Understanding the array as a model of multiplication

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The rectangular array is widely regarded as a key model for developing an understanding of multiplication. It can provide insight into the structure of multiplication and make visible its commutative and distributive properties. Also, as the array evolves into the area model, it can aid the shift from multiplication with whole numbers to multiplication with rationals. However, research literature on primary school children suggest that getting to grips with the structure of the array is far from trivial. Our work with secondary school students suggest that we tend to underestimate these difficulties and move on from the array too quickly. In this paper, we discuss two interviews with Year 7 students (age 12+) in which we asked them to explain why one can use multiplication to evaluate an array.

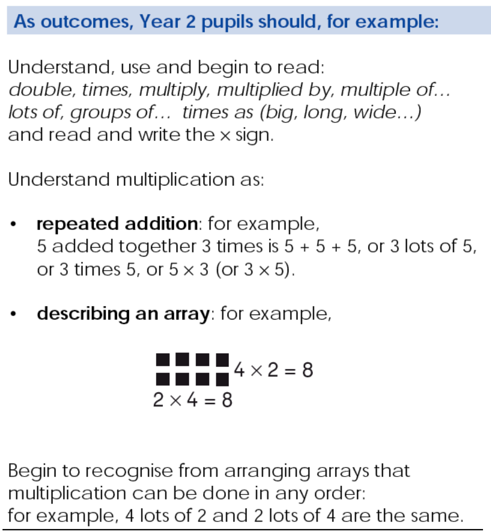
Key words: array; multiplication; model; understanding

Introduction

As part of our recent work on the ICCAMS Maths project (Hodgen, Coe, Brown & Küchemann, 2014) we have been developing teaching materials involving models of multiplication including, in particular, the array. Views about the array as a model to support students' understanding of multiplication often seem inconsistent. On one hand, it is generally agreed (or at least lip service is paid to the idea) that the array provides a powerful model of multiplication: for example, for showing that multiplication is commutative (rotate the array); for showing that it is distributive over addition (partition the array); for revealing the structure of multiplication (be it 'repeated addition' or 'equal grouping' of rows or columns, or be it the 'partnering' or 'Cartesian product' of the elements in a virtual row and column); and, as the array shifts to an area model, for representing the multiplication of rational numbers.

On the other hand, the array does not seem to be used all that widely, and where it is, its cognitive demands seem often to be underestimated.

Regarding its use, the array is mentioned very early on, namely for Year 2 students, in the 1999 National Numeracy Strategy Framework document (Fig 1). A similar statement is given for Year 3. However the array is not mentioned again[1], nor does it appear in the 2001 *Framework document for Key Stage 3* (Years 7, 8, 9). The


As outcomes, Year 2 pupils should, for example:

Understand, use and begin to read:
double, times, multiply, multiplied by, multiple of…
lots of, groups of… times as (big, long, wide…)
and read and write the × sign.

Understand multiplication as:

- **repeated addition**: for example,
  5 added together 3 times is 5 + 5 + 5, or 3 lots of 5, or 3 times 5, or 5 × 3 (or 3 × 5).
- **describing an array**: for example,

  4 × 2 = 8
  2 × 4 = 8

Begin to recognise from arranging arrays that multiplication can be done in any order:
for example, 4 lots of 2 and 2 lots of 4 are the same.

Fig 1: NNS Framework document, p.47

[1] At the presentation of this paper, Anne Watson made the comment that the DfEE was reluctant to allow references to the array as these were deemed to be providing advice on teaching.

'new' National Curriculum document *Mathematics programmes of study for key stages 1 and 2*, published in September 2013, mentions the array just five times, and in rather cryptic and glib statements like this one:

> They make connections between arrays, number patterns, and counting in twos, fives and tens. (p.8)

Again, the equivalent document for Key Stage 3 does not mention the array at all.

The uniform appearance of the array, be it composed of identical elements like 'dots', with gaps between them, or squares or rectangles with no gaps, belies its complexity as a model of multiplication. To us, the structure might seem obvious, but this is not necessarily so for students. Consider the array in Fig 2. Image a student who can see that the array has the same number of elements in each row and the same number in each column, and that, moreover, there are 8 elements in the top row and 5 in the the left hand column so that it can be called a '5 by 8' array. The student might still not see that this provides a model of 5 × 8. At first sight, the 5 and 8 seem to be acting in similar ways, as 'composite units' (eg, Steffe, 1992) that measure the size of each row and each column. However, if the array is to be seen as a model of '5 lots of 8', say (ie multiplication as repeated addition or equal grouping (eg, Anghileri, 1989)), then the 5 has to take on a second meaning, as not just the number of elements in each column but as the number of rows - a 'composite, composite unit'.

