Working group report: a brief history of trigonometry for mathematics educators

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Despite the words: 'Mathematics is a creative and highly inter-connected discipline that has been developed over centuries, providing the solution to some of history's most intriguing problems.' in the Purpose of Study section, the new English mathematics curriculum alludes only to Roman Numerals in the primary programme of study, and there is no further mention of historical or cultural roots of mathematics in the aims, or in the programmes of study. In contrast, the increased expectations for lower and middle attainers in the new curriculum challenge teachers to make more mathematics accessible and memorable to more learners. The history of mathematics can provide an engaging way to do this. There are also many opportunities in post-16 mathematics. Further to our recent article on quadratic equations, we use trigonometry to illustrate some of the ways that history of mathematics can enrich teaching of this topic.

Keywords: history of mathematics; trigonometry; BSHM

Introduction

In the 2014 National Curriculum for mathematics (DfE, 2014) all KS4 (ages 14-16) students are expected to make links to similarity including trigonometric ratios, and apply trigonometric ratios to finding lengths in right angled triangles and 2D figures. Higher attainers are expected to recognise, sketch and interpret graphs of trigonometric functions for angles of any size (in degrees), and extend application of trigonometric ratios to include 3D figures, the sine and cosine rules, and the area of a triangle ($\frac{1}{2}ab\sin C$). The history of trigonometry offers an accessible way of introducing this content that is more likely to capture students' interest.

In our paper on the history of quadratics (Rogers & Pope, 2015) we set out the rationale for using the history of mathematics in education. There are several articles on the NRICH project website aimed specifically at teachers (Rogers, n.d. a-d). In this short article, we introduce some key resources and explore the development of associated mathematical ideas.

A map of the development of trigonometry

The history of trigonometry extends across the world over more than 4000 years. The map below (igure 1) illustrates the various sources. Mathematical and technical knowledge was transmitted, applied and modified by travellers across the continents (Katz, 2007; Robson & Stedall, 2009) and (Fauvel & van Maanen, 2000).

The development of trigonometry from simple naked-eye observations of the heavens to its use in the sophisticated technology of today provides one of the most important and interesting stories about how mathematics lies at the heart of a significant chapter in our cultural evolution.

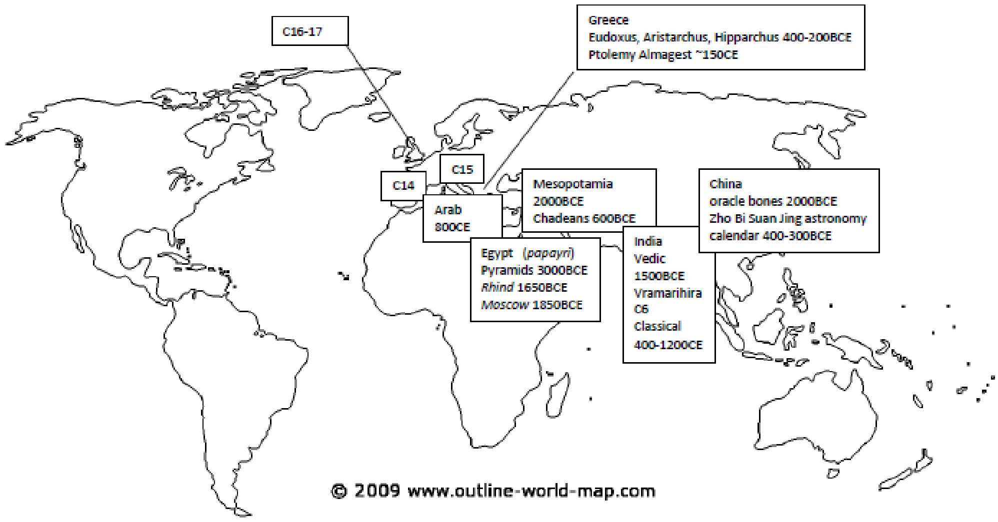


Figure 1. Map showing the historical and cultural roots of trigonometry

Beginnings of trigonometry

Motivated by wonder and curiosity early peoples linked events in the heavens to the seasons and their explanations provided the elements for primitive cosmologies and religious beliefs. By *2300 BCE Egyptian priests had divided the ecliptic into 36 sections of 10 degrees each*. (The sun's apparent daily movement across the sky is about $\frac{1}{360}$ of a circle.) *By 500 BCE the Babylonians had established twelve 30^o divisions of the sky* that became the 12 houses of the zodiac. Accurate and *systematic recording of observations* enabled the use of *interpolations using differencing techniques*, and all early civilisations produced *star charts* of remarkable accuracy. From 2000 BCE the Vedic peoples of India were constructing altars accurately oriented in an E-W direction, and working with the *ratios of the shadow-stick. Accurate Lunar and astronomical calendars were essential for religious and social organisation* and the use of horoscopes. Similar activities have also been recorded in China from about 2000 BCE.

