Developing algebraic language in a problem solving environment: the role of teacher knowledge

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This paper describes a teaching sequence designed by a team of three teachers in Spain to enable a group of 13 to 14-year-old students to develop algebraic language through problem solving. Problems are introduced which provoke the thinking needed to solve systems of linear equations, without formal instruction in standard methods. We consider the mathematics-related knowledge that the teachers used while implementing these tasks, using the Knowledge Quartet (KQ) model to analyse this knowledge. In particular, we show how the connections that the teachers make between different representations of the same concept are key for the students to acquire algebraic language as one way to solve certain problems.

Keywords: algebra; problem solving; teacher knowledge; Knowledge Quartet

Introduction

The research reported in this paper is taking place in a British school (following the English national curriculum) in Barcelona, and involves the four mathematics teachers (including the first author) who teach in the two first years of the secondary stage. The school has three classes in each year group, taught by three teachers teaching at the same time. Instruction is planned jointly by the team, and this situation allows us to investigate how three different teachers implement the same material with comparable classes.

Specifically, this research aims to understand how the teachers use their knowledge to help students to learn to use algebraic language in a problem-solving environment. The teachers involved agree that learning is a process in which the learner has the autonomy to construct their own knowledge, and that teachers act as a guide in that process.

One feature of the mathematics department is that they do not use a textbook, but instead they design their own activities or they adapt existing material. In this scenario there are not themes labelled “algebra”, or “linear equations”. However, the teachers help students to construct the algebraic language inherent in units focussed on solving a problem in a particular context, performing an investigation, or producing a mathematical model.

This paper describes the teachers’ planning, one of the learning activities, and an analysis of one lesson, focusing in particular on how the teacher makes connections between different representations in order to help students to solve a particular type of problem. The Knowledge Quartet (Rowland, Turner, Thwaites & Huckstep, 2009; Rowland, 2014) will provide a language with which to discuss the mathematics teaching practice, with a focus on the teacher and their mathematical knowledge in teaching.

Analytical framework: the Knowledge Quartet (KQ)

The Knowledge Quartet identifies three *categories of situations* in which teachers' mathematics-related knowledge is revealed in the classroom: named *foundation, transformation, connection* and *contingency*. Table 1 outlines these and their contributory codes which arose from grounded analysis of mathematics classroom data (Rowland et al, 2014). Each dimension is composed of a small number of related subcategories (i.e. open codes).

Dimension	Contributory codes
Foundation: knowledge and understanding of mathematics per se and of mathematics-specific pedagogy, beliefs concerning the nature of mathematics, the purposes of mathematics education, and the conditions under which students will best learn mathematics	awareness of purpose; adheres to textbook; concentration on procedures; identifying errors; overt display of subject knowledge; theoretical underpinning of pedagogy; use of mathematical terminology
Transformation: the presentation of ideas to learners in the form of analogies, illustrations, examples, explanations and demonstrations	choice of examples; choice of representation; use of instructional materials; teacher demonstration
Connection: the sequencing of material for instruction, and an awareness of the relative cognitive demands of different topics and tasks	anticipation of complexity; decisions about sequencing; making connections between procedures and between concepts; recognition of conceptual appropriateness
Contingency: the ability to make cogent, reasoned and well-informed responses to unanticipated and unplanned events	deviation from agenda; responding to students' ideas; use of opportunities; teacher insight during instruction

Table 1: The Knowledge Quartet – dimensions and contributory codes

School algebraic language

Our conception of teacher knowledge includes knowing what to teach, and why ('theoretical underpinning of pedagogy': Rowland et al., 2009), and how to design tasks for learning. In the light of the team's 'bottom up' approach to planning instruction, it was necessary to agree what algebra is, and what students should learn. Following a literature review on algebraic language construction, it was decided to use the theoretical background given by Kaput (2000) and the NCTM (2000). In particular, algebraic school language enables students:

1. to communicate generalisations and formalisations;
2. to represent abstract structures and to reason with them;
3. to understand functions and others relations between variables;
4. to carry out mathematical modelling.

Given their agreement about these components, the teachers were clear about the necessary objectives of the activities to be designed to introduce algebraic language to the students. It was decided to investigate one of the activities (described later) that the teachers had prepared to implement in their classrooms, with a particular focus on the second and third algebraic language dimensions listed above.

Data collection and selection

During the 2014-15 academic year, all of the weekly department meetings were video-recorded, and also all the lessons of these four teachers whose objectives included learning some of the algebraic language dimensions.

A set of activities implemented in the second year of secondary school (student age 13 to14) were selected for analysis, in which the objective was to develop the skills needed to solve simultaneous linear equations. The students involved in the research had had no prior instruction on solving equations, but they were accustomed to use algebra to communicate generalisations (Kaput, 2000).

Characteristics of the learning activities

The algebra-oriented learning activities devised for the second year classes had the characteristics of problem solving tasks in the sense of Schoenfeld (1992): i.e. students did not already know procedures to solve the problems directly, but they had sufficient resources and heuristics to attempt to resolve them. The intention was that students learn how to solve linear equations by solving relevant problems and making connections between iconic, algebraic and tabular representations of key information.

