## Using patterns to model multiplication

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Our current work with ‘low attaining’ Year 7 (age 11-12) and Year 8 (age 12-13) students suggests they tend to see multiplication in terms of procedures rather than structure. On the other hand, we have found that they can sometimes make insightful use of models of multiplication once they have become familiar with them. In this paper we report on interviews undertaken by the authors with students from a Year 7 ‘nurture group’, using various patterns. The students showed evidence of being able to construe the multiplicative nature of the patterns. Moreover, in comparing two blocks of 20 dots, they showed evidence of being able to see how the structure of one 20-block is embedded in the other.

**Keywords: multiplication; models; array; dot pattern; low attainers**

## Introduction

The Education Endowment Foundation (EEF) has recently funded the ICCAMS Maths project (Hodgen, Coe, Brown & Küchemann, 2014) to undertake a large-scale randomised control trial to test the effectiveness of the ICCAMS lesson materials. A crucial feature of the trial is that it will involve teachers who have been introduced to the materials by PD leads who have had no direct involvement in developing the ICCAMS lessons or training materials.

As a prelude to this work, we refined or extended some of the lesson materials, in particular by investigating how the use of models can support low attaining Year 7 and 8 students’ understanding of multiplication. During the 2015-2016 academic year, we interviewed small groups of such students and trialled materials in a small number of Year 7 and 8 low attaining mathematics classes. From this experience we gained the impression that many of these students see mathematics as consisting primarily of procedures. So, for example, when we asked students to perform a calculation, we found that they commonly resorted to a column method, even for addition and in situations where one might have wished them to work mentally (e.g. Fig 1). When asked to multiply, they quite commonly made use of the long multiplication algorithm, albeit with varying success and sometimes in idiosyncratic ways (Fig 2). However, even when they could perform the algorithm successfully, they often seemed to struggle to explain why it works. Indeed, when asked for an explanation, students commonly interpreted this as being asked to explain *how* it works. We also found that students would often interpret a multiplicative situation additively, even when this was highly inefficient, as in Fig 3 (where the student is finding the annual cost of monthly payments of £11.60).

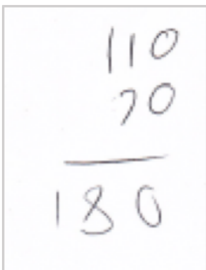


Fig 1

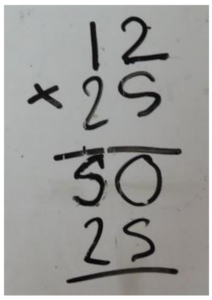


Fig 2

Fig 3

We encouraged the students to use models of multiplication, in the form of stories and diagrams. The students often found this to be very illuminating, though they also encountered unexpected difficulties, in particular with the array model of multiplication.

In one of our lessons, students are asked to compare the expressions 21×53 and 22×53. The immediate response from many students (including some from higher attaining classes) was to say 22×53 is bigger – by just 1. Students found it difficult to examine the structure of the expressions. Instead, they would light on superficial differences or try to evaluate the expressions, thereby losing the structure. We have devised a lesson which precedes this, where students are asked to write stories for expressions showing a binary operation, including multiplication. This can be very challenging at first (Brown & Küchemann, 1977). It is easy enough to write stories for an additive binary expression, by letting both numbers refer to the same kind of element, say apples: 'A box contains 21 apples, another box contains 53 apples. How many apples are there altogether?'. However, this doesn't work for multiplication (unless one is familiar with the Cartesian product). At some point in the lesson a breakthrough usually occurs, either spontaneously from the students or through prompts from the teacher, with the creation of a story such as this:

> A box contains 21 apples. There are 53 such boxes. How many apples altogether?

Once this happens, the class can begin to build a repertoire of stories that can be used in future lessons.

With such a story, students found it relatively easy to see the effect of changing one of the numbers. Thus, changing '21 apples (per box)' to '22 apples' means there is an extra apple in each of the 53 boxes, making 53 extra apples in all, whilst changing '53 boxes' to '54 boxes' means there is an extra box of 21 apples.

