighths.

R now seemed content to go along with this alternative numerical rule, though there is nothing to suggest that he (or indeed D) had any warrant for this, other than perhaps remembering that he had heard the rule before. W would not commit himself to ⅜ or ³⁄₂₄: “I just can’t do fractions”.

We now referred the students back to their first diagram, and asked how it might help with their rated addition method, ie with finding ¼ + ⅛. R now shaded the top-left region (¼) of the rectangular shape (see Fig 5) and, prompted by D, a non-adjacent triangular region (⅛). He then said, with an element of surprise in his voice,

> That’ll be ... So, three eighths!?

We finished the interview at this point, with the students pleased (and relieved?) to have found two methods that worked. They also expressed some surprise and amusement that what now seemed an obvious answer to ¼ + ⅛ had taken them so long!

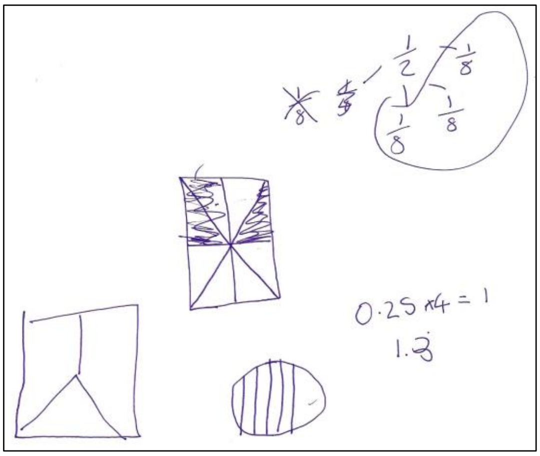

Fig 5

The second interview

Two weeks later, I interviewed two of the students, D and W, again. I once more asked them about ¼ + ⅛, and whether they remembered how we had solved it last time. They were very hesitant. Two minutes into our discussion, W said that the answer ⅙ had come up and that we'd had "add a quarter and a quarter and a quarter, or something ... which was three quarters". I didn't pursue this, but instead asked what method they might use now, to add the fractions. Again they were very hesitant, but eventually W suggested "Draw it out again, and then try to figure out how...". He then drew the left hand figure in Fig 6 (below). However, because of the spurious regions that his 'diagonal' lines produced, I drew the figure again (right), trying to ensure that the partitioning lines met at a single point. The poor quality of W's drawing might have arisen partly out of nervousness, but it also suggests a lack of insight about what a functional drawing should look like.

D then shaded the drawing, in much the way R had done, first shading a quarter of the figure, then an eighth, but without having a gap between them this time. He then ran a finger over the shading and said "So then we've got ... three eighths...". W then said, "Yes, three eighths, and I think we had to add them together ... which was three twentyfourths". After a nervous pause, the conversation continued thus:

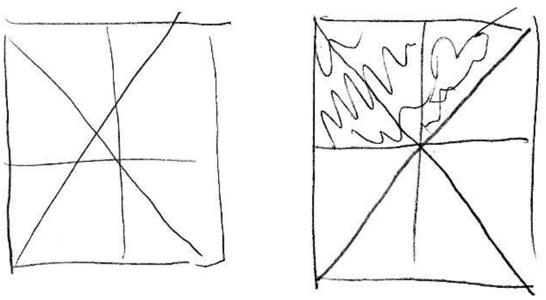

Fig 6

> W: I think we ended up saying add three one-eighths together...
> DEK: Which is what?
> W: three twentyfourths.
> D (hesitantly): Wouldn't it be three eighths?
> W: Oh yeh! You said three eighths [last time], didn't you?
> D: Yes...'cos you don't add the denominators; just leave it out... 'cos they're all the same.
> W: We have to make the quarter the same as the eighth, we have to double it, so it would be two eighths.

Here the students are again making use of two forms of knowledge - geometrical and numerical - but their understanding still seems tentative and they don't fully coordinate the two. They both seem to agree that the drawing shows three lots of one eighth. However, D is hesitant about concluding from this that there are three eighths altogether. He seeks confirmation from a rather tenuous numerical rule along the lines of 'you don't add denominators when they're the same'. W's answer of "three twentyfourths" seems to be based purely on a numerical rule again ('add tops and bottoms') - it has no obvious connection to the drawing or the Recipe scenario. His subsequent rule about having to make quarters into eighths ("we have to double it") also seems to be purely numerical - it doesn't seem to arise from or connect with the drawing even though it is obviously possible to see from the drawing that $\frac{1}{4}$ is equivalent to $\frac{2}{8}$.

Conclusion

At a minimum, the two interviews show that it is not necessarily the case that having solved a problem once (to everyone's satisfaction, it seemed), students will solve the same problem with greater ease a few weeks later. The topic of fractions is complex and students will have 'engaged' with it over many years – though perhaps not always in a way that has helped them make the topic their own. It is not surprising, therefore, that they draw on a range of ideas and that these are persistent even when they are contradictory. Note also, that these ideas may concern not just the 'meaning' of fractions, but what one typically 'does' with them in the classroom - or interview.

It seems unlikely that the top down approach that I have recently witnessed in half a dozen 'Shanghai' lessons in the UK will improve matters. In those lessons, the material was narrowed down to such an extent that it is almost impossible to go wrong but also almost impossible for students, in this forcibly blinkered state, to get a sense of what the maths that is being 'delivered' to them might be about. This approach, which is currently being funded by the UK government, seems to be based (see eg NCETM, 2014) on the essentialist claim that successful learning is achieved by getting students to embark, in unison, in tiny steps prescribed by the teacher, on what might be called The Long March to Maths Mastery.

As with our current three students, it seems likely that the residue of such teaching will be a set of half remembered rules that students have difficulty in knowing when to use and in knowing what they mean in relation to the mathematical task they are dealing with. To avoid such an outcome, it seems to me that students need many opportunities to express and revisit particular ideas, make connections, make sense of contradictions and modify their ideas accordingly.

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References

Cobb, P. (1994). What is mind? Constructivist and sociocultural perspectives on mathematical development. *Educational Researcher*, 23, 7, 13-20.

Hart, K. (1981). Ratio. In K. Hart (Ed.) *Children's Understanding of Mathematics: 11-16*. London: John Murray.

NCETM. (2014). Mastery approaches and the new national curriculum. Retrieved from https://www.ncetm.org.uk/public/files/19990433/Developing_mastery_in_mathematics_october_2014.pdf

Ofsted. (2012). *School Inspection Handbook*. January 2012 No. 090098.

Ofsted. (2015). *School Inspection Handbook*. January 2015 No. 120101.