Making choices when solving quadratic equations

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There are three common algebraic methods for solving quadratic equations in UK classrooms: factorising; completing the square; and using the quadratic formula. However research shows that internationally students tend to choose to use the quadratic formula, even when the quadratic equation is given in factorised form. As teacher educators, we were interested in the methods used by teachers. In this paper we explore the decisions student teachers make when solving quadratic equations using three tasks. We focus in particular on how the form of the equation and the nature of the roots affect these choices.

Keywords: Quadratic Equations; Awareness; Teacher decision making

Introduction

This study was driven by the question "do beginning teachers associate certain methods for solving quadratic equations with certain forms of quadratic equations?" Behind this question lies issues for mathematics teacher education around teachers' awareness of what comes to mind when they work on a particular concept themselves, why particular methods might come to mind for them, and how this shapes the way they teach these concepts and methods (Mason, 1998; Mason & Davis, 2013).. The mathematics education literature widely reports that internationally students often use the quadratic formula despite other methods being available to them (Tall et al. 2014). This includes situations where other methods would be more efficient, such as $(x + 2)(x - 3) = 0$, $(x - 3)^2 = 4$ or even $m^2 = 9$. However, there are many issues with relying on the quadratic formula as a method. Students are prone to making arithmetic errors when using the formula, including substitution errors, errors when working with negative numbers as well as issues around the interpretation of $\sqrt{}$ and $\pm$ (Didis & Erbas, 2015). The quadratic formula does work as a solution method for all quadratic equations unlike factorisation (as predominantly only rational roots, or even only integer roots are involved with factorisation as a solution method), which has resulted in some teachers choosing only to teach the quadratic formula as a method. Indeed, equations that can be factorised may be considered artefacts of school mathematics as students are unlikely to meet equations that can be factorised arising from the material world (Bossé and Nandakumar, 2005). Yet the method of completing the square also provides a solution for all quadratic equations, but is a method seldom used by students in studies of how students solve quadratic equations, though this may be a consequence of the focus on algebraic representations in these studies rather than working with quadratic functions and including graphical representations. Until recently in the UK completing the square was usually only taught to students with high prior attainment and in some instances some lower attaining students were only exposed to factorisation as a method for solving quadratic equations. The recent changes to the National Curriculum in theory will result in a wider range of students being taught a wider range of solution methods.

Methods

This paper focuses on three particular tasks completed by beginning teachers as part of their initial teacher education course. The tasks occurred at different points in the course and the primary purpose of each task related to the aims of the course, rather than a data collection tool. These tasks all involved working with quadratic equations or equations in general and called for some form of written record to be made during or immediately after the task. There were 29 beginning teachers in the study although not all completed all of the tasks. The beginning teachers are all training to teach mathematics to students aged 11-18. The vast majority have mathematics degrees or degrees that include a large amount of mathematics and the majority were educated in England where three methods for solving quadratic equations have historically been included in secondary curricula: factorisation, completing the square, and quadratic formula.

In the first task the students were given three quadratic equations to solve "in as many ways as you can": one equation of the form $ax^2 + bx + c = 0$; one of the form $(ax + b)(cx + d) = 0$; and one of the form $(x - a)^2 = b$. Of the three equations, one had integer solutions, one had fractional solutions and the third had complex solutions. Task 2 was an activity where the beginning teachers worked in five groups with a set of cards with equations, expressions and formulae which they were asked to group according to specific criteria of their choosing, with no instructions as to the number of groups or the labels for the groups. The specific criteria reported in this paper was "according to the most efficient method for solving them" and was restricted to the cards with quadratic equations written on them, within which we include $x^4 - 3x^2 + 2 = 0$. In the third task the beginning teachers were asked to give a written explanation for a school student to follow of how to choose which method to use when solving a quadratic equation.

$(2x-3)(2x-1)=0$ $(x-1)^2=9$ $x^2-4x-1=0$	$(x-1)^2=\frac{1}{4}$ $(x-4)(x+2)=0$ $x^2-4x-1=0$
$(2x-3)(2x-1)=0$ $x^2-2x-8=0$ $(x-2)^2=5$	$4x^2-8x+3=0$ $(x-4)(x+2)=0$ $(x-2)^2=5$

Table 1: Solve these quadratic equations in as many ways as you can.

