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A New Assessment Approach in Mathematics Classrooms in Saudi Arabia

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Assessment is an essential process for gathering information about students' learning and achievement. This process should be integrated with learning and teaching to establish ways for teachers to understand their students' learning and make informed decision about their instruction. In this paper, our focus is on a new approach to mathematics assessment in Saudi Arabia, which has been implemented recently. The new assessment approach is essentially a criterion-referenced assessment which aims to support students' learning rather than measuring their progress solely. It is employed more as part of the students' learning process. We explain why the new assessment approach is introduced, what it is, and how teachers deal with it. Moreover, we also discuss the challenges and implications of implementing new assessment approaches to mathematics teachers, educators and policy makers.

Keywords: criterion-referenced assessment; continuous assessment; students' learning; primary schools

Introduction

Over the last two decades, educational researchers and policy makers in many countries have increasingly realised the need to improve the way assessment is conducted in classrooms because most of the traditional assessment methods usually depend on written tests which are inadequate to enhance students' knowledge, educational process and social aims (e.g., see NCTM, 1995; Black & Wiliam, 1998). To overcome the inadequacies of the traditional assessment methods and reform assessment practice, many educational researchers have expanded the principles of assessment, and they have worked on reforming assessment tools to achieve the desired goals from education. In this sense, assessment in mathematics has to be conducted through different techniques in classrooms, for example, using portfolios, journal writing, project assessment, oral presentation, student self-assessment and performance assessment (e.g., see Black, 1993; Fan, 2011).

According to *Assessment Standards for School Mathematics* issued by the National Council of Teachers of Mathematics (NCTM), assessment is "the process of gathering evidence about a student's knowledge of, ability to use, and disposition toward mathematics and of making inferences from that evidence for a variety of purposes" (NCTM, 1995: 3). Regarding this concept, many studies claim that a transfer or shift from using norm-referenced assessment to criterion-referenced assessment and from written tests to continuous assessment is necessary in order to boost students' learning and to help the teachers improve their performance (Berenson and Carter, 1995; Boud, 2000).

In Saudi Arabia, assessing students' learning is a very high concern of education policy makers, particularly in the subject of mathematics with the intention to improve students' performance in mathematics. Many aspects of education in Saudi Arabia, for example: the national curriculum, classroom instruction and textbooks, have been improved alongside improvements in assessment to make the rationale for introducing new assessment stronger than ever before.

The shift in focus from norm-referenced assessment to criterion-referenced assessment was introduced in primary schools, after many stages of reform and improvement starting from 2000. The basic principle of criterion-referenced assessment is that teachers should teach students what they want their students to achieve and the grade is assigned based on students' standards of performance (Falchikov, 2005). The new assessment policy was completed and introduced in 2007. This paper provides an overview of the new assessment approach in mathematics classroom, which is called mathematics continuous assessment, emphasising assessing students' learning of mathematics through their performance on tasks. In addition, the paper also introduces some initial research work focusing on the challenges facing teachers implementing the new assessment approach.

Why was the new assessment approach introduced?

Traditionally, using classroom written tests, which usually takes place at the end of the term, was the only assessment used in Saudi primary schools, to measure students' achievements in order to award certification after each year. A norm-referenced assessment makes judgments about students and expresses students' grades in rank order, and grading based on norm-referenced assessment tends to emphasise competition among students rather than students' improvement (Torrance and Pryor, 1998). By doing so, young students have suffered by spending hours preparing for the annual tests. In addition, no one can deny the level of anxiety caused by traditional tests (Huberty, 2009); therefore, the student's grade does not always indicate his/her academic achievement. In traditional assessment, teachers assess students through a time-limited paper and pencil test. Accordingly, this form of assessment is to grade students' learning results (Fan, 2011), while the new concept of assessment in school classrooms goes beyond this purpose.

