Improving early secondary school students' abilities to create mathematical proofs: an action research study

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This action research study focuses on improving students' proving abilities, over a four-week period, in the setting of a British school in Madrid. Student participants were given twenty-two conjectures to prove over this time-frame, with varied teaching inputs. Students' proof responses were collected and coded using a proof response framework, with categories including empirical evidence, logical argumentation and visual demonstrations. The coded results were tabulated to look for patterns and to identify any increases in individual proof types. Finally, the timeline of these changes was compared with the timeline of planned teaching inputs to look for key influencing factors. It was found that (1) investigative lessons seem to support students' logical argumentation; (2) teaching on notation appears to support students' algebraic proving skill and (3) when faced with unknown conjectures, students tend to resort to using empirical evidence.

Keywords: action research; mathematical proof; logical argumentation; empirical evidence; algebraic proof

Introduction

Proof in mathematics and its place in the classroom is a fascinating topic often debated by educational researchers. In particular, this classroom based action research study focuses on the abilities of students to create their own mathematical proofs and how teacher inputs can support the developments of students' proving skills. This paper concentrates on three main findings from this practitioner research.

The role of mathematical proof in the classroom

Mathematical proof has a significant role in the development of mathematical thinking and reasoning (Kogce & Yildiz, 2011) such that students should learn to reason and construct proofs as part of understanding mathematics. Proof and logical argumentation provide an essential aspect of mathematical competence because without proof, new ideas and conjectures cannot be validated or contradicted. In addition to verifying conjectures, proofs can help with explanation, communication and discovery.

Hanna (1990) organized her thoughts about proof into three clear categories: formal proof, 'acceptable' proof and teaching proof. She defined formal proof as a finite sequence of statements, starting with an axiom, where each statement is either derived or is an axiom itself, resulting in the statement to be proved. The second type of proof, 'acceptable' proof, has simply to meet the following requirements: "the proof must proceed from specific and accepted premise, must present an argument that is not flawed, and must lead to a result which seems to make sense in the context of other mathematical knowledge" (Hanna, 1990, p. 8). Finally, Hanna identified teaching proof as focused largely on a convincing argument.

Likewise, CadwalladerOlsker (2011) recognises more than one definition of mathematical proof, referring to both formal and practical mathematical proof. The first definition, taken from Rota (1997), essentially confirms Hanna's definitions of formal proof as "a sequence of steps which leads to the desired conclusions" (Rota, 1997, p.183), following explicit mathematical rules. According to CadwalladerOlsker, most practising mathematicians are likely to use practical mathematical proof, described in the same way as Hanna's 'acceptable' proof. This has also been described by Hersh as "what we do to make each other believe our theorems" (Hersh, 1997, p. 49).

Davis and Hersh (1981) introduced and identified the difference between formalist and Platonist views on proof; the former belief relying only on axioms, definitions and theorems to form a deductive proof and the latter focusing on discovery. Accordingly, those holding a Platonist view would prefer to convince themselves or others of a new theorem, however, formalists would not be content until the proof was written formally, following the rules of mathematics (Rota, 1997).

As recognised by researchers (e.g. Hanna, 1990; Hersh, 1997;CadwalladerOlsker, 2011) mathematical proof can span a range of both formal and informal mathematical arguments but, as Hanna first described in 1990, there has been a change of attitude with the teaching of proofs as most mathematicians are now in agreement that "proofs may have different degrees of formal validity and still gain the same degree of acceptance" (Hanna, 1990, p. 7). For the purpose of this study, given the classroom context, I will be focusing on types of 'acceptable' proof argumentation, rather than formal proof.

Students' views on proof

Kogce and Yildiz, (2011) claim proving is considered to be 'unlovely', difficult and frightening by many students. However, Weber (2001) suggests it is not the students' lack of syntactic knowledge, but their inability to use this knowledge in the correct way to construct proof. He suggested that the students' difficulties lie in choosing which theorems are appropriate to use and how to apply them, rather than lacking knowledge of the theorems themselves.

Martin and Harel (1989) suggest students cannot always distinguish between deductive reasoning, empirical evidence and informal argumentation. In agreement, Porteous (1986) pointed out that a very high percentage of secondary age students do not recognise the significance of a deductive proof in geometry, algebra and general mathematical reasoning.