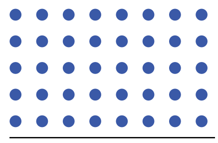


Fig 2

The array can also be interpreted as a model of the Cartesian product. Here we can revert to seeing the 5 and the 8 as both acting as composite units, but not just as measures of each of the given rows and columns, but, crucially, of a 'virtual' row and column as shown in Fig 3. The elements of the array can then be seen as all the possible 'couples' or 'partners' that can be formed between these two virtual sets of elements. The symmetry of this, in terms of the role of the 5 and the 8, is appealing, as perhaps is the fact that here we have a model of multiplication "that does not directly involve the operation of addition" (Anghileri & Johnson, 1992, p.161). Vergnaud (1983) wryly observes that

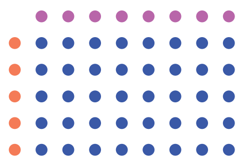


Fig 3

> The Cartesian product is so nice that it has very often been used (in France anyway) to introduce multiplication in the second and third grades of elementary school. (p.135)

It is tempting to say the same thing about the array, including what Vergnaud says next:

> But many children fail to understand multiplication when it is introduced this way. (p.135)

There is considerable evidence to support this claim about the Cartesian product (eg Brown and Küchemann, 1976), which Vergnaud explains on the basis of his analysis that it involves a double proportion.

Similar evidence can be found about the array, especially in studies of young children (eg Outhred & Mitchelmore, 2004; Barnby, Harries and Higgins, 2008). Battista, Clements, Arnoff, Battista & Borrow (1998), working with 7–8 year olds on tasks involving arrays of squares, describe "three levels of sophistication in students' structuring of 2D arrays of squares" (p.58). They make a nice distinction between 'local' and 'global' structuring. As an example of the former, they describe the responses of a student, BH, to two tasks involving a 3 by 4 rectangle. First the student was asked to cover the rectangle with unit square tiles, which he did in the order indicated by the numbers in Fig 4a, below (their Fig 10a), ie top row, bottom row, left side, right side, middle. The tiles were then hidden and the student asked to draw what the arrangement had looked like. BH drew the tiles, one by one, in the same order he had placed them, as shown in Fig 4b. Battista et al describe this as partial row structuring, which

> ...involves local rather than global structuring. That is, when BH structured the array into top, bottom, sides and interior components, he had organised only parts of the array. He did not coordinate his structuring of the squares or components to obtain a uniform organisation that he could apply throughout the array. (p.520)

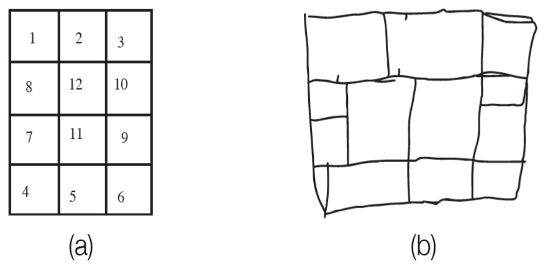


Fig 4

Battista et al make a powerful case that the row-by-column structuring of the array does not reside in the array itself, even though it might appear so to us who have long ago made sense of it, but "must be personally constructed by each individual" (p.531).

Simon and Blume (1994) report on a teaching experiment with adult students (prospective elementary school teachers). In one task, the students were in six groups, each seated around a large rectangular table. Each group was given a small cardboard rectangle and asked to determine how many such rectangles would cover the table. All of the groups used the cardboard rectangle to measure the length and width of their table and then multiplied the two numbers. However, when the teacher asked why they had multiplied the numbers, the students struggled to find a structural explanation:

> Some asserted that "it seemed like the easiest way", or that "in previous math classes you learned the formula for areas". Some responded that they did it because it works; they had seen examples of how the product was the same as the number arrived at by counting up all the rectangles. The teacher pressed them to consider whether there was any reason to expect that their method would always work. Most of the students seemed puzzled by this question... (p.478)

In this paper we report on interviews with two groups of three school students who gave very similar responses to the adult student teachers. Each group came from a Year 7 mixed attainment class (but from different London schools). In one case (Group A) the students were classed as low attaining by their teacher; in the other case (Group B) they were high attaining.

