Workshop report: using concrete materials to learn algebra

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The abstract nature of algebra causes challenge to many students, and attempts to access the subject using concrete approaches are often doomed to ultimate failure by the limited nature of the representations used, for example cups and counters cannot be used to represent negative variables or constants, and algebra tiles confuse length and area. This paper describes a hands-on approach that mimics formal algebraic procedures and accommodates both negative constants and variables, using playing cards in the context of a game. The topics of collecting terms, substitution, expanding and factorising, solving (including variables on both sides), linear graphs and simultaneous equations can all utilise this strategy, and it can be moderated to allow access for less confident students, for example by not using directed number.

Keywords: algebra; manipulatives; game

Introduction

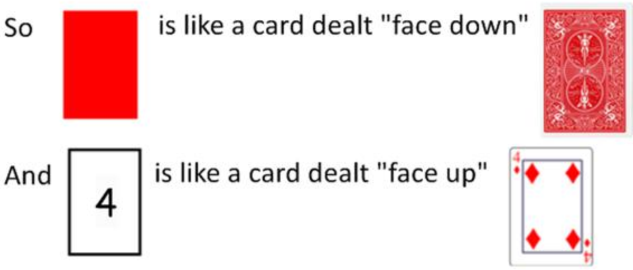
English schoolchildren do not fare well in international comparisons of mathematical attainment, and a contributing factor to overall performance is their weakness in algebra: while English students were 10^{th} of 25 countries in mathematics generally, they were in 17^{th} place for algebra (Mullis, Martin, Foy & Arora, 2012). The government's response to these results is to institute change to the way mathematics is taught in schools, with possibly "much more emphasis on pre-algebra in primary schools" (Gove, 2011, para. 34). Yet while giving children more time to absorb this challenging topic may be helpful, there is an issue about whether primary school (5-11 years old) students are capable of understanding abstract thought – Piaget (1952) said that children of this age are at the concrete operational stage, not yet capable of logical reasoning. This paper looks at a way of introducing algebra in both key stage 2 and key stage 3 (8-14 years old) that uses manipulatives – "objects to think with" (Picciotto & Wah, 1993, section I) in the context of a game.

My doctoral research investigated whether manipulatives, frequently used in introducing number to younger children, can be helpful in demonstrating algebraic concepts at secondary school level. The results were encouraging during the intervention but a post-test revealed little progress, as maths anxiety, arithmetical weakness, the fear of looking stupid and disheartening cognitive conflict took their toll (Curtis, 2015). My response to these emotional barriers was to develop one particular version of the manipulatives to be used as a game, hoping to take advantage of the benefits of social construction of meaning, intuitive approaches to problem-solving, and enhanced emotional experience that games offer (Bragg, 2006).

The algebra game

The key principle of the game is that when dealing cards face down the score exists, but unknown. It is only when the card is turned over that the score becomes known.

This mimics the features of the unknown value of an algebraic letter, and turning over the card mimics the act of substitution. There are four different unknown values every time the game is played, indicated by green, red, yellow and blue cards such that all cards of the same colour have the same value. Cards whose value is known (ie have been turned over) are indicated by white cards whose value is indicated by a number.



The game is introduced by dealing cards and describing each player's hand. This mimics collecting terms: cards of the same colour are collected so that a hand of a red card, a blue card, a 3 and another red card is described as $b + 3 + 2r$ (use of the colour name is condoned in speech, as when writing students very quickly abbreviate without being told to do so). The problem of using letters as abbreviated nouns does not occur as the letter is so closely tied with the value of the card.

The next step is to introduce the idea of the zero pair. This requires an understanding that a positive and negative of the same magnitude will sum to zero. Players can always take zero pairs from the central pile if they wish: a 3 and a -3, or a yellow and a negative yellow (identified as an IOU) – the impact on their score is shown to be zero. This allows a player to give away a card when asked to do so in the course of the game.