It has been suggested by Healy and Hoyles (2000) and Hoyles and Jones (1998) that students generally prefer empirical evidence over deductive arguments. This is further supported by Chazan (1993) who claims that some students will confidently form conclusions about entire sets from only a subset of examples, believing simply that evidence is proof. Healy and de Carvalho (2014) suggest that students tend to produce empirical arguments more frequently than conceptual proofs, but discuss the possibility of students' inexperience of deductive arguments as being a more significant explanation than simply a preference for empirical evidence.

The aim of this study has been to give students the opportunity to improve their understanding of proof argumentation to find out whether students really do simply prefer using empirical evidence or whether, as Healy and de Carvalho (2014) suggest, they simply need greater exposure to deductive proof.

Proof response framework

In order to analyse students' work, it was necessary to create a proof framework for analysis. The framework, based originally on the proof responses used by Healy and Hoyles (2000) in their proof questionnaire, was used to code the students' proof responses in this study. The framework included the following categories: Empirical evidence (E), Proof based on theorems (T), Logical argumentation (L), Algebraic proof (A), Visual demonstration (V), Inductive proof (I), Proof by contradiction (C) and, for completion, 'Don't know' (D).

The first five proof types were based on the student proof examples used by Healy and Hoyles (2000) in their proof questionnaire but the importance of including an inductive proof (I) category became evident in the pilot study of this research, during which a student participant produced an inductive-style proof. Proof by contraction (C), including counter examples, was also included to allow for some false conjectures to be included in this study. This decision was made in order to avoid the students becoming complacent that all the conjectures were true.

Methodology

As the aim of this report was ultimately to improve teaching practice to support students' abilities to create mathematical proofs, I followed one cycle of an action research methodology. This was defined by Lewin (1946) as a powerful tool to achieve positive changes in the classroom. Whilst this methodology has limitations, and in this case focuses only on my own classroom, the motivation for this study is to improve the abilities of students in this study through developing teaching practice so I felt this was appropriate methodology around which to base my research model.

The problem identified was that the students, whilst extremely articulate in their conjecturing, seemed reluctant when asked to prove their ideas. I also found that the quality of their written proofs was poor, often only comprising of a list of numerical examples, in agreement with Chazan (1993) who claimed students often use empirical evidence in place of proof. The issue then, was how to encourage students to use more sophisticated proof argumentation over empirical evidence. In agreement with Weber (2001), I felt that my students had the knowledge they needed to write convincing proofs, but that they needed support in how to create these proofs. My aim was therefore to provide teaching inputs, informed by the initial student proof responses, to support my students in developing their proving abilities.

Research model

In the final research model 50 student participants from high attaining year 7 and 8 classes were given a series of conjectures to prove over a series of four weeks. Teaching interventions were planned, as a result of initial student proof responses, and an initial questionnaire, and were delivered throughout this time-frame. Students' proof responses were collected and coded, using the proof response framework described previously. The coded results were then tabulated and compared with a timeline of teaching interventions to look for key influencing factors.

Proof tasks included conjectures based on mainly numerical and geometrical properties but also some based on the outcomes of investigations. In order to prevent complacency some of the conjectures were true statements, some were false and some were true only for certain cases. Additionally both known and unknown conjectures were included. The latter was a decision made partly to help maintain student motivation throughout the project as students are unlikely to be motivated to create proofs if the only purpose is to demonstrate something they already know to be true (Hoyles & Jones, 1998).

The planned teaching inputs given throughout the research time-frame included: no input (no), written examples (ex), peer support (pe), investigation styles lessons (in), and whole class discussions (di). The timeline of these inputs with respect to the proof tasks can be seen in figure 1. There were also unplanned inputs, occurring when students worked on tasks at home and which could not be measured. These included: parental or sibling support, peer support (outside of lessons), internet searches and other sources of information, such as mathematical books.

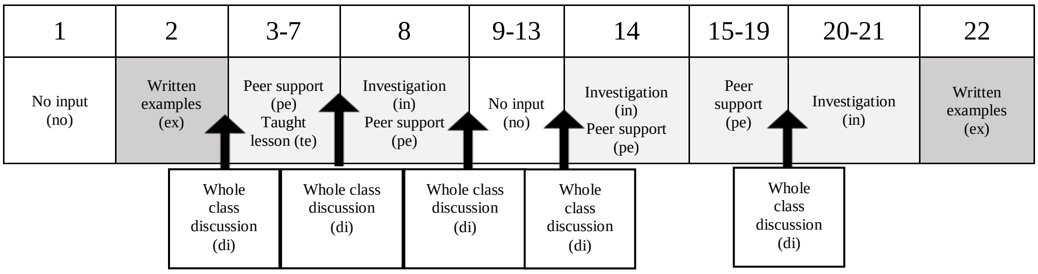

Figure 1. Timeline of teaching interventions, with respect to conjectures.