The pizzas and drinks problem

Figure 1 presents one of the problems within a sequence of tasks, presented iconically. Students are required to find the price of one pizza and one drink using the given information. Before presenting the problem as a whole, the teachers first showed only one of the two conditions (e.g. that 3 pizzas and 3 drinks cost 12€), and asked the students what more they could say about it. For example, some students said that if they knew the price of one pizza, they could find the price of a drink; and conversely. Other students then gave a list of possible price-solution pairs, representing them in a table or a graph.

Figure 1: The pizzas and drinks problem: What is the price of one drink? Of one pizza?

Figure 2 shows how students solved the problem in different ways. Student 1 used proportional reasoning in the first condition, dividing by two, so that two pizzas and three drinks cost 8.80€. Then she compared this information with the second condition, and concluded that one pizza cost 12€ - 8.8€ = 3.20€. It then follows that the price of one drink is 0.80€. She expressed the solution in the iconic way, and she explained the solution orally to the class. Student 2 presented her solution in words, and began by using the second condition to conclude that one drink and one pizza together cost 4€. Then, in the first condition, she could identify four drink-plus-pizza 'groups', costing 16€. Therefore she knew that two drinks cost 17.60€ - 16€ = 1.60€, and so one drink costs 0.80€. Student 3's solution uses the same reasoning, but with iconic representation.

Problems such as this one were designed to promote mathematical activities close to the students' own common sense and everyday meanings. The teachers also wanted their students not to be constrained by rigid use of mathematical tools and to give students the opportunity to construct their own ways to solve problems (Espinoza, Barbé & Gálvez, 2009).

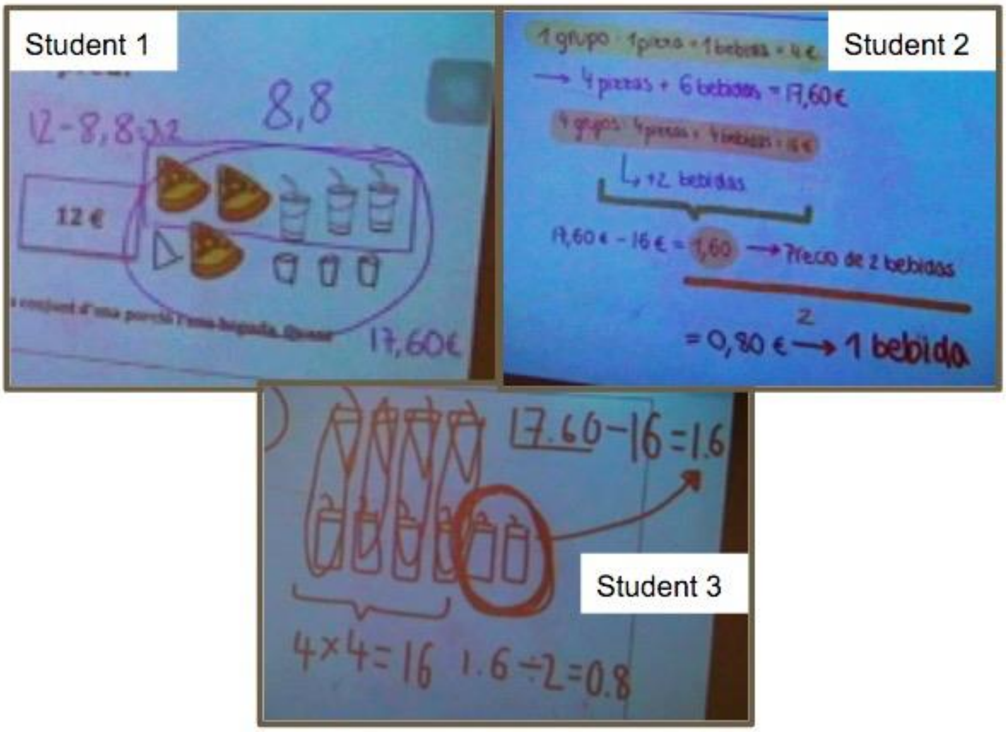

Figure 2: Three different solutions of the pizzas and drinks problem

The 'Chewbacca problem'

After several lessons working on problems like the one described above, the teachers devised a 'test' for the students, consisting of the three problems shown in Figure 3. This was the first occasion on which students were required to work with systems of equations (problems 2 and 3) conventionally represented in algebraic symbolism. The icons used in the first problem are difficult to draw. Therefore it might appeal to the students to change the icons for something else: another icon they could draw quickly, or perhaps a letter. This in turn might help them to see that the second problem is isomorphic to the first.