The Interviews

The interview with Group A lasted nearly an hour. The aim was to trial tasks for a potential lesson, in which we imagine cutting and re-joining a 12 cm by 25 cm rectangular piece of card to make a 3 cm by 100 cm rectangle and a 60 cm by 5 cm rectangle.

We began the interview by showing the diagram in Fig 5 for just 3 or 4 seconds. We then asked the students to give a "rough guess" for the number of "little squares" covered by the white rectangle. The students, who we shall call Fiona, Mari and Neil, responded with "Around 50", "Yes, 50" and "Around 55". These are quite low estimates, given that the actual number is 300 (the rectangle turns out to be 12 unit squares high and 25 unit squares wide). This suggests that these estimates were more likely to have been made by simply thinking of a 'large' number than derived in

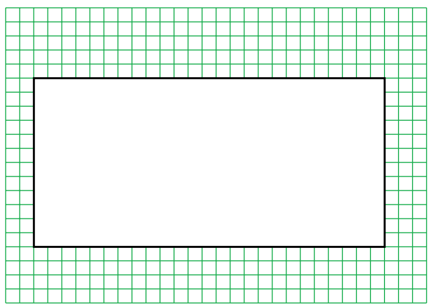


Fig 5

some way from estimates of the rectangle's dimensions. This is perhaps not surprising, since the students were not given time to scrutinise the diagram in detail and since they were being encouraged to guess. The interview continued as follows ('Int' is the interviewer).

> 1.56 Int: If you had longer to look at it, what would you do to try and get the exact number?
> Neil: Count how many squares up and how many goes along and times it [talks over Mari who says the same kind of thing].
> 2.10 Int: Why would that work...
> Neil: Because instead of adding - Mari: Because it's quicker, timesing.
> Int: Why not add or - Neil: Add is longer - Mari: Add takes more time.
> 2.33 Int: Times is quicker. But why is that the right thing to do? Why does it give you the right answer rather than just a quick wrong one?
> Mari: Because if you add it will take too long and you'll lose count a bit, yeh, but if you do how many squares going up and then across you'll know how much to times it by and then you'll get the correct answer.
> 2.56 Int: I'm still not clear why it's correct. I mean, I can understand that it's quick, but it could be quick and wrong, couldn't it?
> [All three start to answer. Int asks Fiona.]
> 3.08 Fiona: Basically I think that it might be right. I think it's better timesing because if you did it without the squares it would be the same if you see what I mean but if you added you might lose count or you might forget where you were, and with timesing it's easier and you don't have to count any squares or anything.
> 3.33 Int: OK, it's nice and easy, if you know your tables, or if you've got a calculator... Neil, do you have anything more to say why it works actually?
> 3.44 Neil: If you add you have kinda like all the lines going down but if you times it fills in basically a whole big square of .... of little squares.
> 3.56 Mari: I've got something to add... If you add you could think you've done one square but you've actually missed it and you just end up getting the wrong answer.
> 4.12 Int: Yes ....

Throughout this two minute sequence, the students seemed to persist in the belief that we were asking about the functionality of their method (it's quicker and more reliable), rather than about its validity (why does it work?). Is this because they think of mathematics as being primarily about procedures and they are not used to giving explanations in terms of mathematical structure? The nearest hint that we get of mathematical insight seems to occur at 3.44 where Neil talks of filling the rectangle (big square) with little squares. He also mentions 'lines going down' so perhaps he is thinking of columns of squares.

It seems likely that the rule that the students were using is something they learnt (at primary school) in the context of area rather than more broadly with regard to arrays. 'Area = length × width' (or some such) is such a simple little rule, it is not surprising that teachers latch on to it and that students remember it (and even apply it correctly much of the time). However, the 'length × width' rule is especially difficult to interpret in the context of area - we are multiplying two lengths and creating an entirely new entity, area. What is area? Do students realise that it is (usually) measured in unit squares, and is about covering a region? It is thus perhaps not surprising that students find it difficult to hold on to any meaning, if they ever had the chance to construct one, that they had for their easily remembered rule.

In the previous school term these students experienced a lesson on arrays that their teacher was trialling for us. The arrays there were relatively small (5 by 9 and 6 by 17) and were composed of dots rather than space-filling squares. Also, the class and the teacher spent some time talking about rows and columns. It seems that this experience did not impact on the current task, either because the experience was too long ago and/or too brief, or because arrays of dots are not sufficiently similar to arrays of squares for these students. Nonetheless, it would have been interesting to return to these tasks and to see whether they could construe their current rule in terms of numbers of rows or columns. In the event, we decided instead to stop our line of questioning about the rule, and, after a brief look at the diagram again, we moved on to our other prepared interview tasks.