Findings

In this article, I have chosen to focus solely on three main observations which arose from the data: (1) investigative lessons seem to support students' logical argumentation; (2) teaching on notation appears to support students' algebraic proving skill and (3) when faced with unknown conjectures, students tend to resort to using empirical evidence.

Discussion

The analysis of logical argumentation (L) proof responses showed a general pattern of increase over time, but this was a weak trend. Instead, I chose to look at the conjectures which received the highest logical argumentation responses: conjectures 8, 14, 20 and 21. These conjectures received considerably higher proportions of (L) responses (48%, 28%, 75% and 60%, respectively). These conjectures were all given during investigation lessons (in) where students worked with peers (pe) and participated in a whole class discussion (di). Analysing the timeline of teaching inputs, the input common to only these conjectures was the investigation lesson (in). Given that the student participants are used to working in a conjecturing environment where they are often asked to explain their ideas in class discussions, it is retrospectively unsurprising that when working in this type of lesson, students have produced logical arguments. This finding does particularly highlight the fact that these students were not able to produce these types of arguments outside of this environment, when conjectures were given to them outside of investigations. This emphasises the potentially problematic impact of situated learning and highlights the importance of ensuring students can apply their learning in different contexts.

The analysis of algebraic proof (A) responses produced an interesting observation: teaching on algebraic notation seemed to support students' algebraic proving abilities. This highlights an issue identified by Weber (2001) and Healy and de Carvalho (2014), that the reason students cannot produce more sophisticated deductive proof is not a lack of ability but a lack of experience with this type of proof. After the taught lesson (te), which included the teaching of algebraic notation, there was a large increase in the number of students who produced (A) responses. For conjectures 1 to 6, two or fewer students responded with (A) and, with the exception of five conjectures, all other conjectures received from 9% to 39% (A) responses. This suggests that the students in this study simply needed to be taught some arbitrary algebraic notation in order to be able to create their own algebraic proofs. Furthermore, this observation suggests that students are more algebraically capable than is often assumed and that learning to apply algebraic notation to producing algebraic proof seems to be a far more interesting application of key stage three algebra, than simply routinely solving equations or collecting like terms.

The analysis of empirical evidence (E) showed a general pattern of decrease in the number of students using only empirical evidence over the course of the study. This suggests, in support of the views of Healy and de Carvalho (2014) that the students in this study used (E) initially because they lacked experience of other proof types. The change in approach by many students over the course of this study also suggests that students do not necessarily prefer to produce empirical evidence, as authors such as Hoyles and Jones (1998) had previously claimed. Another observation which arose was that when faced with an unknown conjecture, students more frequently responded with (E), in agreement with Healy and Hoyles (2000) who similarly reported that students were generally “better” at proving conjectures which were known to them. This observation is still not a confirmation that students prefer to use empirical evidence but simply demonstrates that when unsure, students will naturally resort to their familiar strategy of using examples. I believe that specialising with a few examples to get a sense of whether the conjecture holds is an approach that most mathematicians would also use, however it seems that the students in this study are not then moving onto the next step of generalising. As the overall proportion of (E) responses has decreased, particularly for known conjectures, it appears that experiencing other proof types has had a positive impact but that in my teaching of proof I have failed to emphasise the importance of generalising to show a conjecture holds for all cases. I suggest that if students were fully aware of the need not only to convince themselves through specialising, but also of the importance of producing a rigorous proof which holds for all cases, then they would not stop after writing down some examples.

Implication for future research

This study reports on one cycle of action research, undertaken with a group of students in key stage three. Over the course of this study there were many variables involved, including planned teaching inputs and outside factors, such as parental support. It has therefore been difficult to explicitly pinpoint which teaching inputs have had the most significant impact on the ability of these students to produce mathematical proofs. Whilst the initial findings from this study are interesting, future investigation is required in order to test the validity of these claims. These findings can be treated as hypotheses for subsequent cycles of this action research study. may allow for influencing factors to be isolated more easily.

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