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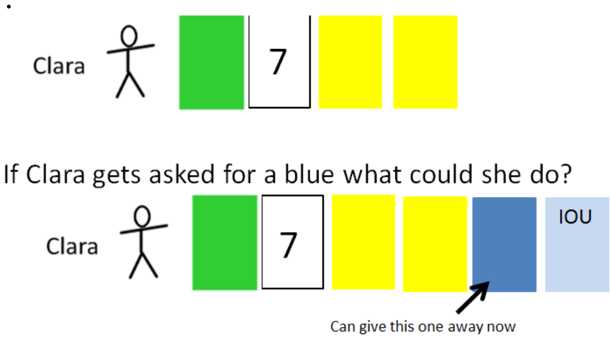
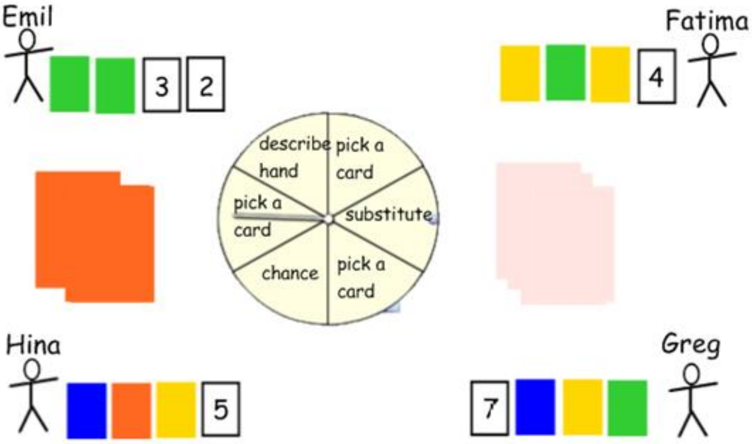


This may be a concept that is too challenging for players not acquainted with directed number, and can be omitted for a less sophisticated version of the game. This step is the only action that does not directly mimic an algebraic procedure, but it is justifiable algebraically and sets the scene for collecting positive and negative versions of the same term.

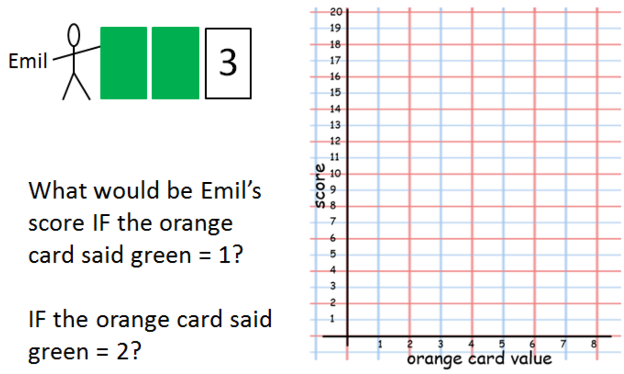
The game takes the form of a set of playing cards that is dealt between players, with the additional features of a spinner which determines each player's action, substitute cards which identify the value of each colour within the round, and chance cards that introduce dramatic interventions. Each player spins the spinner which has the options of landing on 'pick a card' (probability = $^{3}/_{6}$), 'describe your hand' (probability = $^{1}/_{6}$), 'chance' (probability = $^{1}/_{6}$), and 'substitute' (probability = $^{1}/_{6}$). 'Pick a card' instructs players to take a card from the central pile; 'describe your hand' requires players to describe their hand in its simplest term (collecting terms appropriately) with the threat that a player who correctly challenges their description can take their hand; 'chance' directs players to pick a card from the chance pack, which have instructions like 'Take a *b* from each player'; and 'substitute' signals the end of the game when players use one of the substitute cards to indicate the values of their cards. These substitute cards can either only give positive values of *b*, *g*, *r* and *y* or can include negative values for some – again, it may be better to remove negative value substitute cards for players who are not confident with directed number. Substitution directly mimics the algebraic action.



When the substitute card is used players calculate their scores now that the cards have been 'turned over' and the winner of the round is recorded. They then play another round. The winner of the game is the person who wins most rounds.