In the Stories lesson, and in the lesson comparing the multiplicative expressions, students were encouraged to draw diagrams alongside their stories (Fig 4). These tended to represent multiplication as 'grouping' which is perhaps not surprising as this is probably the simplest interpretation of multiplication and well-suited for multiplication of whole numbers (Anghileri & Johnson, 1992). Again, these diagrams make it relatively easy to see the effect of changing one of the numbers.

In the lesson notes for the expressions lessons, we suggest that teachers should at some stage model the initial expression 21×53 with an array, since the effect of increasing one of the numbers by 1 has the very clear effect of adding an extra row or column to the array.

Given the salience of the array, we had also expected that some students would choose this spontaneously, especially as we had been trialling other lesson materials involving the array with these students. However, to our surprise this rarely occurred.

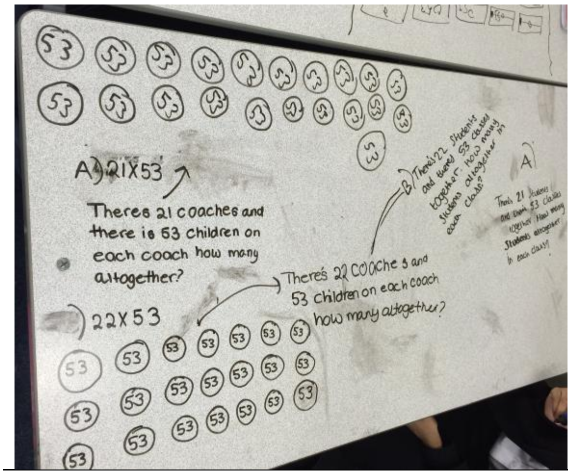

Fig 4

From our observations of these array lessons and related group interviews, it seems many of these low attaining students know that one can find the number of elements in an array by multiplying its dimensions. However, this is not necessarily true for all students. For example, Fig 5 represents a student's attempt to estimate the number of jelly beans that would fit on a piece of paper. Here it is possible that the student's difficulties are more fundamental, in that they may be concerned with being able to visualise the ordered nature of an array.

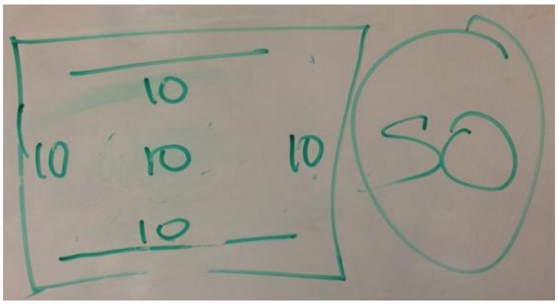


Fig 5

Students who knew about multiplying the dimensions of an array to find the number of elements, often talked about 'area' and 'length times breadth', which suggests they were recalling a learnt procedure. When we asked why this worked, they often failed to give an appropriate answer and interpreted our question as being about the arithmetic procedure. Thus it is possible that students were less familiar with the structure of an array than one might have expected from their knowledge of the area formula and from our perhaps naive assumption that they would have become familiar with the array in primary school. A further reason why arrays might be difficult for students, despite seeming to be so salient to us, is that students might well get confused by the fact that the number of elements in a row [or column] is given by the number of columns [or rows] in the array.

## Interviews

We also found that students had a tendency to partition arrays, and thus structure them in unexpected ways, rather than simply in terms of their dimensions. In an interview with three students from a Year 7 nurture group, we drew a row of 6 dots as a precursor to modelling 9×6. One of the students intervened to describe how she would model 11×6 by drawing a total of 11 such rows. To simplify the task, we then asked her to model 4×6, starting with our row of 6 dots. To our surprise, instead of simply adding three more rows, the student drew a vertical line after our 4th dot, then added two more dots to the row (thus making two lots of 4 dots) and then drew this twice more, as shown in Fig 6. This is perfectly nice, but when we tried to nudge her into seeing that her overall array is also a model for 3×8, we didn't succeed.

Fig 6

A communication issue also arose here, which might seem trivial to us but can cause considerable confusion to students. We had asked the student to model "4 times 6", by which we had in mind "4 rows (or lots) of 6". The student used the same phrase for her drawing, when she said, while pointing to the sub-rows of 4, "All equals 4, 4, 4 [pause] and then 4 times 6". It would be difficult to construe "4 lots of 6" in her drawing, and for her the phrase had a different, albeit mathematically equivalent, meaning along the lines of "4, 6 times".