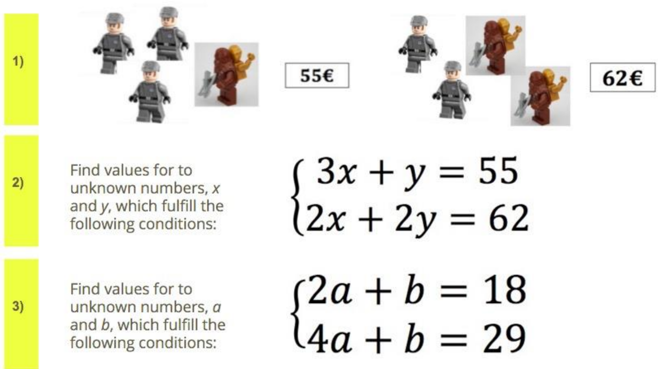

Figure 3: The test items given to the students

Lesson analysis

We now analyse how one of the teachers helped students to solve the third problem shown in Figure 3. The students had had twenty minutes to solve the three problems individually, and this was followed by a whole-class discussion about how to solve the first problem, and concluding that problem 2 is the ‘same’ problem as problem 1. The following analysis will use the dimensions and codes of the Knowledge Quartet (Rowland et al., 2009). On the basis of this analysis, we propose an additional code in the *Connection* dimension of the KQ: *making connections between representations*.

Decisions about sequencing - Making connections between representations

The teacher began writing the equations in problem 3 on the whiteboard, and tried to connect that problem with the problems that they solved using iconic representations:

> Teacher: We saw in previous lessons how through some information, prices, etc., we could deduce the price of each thing, didn’t we? And we saw that this could be identified with another kind of representation [referring to the algebraic representation].

In particular, he connected problem 3 with problem 1, because (as we shall show) he wanted to give meaning to the letters ‘*a*’ and ‘*b*’, not because he wanted to connect the procedures.

Choice of representations - Theoretical underpinning of pedagogy

The teacher wanted to introduce a new procedure to solve the problems. He had read the student solutions to the test, and seen how one of the students solved the problem. He asked that student to go to the whiteboard and explain his solution. The student called the method “by trying”. For example, he tried replacing ‘*a*’ by 4. The teacher asked what ‘*a*’ meant to him, and several students said the price of a pizza, or a Chewbacca, for example. It appeared that that teacher believed that students learn by constructing their own knowledge. He asked questions about the meaning of the letters, connecting that problem with the problems about prices.

Identifying errors - Theoretical underpinning of pedagogy

As the student at the whiteboard replaced ‘*a*’ by 4, he deduced from the first equation that ‘*b*’ would then be 10, and wrote: “4x2 = 8+10 = 18”. The teacher decided to correct the error in the use of the equals symbol, believing that correct use of “=” is important when students solve equations.

Deviation from agenda

Before the equals symbol correction incident, the student had written on the whiteboard:

4x2 = 8+10 = 18
4x4 = 16 + ___ = 29

He had found that ‘*b*’ would need to be 10 in the first equation, but that *b*=10 would not satisfy second equation. Earlier, in correcting the written test scripts, the teacher had thought that student was simply trying two random numbers for ‘*a*’ and ‘*b*’. The student’s method was more powerful than the teacher had thought. What the student called “by trying” was in effect a form of trail-and-improvement: “I was trying values for ‘*a*’ and calculating the corresponding value of ‘*b*’ in the first equation. If that value of ‘a’ and ‘b’ satisfy the second equation, then I have the solution”.

This was clearly a *contingency* episode. The teacher had interpreted the student’s written resolution incorrectly, was surprised in the classroom, and had to revise his thinking accordingly.

Making connections between representations - Use of instructional materials

The teacher could have used the student intervention to present the representation of solution-pairs in two tables. Instead, he decided to ask the student to write the values that he tried in his own way. Figure 4 compares the two representations.

Teacher demonstration

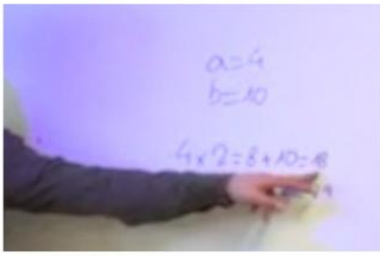

Table of values

2a+b = 18

a	b
4	10
5.5	7

4a+b=29

a	b
4	13
5.5	7

Figure 4: Teacher representation vs Table of values representation

Conclusion

In solving the 'pizzas and drinks' problem, the students used different ways of reasoning, and also various representations: iconic (students 1 and 3) and symbolic (student 2).

The teacher uses these types of problems to provoke student reflection about the meaning of the letters in linear equations, and tries to show a different process that also works to solve those problems, thereby helping them to connect different representations by considering the meaning of the letters. It would have been advantageous if the teacher had understood the student's method more readily, and exploited the situation to show yet another connection between representations: the algebraic formulation of the equations with the tables of solution-pairs.

The findings of this research show how these teachers use problem solving activities in the classroom, to make connections between representations, supporting students in the construction and use of algebraic language. This component of teacher knowledge is absent so far in the Knowledge Quartet: therefore we propose a new code within the *Connection* dimension of the KQ, namely: *making connections between representations*.

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