Could an iterative approach to relational and procedural tasks aid depth of understanding Mathematics?

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This paper provides an account of ongoing research undertaken as part of my Teaching A-Level Maths course run through Leeds University and Mathematics in Education and Industry (MEI). I chose to research a sequence of A-Level lessons (16-18 year olds) where I was re-teaching the topic of transformations of graphs as my students did not have a deep understanding of the concepts involved. Whilst undertaking the research for this topic, a supplementary question arose which is 'When presenting tasks to learners designed to increase procedural skills and/or conceptual understanding, does the order in which they are taught matter?' The research shows the students in question had a perceptible growth in their procedural and conceptual understanding of the topic. The ongoing aim of the research is to see how iteratively presenting tasks could aid cognition of mathematics at all ages and attainment levels.

Keywords: Procedural; conceptual; iterative; transformations; functions

Introduction

The importance of teaching of Mathematics at all ages through a variety tasks designed for procedural and/or conceptual learning has been widely discussed through various educational research papers over the last 25 years. Whilst some educationalists prefer one technique over the other, there is a growing consensus that both have a valid place in the A-Level classroom. In this paper I discuss how the increased confidence and growth in understanding of a class of year 13 students was due, in part, to being offered a wide variety of tasks designed to enable students to develop both their procedural skills and their conceptual understanding of the topic in question. The paper will draw upon research which suggests that when procedural and conceptual tasks are provided in an iterative way to students, cognitive development is fostered. The three specific lessons taught were on combining transformations of functions and the effect that these transformations have on their graphical representations. The paper will show how there was a perceptible growth in both confidence and understanding with regard to multiple representations of the functions, the actions of combined transformations and their corresponding graphs.

Background information

The nine students, who are of average A-Level attainment had already been introduced to the topic of transformations of graphs about eight weeks prior to this study and a significant number of them had expressed concerns about their lack of confidence with both the *rules* to follow and how to answer examination style questions. Their summative assessments reflected these concerns. Previous lessons had focused initially on the procedures to follow: a set of rules had been given to the students and they were asked to use a graphical computer package to test whether the rules worked at all times, sometimes or never. Pupils were then required to complete a number of questions, using the software to verify their answers. At the time, all of the pupils got the correct answers and were told to 'learn the rules'. However, when questioned a few weeks later, it was evident that few had actually remembered the rules properly. Some were confused by the rules they had been given, whereas others had learnt the rules incorrectly and were applying them in the wrong order. This fundamental lack of understanding of the topic, linked with the findings from the examiner's reports from the previous eight years highlighting concerns that this topic is not fully understood by pupils and often answered badly, has provided further impetus for this study.

Transformations of graphs

When exploring transformations of graphs, there are many difficulties which students need to overcome: the notation, the actual drawing and recognition of the graphs themselves and multiple representations of algebraic and graphical functions. These, coupled with the fact that they can perform the transformations in more than one order and get different answers, has led students to succeed rarely at this topic Students find the "order" in which to perform the transformations to be the most challenging aspect as most students can recognise which transformations are taking place

(Borba & Confre, 1996, p.316) found "traditionally, transformations of functions have been taught with a strong emphasis on algebraic symbolism and in relative isolation from the visual transformational topics in geometry. The typical approach to transformation in most conventional textbooks varies the coefficients of a function and examines the resulting changes in the graph." This was very similar to the questions that my class had been asked and Borba & Confre (1996) call this a "template approach" and documents student difficulties including a "tendency to memorise rules without understanding their genesis and failure to make the subtle distinctions among different symbolic forms" (p.320). In the case of the students in question, the lack of memorisation was also a significant hurdle due to the complexity of the task.