It thus occurred to us that, as well as working with the simple array, it might be useful to present students with more concrete and already partitioned examples such as the sheets of stamps shown in Fig 7, and also to explore other well-structured visual representations of multiplication such as the one in Fig 8, which makes use of the dot patterns for 4, 5 and 6 that are familiar from the faces of dice.


Fig 7

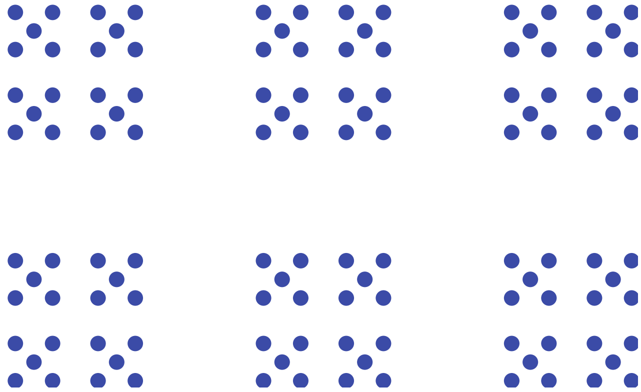

Fig 8

So in another interview with three students from the Year 7 nurture group, we introduced the idea that we could check the given number (50) of second class stamps (Fig 7) by keying the expression 3×5×3+5 into a calculator. It took a while for all three students to make sense of this, but once they had, it took less than 30 seconds for them to construct a similar expression (3×8×4+4) for the 100 first class stamps.

We presented the dice pattern in Fig 8 (above) to another three students from the Year 7 nurture group, towards the end of a 55 minute interview. We started by asking how we could "use a calculator" to find the total number of dots. What we had in mind, of course, was to express the number of dots as 5×4×6 (or 6×4×5), though this was not immediately obvious to the students – and why should it have been? One student, E, used repeated addition through most of the interview but the other two students, C and M, approached the task in an essentially multiplicative way from the outset, and eventually come up with 5×4×6, as well as the products 6×20, 5×24, 10×12, and 60×2. They were able to explain these in terms of the dot pattern (e.g., there are 24 lots of 5-blocks, 12 lots of pairs of 5-blocks). However, they did not seem to relate these to 5×4×6; indeed, they seemed to see each product in isolation.

At this point the bell rang, signalling the end of the period. However, we decided quickly to show the students another, related, task (Fig 9, below):

How many dots using my calculator? What would I multiply?

C responded by counting the number of 4-blocks in the top-left 20-block. "I reckon 4 times 5." She entered '4×5=' into the calculator, which gave 20. She then started to count the 20-blocks and keyed-in '×6=', which gave 120. At our request, she then wrote '120' on the task sheet, with '4×5×6' underneath. So the hoped-for product was construed very quickly this time and this seemed a good moment to finish the interview. However, one of the interviewers now placed the two pages showing Figs 8 and 9 next to each other. She pointed to the top-left 20-block in each figure and asked:

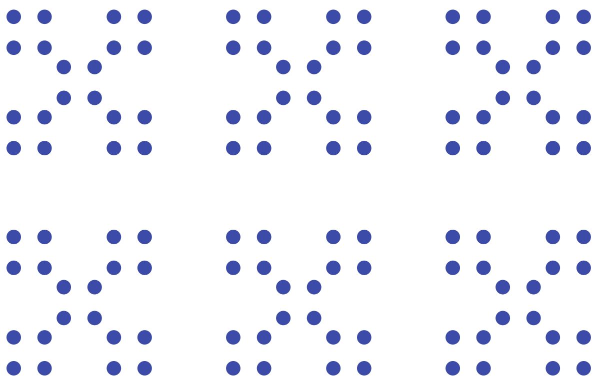

Fig 9

If you looked at the first one here and the first one here...?

C responded with "There's an extra one square", while pointing to the middle block of the five 4-blocks in the top-left 20-block on one page, and to the lack of a fifth 5-block in the top-left 20-block on the other page. M interrupted with, "One of the dots in the middle [indicating the central dots of the four 5-blocks] is one of the dots for the middle one [points to the middle block of the five 4-blocks on the other page]".