Greek scholars were the first to develop and elaborate a *mathematical theory of the cosmos*, (from about 500 BCE) and were the first to begin *systematic map-making* of their world. Using Babylonian data and observational techniques they began to estimate the *distances and sizes of the Sun and Moon*. The stars were seen as fixed on the inside of a sphere, and so *tables of chords* were made for measuring distances between stars, and *spherical geometry* was developed to deal with these problems.

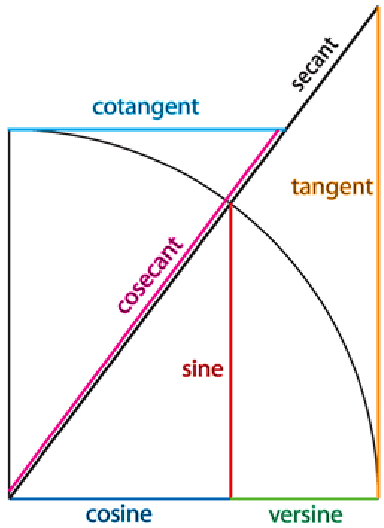
Ptolemy's *Almagest* (150 CE) contained all the then known astronomical knowledge: geometrical and numerical procedures, *the longitude and latitude of heavenly bodies*, information about *parallax, the distance and relative sizes of the Sun and the Moon, lunar theory, solar motion, and the occurrence of eclipses, transits, and occultations.* By this time, a number of algorithms were used for finding *sums and differences of angles* and a new, more accurate table of chords was produced, with the calculations using *Babylonian sexagesimal units*.

The Arab civilisation marks its beginning from 622 CE, and a century later, the Arabs had established themselves as a powerful unified force across large parts of the Middle East and by the 9th century, were collecting and translating all kinds of knowledge from Mesopotamia, India, China and Greece. From India they used the idea of the *Sine* and *second order differencing* to rewrite much of the Greek data as *Sine tables*.

For the Arabs, the key importance of trigonometry was religious. The first sight of the crescent moon to establish the new month, finding the direction of Mecca, and the correct times for daily prayers were essential.

Abul Wafa (940-998) was the first to study trigonometric identities systematically. By establishing relationships between sums and differences, and fractions and multiples of angles, more efficient astronomical calculations could be conducted and more accurate tables could be established. The sine, versine and cosine had been developed in the context of astronomical problems, whereas the tangent and cotangent were developed from the study of shadows of the gnomon. Abul Wafa established the relations between the six fundamental trigonometric functions for the first time. Thus he was able to demonstrate many new identities using these new relations such as: $sec^2\theta = 1 + tan^2\theta$ and $cosec^2\theta = 1 + cot^2\theta$. These relations are the basis of our GPS systems today.

Figure 2. The versine is used by railway engineers and opticians for measuring curvature.



By about 1300 CE, Arab scholars had established trigonometry as an independent science, with many applications in surveying, navigation and map-making as well as astronomy.

All the aspects of the trigonometry studied in school on the left hand side of the table below (Watson, 2010) were well-established and well practiced by this time.

Triangles Right angles Different orientations Compound shapes Angles Measurement of angle & length Ratio Proportion Similarity Expressing ratio as a number Understanding new notation	Wave functions Transformation of functions Inverse functions Rearranging formulae when a function is involved Levels of accuracy and rounding Use of letters to mean labels, unknowns or variables Components of vectors Surveying Polar coordinates and parametric curves

Table 1 Trigonometric topics

Regiomontanus (Johannes Muller 1436-1476) was the first European scholar to write about trigonometry in his *De Triangulis Omni Modis* (About all kinds of Triangles) of 1464. There is a direct intellectual link to the work of Copernicus who changed the whole world view.

Navigation

Finding your way across the Mediterranean was not too difficult once you had learnt the directions of the seasonal winds. However, the Chinese, Indians, and parts of the Arab civilization had no 'internal sea', and as the Portuguese sailed around Africa to the East Indies, and Europeans ventured to the Americas, the heavens looked quite different, and accurate navigation was essential for Trade, Conquest and Empire-building.

The old devices of *Quadrant* and *Cross-staff* could be used on land for finding the altitudes of stars, but were exceedingly difficult to use on the deck of a ship.