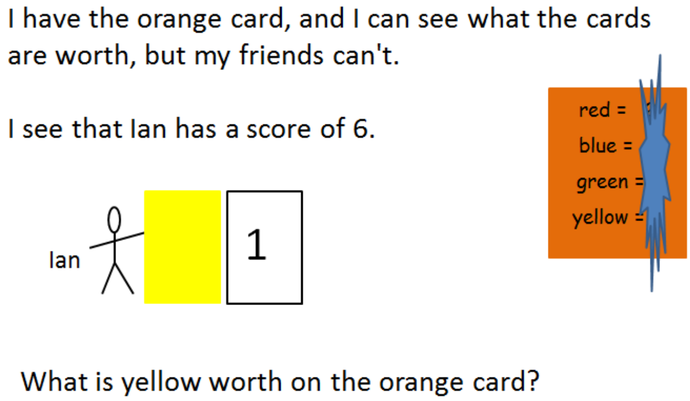
This introduction takes one lesson, allowing plenty of time for several rounds of the game to be played. I have used this with key stage 2 and key stage 3 children who enjoy playing the game, particularly when using the more brutal 'chance' cards. Students collect terms spontaneously, avoiding the common problem of writing $3r + b - 2r$ as $5r$ – b (i.e. not recognising negative terms, but seeing the – as relating to what follows the *r* term) instead correctly seeing the expression as two zero pairs of red/IOU cards, a red card and a blue, ie $r + b$.

The game is then used as context for other algebraic topics. Linear graphs are introduced as a way of finding all possible scores for a hand before the substitution value is known:



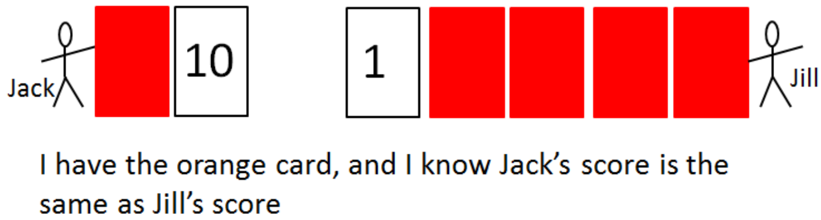
This may be presented as a bar graph to introduce the idea of the relationship between the horizontal and vertical axis, before removing the bar and focusing on the position at the top as a co-ordinate. This act of repeated substitution directly mimics the algebraic procedure.

Solving equations is seen through the context of someone having sight of the substitution card and so knowing a player's score:

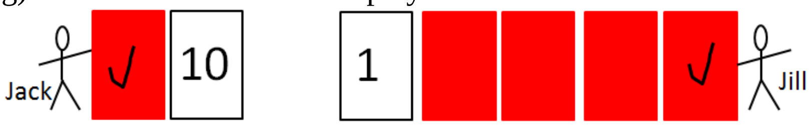


This is then extended to an equation of the form a*x* + b = *y*. The solution is usually arrived at by trial and improvement, randomly guessing a value to substitute for the yellow card and evaluating the answer. However in doing this students quickly realise that they do not need to keep adding the b to the a*x* value if they subtract it from *y* and use *y* - b to evaluate their a*x* result. Thus they discover the inverse method for themselves. Of course, it could be taught directly.

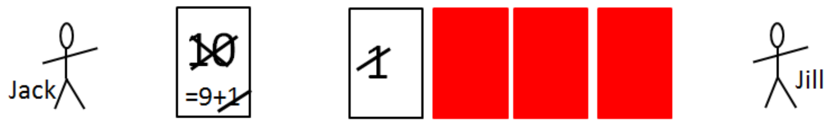
This is extended to solving equations with the variable on both sides by creating the context of someone having sight of the substitution card and seeing that two players' scores are the same as each other's:



The technique for solving these problems is described in terms of matching (and then ignoring) what is the same for both players:



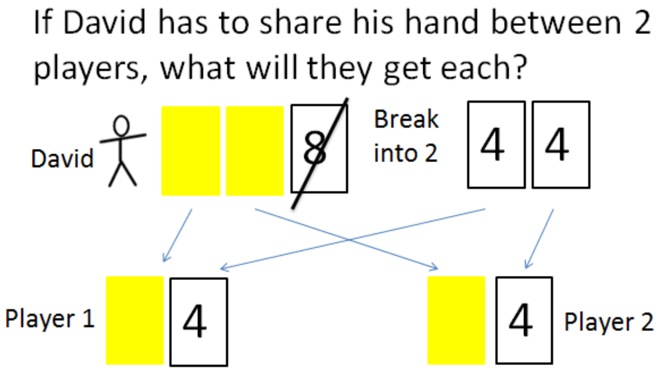
If the hands are the same as each other, and we can match one red card in each hand, the rest of the hand without the red card must be the same as each other…