Procedural and conceptual understanding

The widespread use of the terms conceptual knowledge and procedural knowledge can be attributed to Hiebert (1986). He defines conceptual knowledge as "knowledge rich in relationships" (p.3) Whereas, procedural knowledge is defined as "knowledge consist(ing) of rules or procedures for solving mathematical problems (p.7). Star (2005, p.408) however, argues that Hiebert's definitions of conceptual and procedural knowledge are too narrow and states, "conceptual knowledge has come to encompass not only what *is* known (knowledge of concepts) but also one way that concepts *can be* known (e.g., deeply and with rich connections). Similarly, the term procedural knowledge indicates not only what *is* known (knowledge of procedures) but also one way that procedures (algorithms) *can be* known (e.g., superficially and without rich connection.)" This added level of meta-cognition somewhat blurs the lines between procedures and concepts, suggesting that learners use both skills when they are aware of the skills they are using. Haapasalo (2003) builds upon these ideas further by explaining that a wider definition of procedural and conceptual knowledge implies a shift away from the traditional view that procedural understanding is *based upon* conceptual knowledge or vice versa. He says, "Any *P-C* ***(***procedural-conceptual) distinction is, at least, person, content and context dependent. With respect to educational context, it depends on the pedagogical theory guiding the teaching/learning process." (p.9) The idea that Haapasalo is suggesting is that procedural and conceptual understanding is linked in a more complex way than previously described by Hiebert.

The typical diet of tasks that have been presented to the students in the past have enabled them to become proficient in answering mathematical questions by applying rules, algorithms, techniques and shortcuts by which the correct answer can be found. Students comment that the style of teaching at A Level is *different* as they are no longer being told "what to do". The students' test scores in formative assessments show that the areas which students in the study find hardest to comprehend are those that would undoubtedly benefit from having a multiple representational understanding: graphical representations of functions, coordinate geometry and applying calculus for optimisation. Anecdotally, all the pupils in the study preferred an algebraic approach to solving coordinate geometry questions rather than drawing a sketch and *seeing* what was happening. Similarly, many were reluctant to draw sketches of normal distribution curves when studying statistics.

Many educationalists have discussed how the cognitive processes between procedural (instrumental) and conceptual (relational) learning have enabled students to have a greater understanding. (Rittle-Johnson & Koedinger, 2009, p.483) state "Children often must learn both fundamental concepts and correct procedures for solving problems in a domain. There is now general consensus that knowledge of concepts and procedures are both important, that they influence one another during learning and development." Conceptual understanding can also be aided through small sketches / drawings and in fact graphs of functions, are themselves, conceptual relationships.

The rationale behind teaching pupils by allowing them to see mathematics in multiple ways is not in question. The issue that has been raised of these students is that, until recently, they have not been used to conceptual learning, and they appear to *prefer* procedural skills that enable them to get to the correct result, with minimal effort. When a procedural approach had been applied to multiple transformations of functions, students' previous homework scores had showed that a purely procedural approach alone has not been successful. Many students scored no marks on the homework and despite following the 'rules', wrong answers were put in most cases. Students only scored marks on the types of questions where the order of transformation did not matter and where the student was asked to describe the position of a given coordinate following a series of transformations. Students were able to calculate the new coordinates but were unable to draw the graphs accurately. The topic itself lends itself to a deeper, conceptual understanding as the graphical representations of functions are, themselves, conceptual constructs. The questions that have arisen are: faced with pupils who have previously been taught procedurally, how should tasks be presented to students which enable them to have a deeper understanding of the topic, i.e. learn conceptually, when this approach is not one they are used to? does the order in which the tasks are presented matter? can students develop meta-cognitive processes to learn conceptually?