We all reacted with delight (and some surprise) at this insight and, after an appreciative pause, wound up the interview!

In our view, this last insight gained by this student (and to some degree by the other students too) is a considerable achievement. It can be argued that we have provided a structure for the students with our carefully matched dot patterns. However, they still have to discern the structure themselves. M is saying more than that the two kinds of 20-block are equal because 5×4=20 and 4×5=20, or because of some kind of awareness that multiplication is commutative. In effect, M is structuring the 20-block formed of the four 5-blocks in two ways simultaneously, by double counting (see Küchemann, 2016): the four 5-blocks can be thought of as four 4-blocks with an extra 4-block composed of the centre dots, as shown in blocks A and B of the schematic sequence below (Fig 10); further, this 4-block corresponds to the central 4-block of the five 4-blocks that form the other 20-block, as shown by blocks C and D in the sequence below.

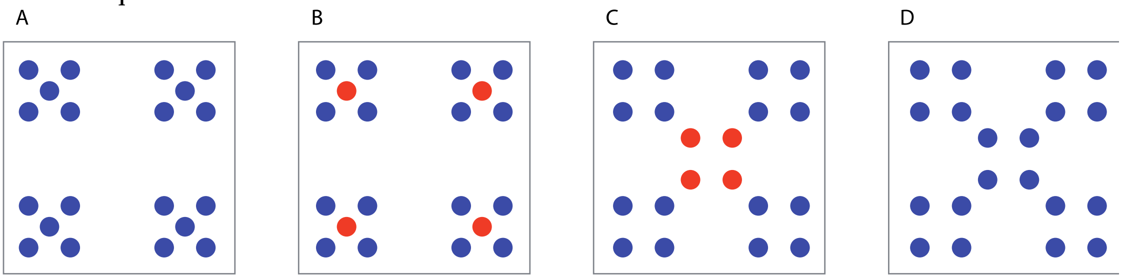


Fig 10

## Discussion

The array is an important model of multiplication that can help students extend the meaning of multiplication beyond repeated addition. It is also important because it can lead students to the area model which in turn can be used to model the multiplication of rational rather than just whole numbers.

However, our work with low attaining students suggests that making use of the array is not straightforward. Students may know that they can find the total number of elements in an array by multiplying the number of elements in a row by the number in a column; however this does not mean that they necessarily see the structure of the array in terms of rows or columns. And even when they do, there can be ambiguities in how the array is described. Consider an array formed of 3 rows of 8 dots (as in Fig 6). We can describe it as "3 times 8", but do we mean "3 rows of 8" or "a column of 3, 8 times"? Or we can describe it as "8 times 3", but do we mean "8 columns of 3" or "a row of 8, 3 times"? These meanings are *equivalent* of course, by the very nature of multiplication, but they are not the *same* and can thus lead to confusion in a class discussion.

We found that students had a tendency to construct or evaluate arrays using partitioning. This has an appealing richness about it and may also at some stage help students gain insight to procedures like long multiplication. We thus felt it was worth working with these more complex arrangements, whether or not students had a ready grasp of the idea that the array can be structured into a number of equal rows or equal columns.

In the event, the 'stamp sheets' task turned out to be extremely effective - for several reasons, perhaps: it seemed richer than evaluating a simple array; it was 'realistic'; it involved an effective partitioning; it lead to an accessible but satisfyingly complex expression which demonstrated the power of multiplication.

The dot pattern tasks based on the familiar arrangements seen on dice, shared some of these characteristics and seemed similarly effective. The tasks involve the array but, we would argue, are sufficiently different from the standard array (partitioned or not) to offer a fresh insight into multiplication and thus help students move beyond repeated addition. In the interview, two of the three students were able to structure the patterns multiplicatively. They came up with a variety of multiplicative expressions which they could relate back to the patterns though not yet directly to each other. Remarkably, though, the students showed genuine insight into how a "4 lots of 5" pattern could be re-structured into "5 lots of 4", which is a substantial mathematical achievement.

## Acknowledgement

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## Note

An animation showing other 'complementary' dot patterns like those in Fig 10 can be found here: https://www.youtube.com/watch?v=JcqrxReOzoE

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