Results

Task 1 - quadratic equations

The beginning teachers solved all the given equations correctly using a variety of methods. In the analysis we focus on the three most common methods: factorising; completing the square; and using the quadratic formula. We also distinguish between the first method used and the variety of methods used.

When the equation was given in factorised form the vast majority used the factors to solve the equation in the first instance. Similarly when the equation was given in square form, the majority solved the equation by taking square roots first. However with both these types of equations there were beginning teachers who used the quadratic formula following algebraic manipulation as their first method. There were also several beginning teacher who completed the square first when the equation was given in the standard form.

		First		
		Factorised	Completed the square	Quadratic Formula
Factorised form	$(2x-3)(2x-1)=0$	14	0	0
	$(x-4)(x+2)=0$	10	0	2
		24	**0**	**2**
Square form	$(x-1)^2=\frac{1}{4}$	0	4	3
	$(x-1)^2=9$	0	6	1
	$(x-2)^2=5$	0	7	4
		0	**17**	**8**
Standard form	$4x^2-8x+3=0$	1	2	3
	$x^2-4x-1=0$	0	3	11
	$x^2-2x-8=0$	5	0	2
		6	**5**	**16**

Table 2: First methods used by form of the quadratic equation

When we consider all the methods used, the quadratic formula was used in the vast majority of the questions. Completing the square was also very common when the equation was not given in factorised form, suggesting that completing the square is used more generally as an algorithm, rather than only when the form of the quadratic equation is highly suggestive of being associated with the square of a linear factor.

	Total				n
	Factorised	Completed the square	Quadratic Formula	Other	
Factorised form $(2x-3)(2x-1)=0$ $(x-4)(x+2)=0$	14 13 **27**	7 6 **13**	12 11 **23**	8 7 **15**	14 13 **27**
Square form $(x-1)^2=\frac{1}{4}$ $(x-1)^2=9$ $(x-2)^2=5$	3 7 0 **10**	7 6 8 **21**	7 6 11 **24**	3 4 3 **10**	7 7 13 **27**
Standard form $4x^2-8x+3=0$ $x^2-4x-1=0$ $x^2-2x-8=0$	4 4 6 **16**	4 11 6 **21**	5 14 7 **26**	4 6 2 **12**	6 14 7 **27**

Table 3: Number of methods used by form of the quadratic equation

Task 2 - sorting quadratic equations by efficiency of methods

In general the equations were sorted into categories related to their form. Equations that had already been factorised in some way were grouped under factorisation as being the most efficient method of solution. Equations given in square form were grouped as completing the square. Having said this, there are some interesting categorisations made by some groups.

$(x + 3)(x + 4) = 2$ was categorised by all groups as an equation that could most efficiently be solved using factorising despite the fact that this would involve expanding the brackets, rearranging the equation and then factorising. Arguably completing the square to express this equation as $(x + 3½)^2 = 2¼$ is just as efficient a method, requiring the same number of steps, though it does involve working with fractional values rather than integers.

$x^2 + 2x - 1 = 0$ was categorised by different groups with factorising, completing the square, and the quadratic formula all given as the most efficient methods for solving the equation. We noted that given this equation has irrational solutions it is surprising that factorisation was chosen. $x^2 - 2x + 3 = 0$ was categorised similarly and here the solutions are complex. It is possible that the equations were categorised on the basis that the coefficients were 'nice numbers' as beginning teachers were not asked to actually solve the equation in this task, but rather just to state an efficient method that they would (try to) use.