Borba & Confre set out their research by stating that the *order* in which they wish to teach the students is going to be fundamental to the way that the student learns. They postulate a general approach to teaching transformations that begins with graphical visualization. It is followed by generation of tabular data from the graphs and investigation of how this data changes under different transformations, working towards the development of traditional algebraic symbolism in f(x) form. As this approach was almost similar in structure and approach to the sessions I had taught previously, I felt it necessary to structure the tasks in a different way. Rittle-Johnson & Koedinger (2009) explored the idea of presenting tasks "iteratively". Tasks were designed to alternate between conceptual tasks and procedural skills, postulating that "experimentally manipulating the order of instruction to follow an iterative sequence can improve learning, compared to a concepts-before-procedures sequence." (p.496). Although Rittle-Johnson & Koedinger's study was based on arithmetic learning, I was keen to trial this iterative approach on the sequence of lessons with an algebraic and geometric focus as their view that "Children often must learn both fundamental concepts and correct procedures for solving problems in a domain. There is now general consensus that knowledge of concepts and procedures are both important, that they influence one another during learning and development" (p.483) fitted with my own views at that time.

Lesson structure and resources used

In order to increase the understanding of the students in this class, a series of activities were used in order to develop their knowledge of transformations of graphs. The activities were structured in a traditional way, starting by identifying prior knowledge and building upon that by introducing new concepts and adding levels of complexity one step at a time. However, each activity in the study would oscillate between a procedural task and a conceptual task, based on Rittle-Johnson's (2009) theory that iteration builds deeper understanding.

Task 1 was based around students completing a worksheet with 8 single transformations of functions. This was building on their prior knowledge and was designed as a procedural, but one where students were encouraged to discuss their answers in dyads.

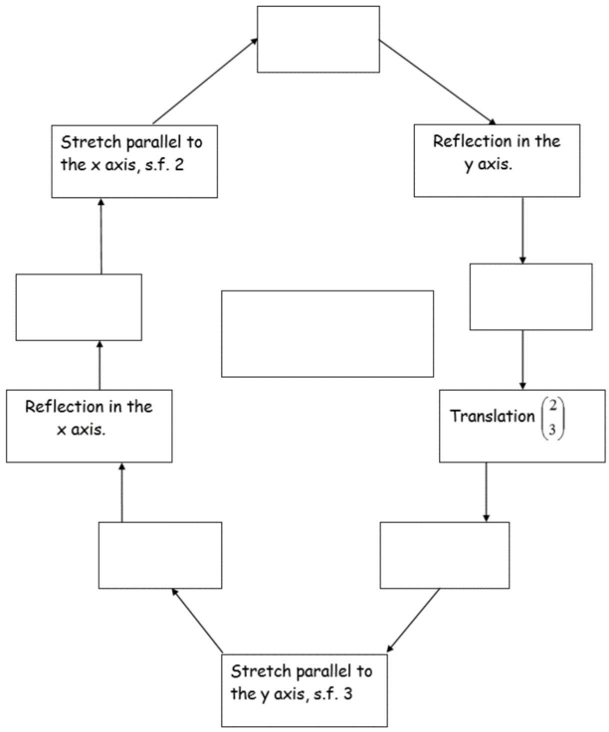
Task 2 involved learners exploring the physical aspects of transformations. This task is one designed to build conceptual understanding. Students were asked to perform a series of transformations to a right angle triangle which could be loosely placed on a coordinate grid. They were then asked to explore what happened if multiple transformations took place. Did the order matter? Did it always matter? Could some transformations be done in any order? Which ones? Students were given various enlarged triangles to aid with the transformation.

Task 3 was again procedural. Students were given traditional questions asking students to describe how the graph of $y= f(x)$ is transformed to give various graphs. For example, students were required to explain the steps taken to transform $y = f(x)$ into $4 – f(x)$ or how $y = |x|$ becomes $y = 2 - |3x|$

Task 4 was given to the students as a conceptual way of considering a single point on the graph and following that point through two successive transformations. This could be done algebraically as well as geometrically using Geogebra to help. Students were given a function, for example, $y = 1/x$ and were then asked to consider a co-ordinate of their choice that lies on the graph. Students were then asked what the new function would be and what the new corresponding coordinate would be after a series of transformations. Students were again asked to work in dyads to foster conversation and mutual support.