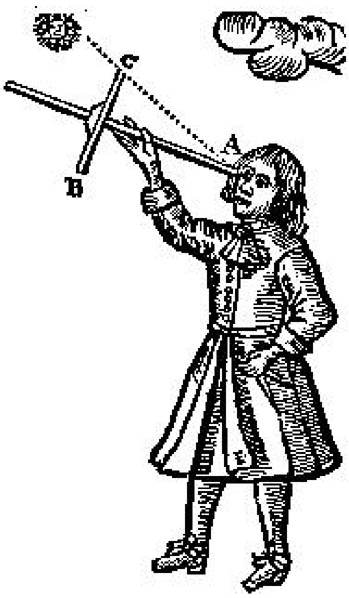


Figure 3. Using a cross-staff to measure angles

Measuring Angles with a Cross Staff

John Sellers (1692) *Practical Navigation*. The direction AB would be aimed at the horizon and the cross BC would slide back and forth along the main staff . There were many variations.
The origins are obscure; but it was used at least as far back as the Chaldeans, referred to in the Bible, a people who settled in the area of the estuaries of the Tigris and Euphrates. They were renowned for their knowledge of astronomy and astrology, and were absorbed by the Assyrian empire in the 6th century BCE. The Bible reference is from the Book of Daniel (5,30), which is the famous story about King Belchazzar and the writing on the wall. (see Rembrandt http://www.nationalgallery.org.uk/paintings/rembrandt-belshazzars-feast)

As trigonometric knowledge was further developed and slowly passed on to European scholars, probably the most important technical innovation from Arabia was the *Astrolabe*, a sophisticated instrument of toothed wheels that emulated the movements of the heavenly bodies.

Figure 4. Hispano-Moorish Astrolabe c. 1260.

In this Astrolabe the celestial sphere is projected onto the plane of the equator. There are other interchangeable plates depending on the latitude of your location. Starting from the top centre clockwise you can see the Arabic numerals for every 5^0 interval to 360^0. The Latin inscriptons in the inside circle are the names of the zodiac. The pointer (the alidade) is for measuring heights of stars. From this 'Mediaeval GPS system' you can tell the time and find your position anywhere on the Earth. This image is from the Museum of the History of Science, Oxford. http://www.mhs.ox.ac.uk/

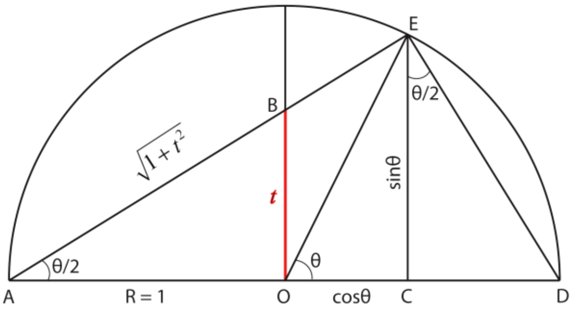
Governments invested in producing *Trigonometric tables,* and the astrolabe was overtaken by simpler instruments that were easier to handle; like the *Armillary Sphere* and importantly, the *Sextant* which, combined with the *Telescope* and *Compass* became the standard equipment on board ship. While the altitude of stars and the sun could easily be found, the problem of *Longitude* was more serious. There was no way of finding the accurate time from your home port to a place in the middle of the ocean until John Harrison invented the *Marine Chronometer* in the late 18th century.

In the 16th century, Francois Viete was the first mathematician to use specially derived double-angle and triple-angle formulae for the solution of quadratic and cubic equations, and from the 17th century with the development of algebra and the advent of the calculus, functional concepts and transformations led to the developments found in the right hand side of Table 1.

There have been many refinements built into measuring and surveying devices and into navigation systems for ships, aircraft and satellites today, but the basic trigonometry is still there and it is valuable for learners to appreciate where it came from and how it was and is still used. The opportunities for motivation and cross-subject and inter-cultural connections are considerable.

Conclusion

The social and cultural aspects of the origins of mathematics lie in solving important practical problems. Trigonometry conceived as a subject area brings together many basic aspects of mathematics that were developed for problem solving over the generations. The geometry of triangles, measurement of angles, heights, distances and time, rely on geometrical theorems that form a basis for trigonometric formulae, seen in the unit circle derivation of Pythagoras's theorem as $sin^2\theta + cos^2\theta=1$



Following Abul Wafa, the great scholar al-Biruni (973-1048) showed how many functions could be related, and he introduced the idea of a parametric representation as shown in this modern diagram.

Figure 3. Functions related in Al-Biruni's *Keys to the Science of Astronomy.*

These brief examples show that trigonometry is a coherent subject area with an interesting past and inspiring future; not just a collection of disconnected formulae to remember.

Context

This short article arises from the BSRLM History of Mathematics working group as part of its preparation for an Anglo-Danish History of Mathematics in Education conference to be held at Bath Spa University, 22-24 August 2016, see www.atm.org.uk.

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