Task 3 - Explaining how to choose a method

This task involved the beginning teachers giving some form of explanation or set of instructions for choosing which method to use when solving a quadratic equation. The majority gave written explanations but a few gave their explanation using some form of flow chart. Overall using the quadratic formula was mentioned the most times in these explanations (34 times) and completing the square the least (24 times). Completing the square was also missing entirely from 2 explanations. Factorisation was the most common method to be mentioned first (13 times) and using the quadratic formula was only mentioned first once, in the form of a poem (see below). Several of the beginning teachers also talked about rearranging the equations into the standard form, referring to the form $ax^2 + bx + c = 0$. Figure 1 illustrates how the explanations moved from one method to another, with the orange and grey arrows indicating the number of times students moved from the method at the beginning of the arrow to the method at the end of the arrow, and the red numbers indicating the number of beginning teachers who started with the particular method.

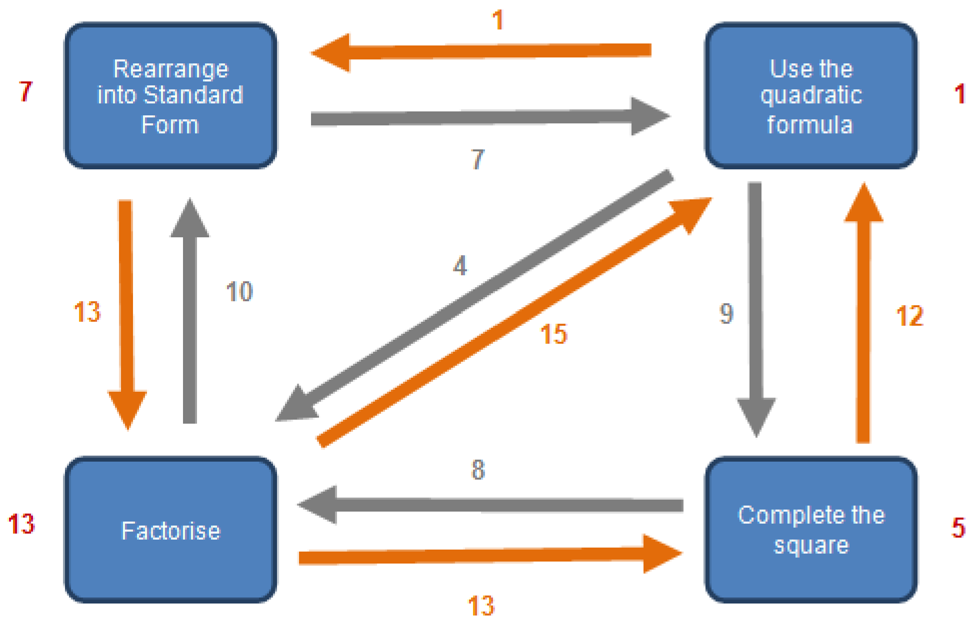

Figure 1: Movement between methods.

Several beginning teachers also gave specific conditions under which particular methods should be used. Factorisation was suggested if the equation was already in a factorised form (11 explanations) but also some restricted the method to those equations where all the coefficients are integers. When considering completing the square, 8 explanations specified explicitly that the x^2 coefficient needed to be 1 and 6 of these also specified that the x coefficient also needed to be even. The most common conditions for using the quadratic formula were that the x^2 coefficient was not equal to one, and "if all else fails".

Conclusion

The results of these tasks suggest that using the quadratic formula is not, as the international literature might suggest, a first resort for these beginning teachers but rather might be seen as a last resort when all other methods have been discounted.

Beginning teachers do associate particular methods for solving quadratic equations with particular forms of the quadratic equations. Whilst they are aware that the quadratic formula will always work as a solution method for any quadratic equation, there was a clear preference for factorising when the factors were easy to identify or looked like they could be, or when the equations was already in factorised form. Completing the square was also often used by the beginning teachers themselves when solving quadratic equations, but interestingly was not emphasised in their explanations to students as a method to use. It was beyond the scope of this study to explore whether there were pedagogical reasons for foregrounding factorisation and use of the quadratic formula, but the study does exemplify a potential difference between what comes to mind for a beginning teacher when working on a concept themselves and what comes to mind when they teach the concept.

> "If you are feeling sad and blue and you do not know what to do, let me tell you;
> use the quadratic formula.
>
> If you are feeling contemplative and ponderous, I would suggest you attempt completing the square"

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