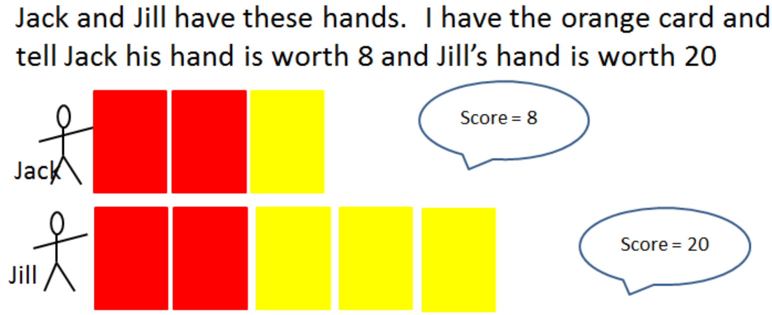
We can match the known value of 1 for each player, so the remainder must be equal…

If three red cards are the same as a score of 9, each red card must be worth $^9/_3 = 3$. This technique mimics the act of subtracting the smaller coefficient of the variable from both sides, subtracting the constant that is on the remaining variable's side and dividing the remaining constant by the coefficient of the variable. This representation reinforces the meaning of the = sign as equivalent rather than 'makes' as in 2+3 'makes 5', which is a substantial stumbling block for novices.

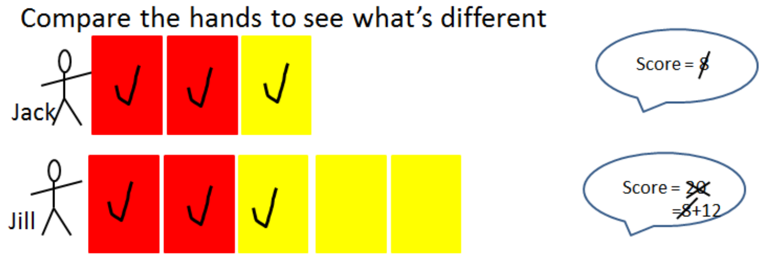
Factorising and expanding are dealt with as duplicates of hands in the context of the chance cards 'double your hand' or 'share your hand between 2 players'. This allows students to recognise that, when expanding, not only the variable must be multiplied but also the constant. Factorising becomes a matter of sharing out the cards in a hand, and swapping a constant for its factors. The concrete nature of the cards means that children find it easier to see common factors.



Simultaneous equations can be described as two hands in the same round, with two different colour cards:



Again this is solved by matching what is the same in both hands both in terms of variable and value:



Two yellow cards are responsible for the remaining score of 12, so are worth 6 each, and substituting 6 on a yellow card means the red card must be worth 1. Again this mimics the process of subtracting one equation from another (once the coefficient of one of the variables is the same in each equation), solving the resulting equation in one variable and then substituting to find the other.

It is suggested that students are given the time to play the game in subsequent lessons even though it is only setting the scene for how the cards behave, as it will reinforce the familiarity of the concept.

Limitations

The cards do not represent the multiplication of an unknown by an unknown. I am convinced that this is a much harder concept than the use of an unknown in a one dimensional way, and its common introduction immediately after covering collecting terms causes confusion between $2x$ and x^2 that for some students may never be resolved. I suggest that multiplication by unknowns is left until the foundations as indicated above are absolutely secure.

Similarly, sequences could be represented using cards, but there is the potential for confusion in that in sequences one is looking for the 'quantity of cards' rather than the value of cards of a known quantity. As it is challenging for students to work with concrete materials when the ignorance of the quantity needed prevents their being laid out, I again suggest leaving this topic until the foundations are secure and students are able to cope with abstract representations.

Conclusion

The students with whom I have worked on this have found the ideas very easy to grasp, such that 9 year olds were solving equations such as $4x + 8 = 6x + 2$ after only three hours of lessons (no negative values were used in these lessons). More importantly, as no algorithms were imposed the cards allowed students to make sense of their calculations so that they were able to collaborate and evaluate their answers meaningfully. Future work will investigate attitudes to this representation, but anecdotal evidence indicates that the usual groans that greet the announcement of an algebra lesson are replaced with "oh, miss, can we play the game?"

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