Task 5 was a further consolidation and procedural task given to the students to work on independently. There were two main types of question given. Firstly, students were asked to find and simplify an equation following two transformations (translation by 3 positive units in the x direction followed by a scale factor 3 stretch parallel to the y axis). The second type of question involved learners sketching graphs, labelling any asymptotes, turning points or points of intersection with either coordinate axis. Types of equations students were asked to sketch were: $1 - 2\cos(x^o)$ between 0^o and 360^o

Task 6 proved to be the most challenging conceptually. Students were given a circular grid with ten sections. Five of the sections had a transformation written inside, followed by a blank section. Students were encouraged to start with y= sin(x) in any blank position and follow the transformations in order, moving around the circular grid. The aim was to find out which position to start in to achieve the function written on the board. Students were again asked to discuss their findings as they went along, identifying mistakes and discussing why y=sin(x) could NOT start in a particular position to give the desired outcome.



Task 7 was the final task and was given in order to test their understanding of exam style questions. Students were asked to work through a pack of past questions from exam boards. As well answering the question, pupils were asked to reflect upon where candidates sitting the paper may have gone wrong. Pupils were then shown the examiner's reports for each question to see if what they had suggested was correct. This had a dual purpose of aiding meta-cognition as well as highlighting any errors the students had actually made when completing the questions.

Reflections and thoughts for the future

After following a strictly iterative sequence of lessons as researched by Rittle-Johnson, I am drawn to the same conclusion that she was, "…children first acquire procedural knowledge and then gain conceptual knowledge from reflecting on the procedures." Students in the current study had some knowledge of procedures previous to being retaught and some knowledge of procedures may be necessary to fully benefit from the conceptual tasks within the sequence of lessons. Star (2005) addresses a similar question apropos to algebra as to "whether developing skills with symbols leads to conceptual understanding, or whether the presence of basic understanding should precede symbolic representation and skill practice?" (p.404) After teaching these lessons, It appears that a mixture of procedural and conceptual tasks is beneficial to pupil's cognitive development within the A-Level classroom, but the mixture of activities which enable pupils to collaborate is of equal importance. The collaborative nature of tasks, particularly with A-Level students who by nature have a higher level of vocabulary, enables them to learn through discussion. Haapasalo (2003) succinctly writes "an appropriate assumption seems to be that *doing* should be cognitively and psychologically meaningful for the pupil." (p.17)

There is some evidence that at the end of this series of lessons the pupils had gained a deeper and more thorough understanding of the topic being taught. They, themselves, verbally concluded at the end of the sessions that they *had a better understanding of transformations of graphs* and *this was a really useful set of exercises*. Moreover, their formal test results at the end of this topic shown significant improvements from when they had done the questions first time round. In fact, although some of the formative assessment questions were not even attempted by students first time round, this time they gained full marks.

The mixture of conceptual and procedural tasks presented to the students appeared to have a significant role in this cognitive development, however, it is unclear as to whether the strict iterative order made any significant impact. After all, this study of nine pupils was not conducted in a particularly scientific way, nor can I ever know what they could achieve if I had taught them using 'chunks' of procedural based tasks followed by 'chunks' of conceptually biased ones. Certainly, for this topic, I consciously constructed the topics to provide an oscillation between a focus on procedures and concepts. I added in the elements of meta-cognition, enabling pupils to work in pairs and small groups, to verbalise their thoughts to each other as well as the group. Pupils had the opportunity to use computer aided graphical representations of the functions being transformed, which I am sure aided their learning. The re-teaching of these tasks enabled a deeper learning experience to take place, but whether their increased confidence was down to one or all of the factors mentioned above, I cannot say for sure. In order to answer the question posed at the outset though, Does the order matter? I am convinced, as with multiple transformations of graphs, the sequence in which the operations/tasks are ordered does make a difference to the overall picture of knowledge being generated. I would certainly ensure pupils in the future have some basic procedural knowledge and then use this as a platform onto which they should build their layers of further understanding; some conceptual, some procedural, using methods of re-enforcement along the way.

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