The Group B students were interviewed just after an array lesson that their teacher was trialling for us. This involved a task similar to the Group A interview task, about a rectangular card (9 by 17 this time) covering a grid of squares (Fig 6). The lesson was highly structured with a focus on partitioning the card into rows or columns of unit squares. Despite this experience, and despite their high attainment, these students, who we shall name Nikki, Yara and Theresa, gave very similar responses to Group A:

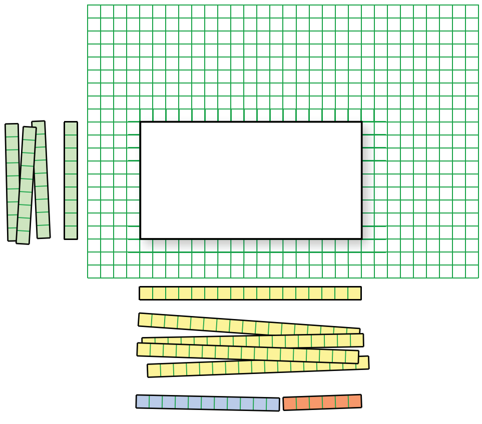


Fig 6

0.30 Int: We were finding the number of squares underneath our white piece of card.... People said all you really need to do is multiply the two sides of the rectangle. Why does that work?

0.58: Nikki: Because to find the area of a rectangle or a square you have to always do length times width.

1.08 Int: That's a rule which we learn and which is true, but why is that rule true?

1.15 Nikki: Because if you count up all the squares it would actually give the same answer as the area. But if you multiply it saves more time.

1.30 Int: OK, so yes, if the rule is true, and we know it is, then it must give you the same answer as if you counted every one. So the rule is quick but why... Could you explain why it must work?

1.48 Yara: It works because they have like the same length and width...

Int: Could you say a bit more?

2.02 Yara: If you like times it you get the same answer.

2.10 Nikki: It would normally work for a square or rectangle... It will work for a parallelogram, but won't work for a trapezium I guess. It has to be - the length for both sides has to be the same, and the width for both sides has to be the same.

[We spend some time discussing how to find the area of a trapezium, at the end of which Nikki draws a rectangle which she divides into 6 by 4 squares.]

3.52 Int: That's a nice drawing. How many squares are there in that rectangle?

04.00 Nikki: So there will be 1, 2, 3, 4, 5, 6 [counts along top row, writes 6]. Times by 1, 2, 3, 4 [counts down the left hand column, writes 6×4=24], which is 24.

Int: OK. Can you use that to explain to me why 6 times 4 is a quick way of getting it.

4.24 Nikki: Um, so, if you count up all the squares it will still be 24, and if you multiply the length, which is this [gestures along top] and the width, which is that [gestures along left edge], you would still get 24 because you're... The area is basically all the ones inside and if you multiply it shows the area which is inside the square, I mean the rectangle.

4.50 Int: Yes, we're trying to count all the squares inside the rectangle, and you're telling me that 6 times 4 will give me all the squares. But why will it give me all the squares?

5.00 Nikki: Because if you multiply the length and the width it will basically give the area, and um... The area means that the area inside like a 2D shape, or a 3D shape, um... so... the area in this shape would basically be 24 because, yeah, it says how much is inside.

5.25 Int: Yes, we know it's 24 because we can count every one, but I still don't quite understand why 6 times 4 is a way of doing that.

5.38: Nikki: It saves on time.

Int: It saves on time but why does it work?

5.42 Nikki: Um... <frustrated laughter>

5.50 Int: This is hard, because it is so obvious to you, it is hard to put it in words..
5.55 Nikki: It's basically the area, that's all I can say. And the area shows what's inside, and that shows how much is inside.
6.06 Int: OK, that's true, it's fine. Can someone else put it in their own words? Say I was a 5 year old and I need to know why this works... Any thoughts? [long pause] [No reply is forthcoming and we move on to something else.]

Conclusion

It is possible that our interview tasks, because they involved a blank rectangle and unit squares, pushed students towards using the rule for calculating area. The rule is easy to remember which may lead some teachers to believe that it is also easy to understand. Might it be that even our high attaining students have never been encouraged to explore the rule properly? If we had used dots instead of squares, would any of the students have given a structural explanation? We clearly need to investigate these issues further. But from what we already know, it seems likely that students would benefit from using arrays, and especially arrays of dots or other discrete elements, far more at both the primary and secondary level than seems to occur at present.

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