The main criticism of traditional tests is that they do not provide teachers with adequate feedback about their students' progression and hence, the performance of teachers and students remains stable at a superficial or manipulated level. Many educators have criticised examining students' knowledge by using traditional tests, because those tests often encourage teachers to 'teach to the test' rather than optimising students' learning. This affects the quality of education and students' learning. Moreover, students' ability to answer test questions in a limited time does not always reflect their strengths (Harris & Bell. 1994; Falchikov, 2005). As Harlen (2000) pointed out:

> Children have a role in assessment for this purpose since it is, after all, the children who do the learning. No one else can really change their ideas or develop their skills. Thus, the more they are involved in knowing what they should be trying to do, the more likely it is that their motivation and effort are enlisted in advancing their learning (P.112).

Improving educational quality and students' abilities to learn were the main aim of the Ministry of Education in Saudi Arabia (Ministry of Education, 2007). Moreover, there is a strong need to solve essential issues like students' and teachers' low performance particularly in mathematics. According to the mathematics curriculum, a major aim of mathematics education in Saudi Arabia is to develop students' higher-order thinking and communication skills (Ministry of Education, 2007). Thus, policy makers in the Ministry have realised the need and importance of improving students' learning through adopting a new assessment approach in classroom. They have introduced continuous assessment, which is based on the process of describing, collecting, and interpreting information about students' learning.

What is the new assessment approach?

According to the education policy makers, assessment should be viewed as a dynamic relationship between students and teachers in order to support students' learning. To meet this desire the Ministry of Education established continuous assessment in primary schools. This assessment involves different classroom exercises to gather information about the students. Continuous assessment is defined as an assessment approach which should describe the range of sources and methods that teachers use to gather and interpret information about learners; and this information is then used to help teachers understand their learners and to plan their classes (Airasian, 1991). According to Heywood (2000), the term coursework or continuous assessment is formative assessment because the grades are fed back to the students after their work.

The system of continuous assessment enables teachers to assess students in a variety of ways over time. It means that teachers will know more about their students' learning, so they can provide students with suitable feedback based on their performance (Ministry of Education, 2007).

Moreover, assessing students through a set of specific criteria can help parents to understand, measure and support their children's learning (Salvia, Ysseldyke and Bolt, 2007). Hence, they will have a fuller picture about what kind of progress has been achieved.

The continuous assessment format

In order to help teachers make the judgments about students' achievement they are expected to implement continuous assessment, which is based on a set of criteria that have been defined and determined by educators in the Ministry of Education. The grades are supposed to reflect students' achievement based on pre-determined criteria. The set of criteria have been based on the knowledge that we want students to master, by specifying the knowledge and skills at each grade level in primary school (Ministry of Education, 2007).

The curriculum skills were categorised into core skills and non-core skills. In other words, students have to pass the core skills in one grade level in order to progress to the next grade level. Creating the assessment criteria means that all students in Saudi Arabia are assessed using the same criteria, which suggests that measuring students' performance ought to be more consistent and reliable.

Under the new assessment approach, schools teachers will use rubrics to assess their students. Rubrics are a set of criteria or scoring guides that describe students' levels of performance or understanding. Rubrics are authentic assessment tools to evaluate students' academic performance (Stevens and Levi, 2005). Working through rubrics provides a more effective reflection of what students know and can do, and sets out the skills that must be mastered. Rubrics also provide teachers and students with goals for further progress and reduce students' and parents' complaints about achieved grades. (Montgomery, 2002).

How can teachers deal with this assessment?

Teachers will use the achievement criteria to make judgments about the students' learning. They draw on assessment data that they have collected during the term. Their judgments about the students' learning and formal reporting about students' learning are sources of feedback to students and their parents.

Table 1 below shows a sample of the rubrics used by teachers to assess students. The example is taken from the second grade (there are six grades in primary schools in Saudi Arabia).

Table 1: A Sample of the rubrics to assess students in Saudi Arabia mathematics classrooms.

Skills	Criteria	Assessment tools
Solving simple problems, involving multiplication and division	1. Use repeated addition, arrays, and counting in steps to do multiplication. 2. Use repeated subtraction, equal sharing, and forming equal groups with remainders to do division. 3. Know multiplication facts for 2 to 10.	
Recognising triangle shape.	1. Identify triangle shape. 2. Give examples of triangles that can be found in our world. 3. Draw a triangle.	

Regarding the above table teachers need to follow the criteria, in this case, when a student fails to pass one of the previous criteria, he/she cannot pass that skill. Teachers involve using a variety of tools such as observation, short tests and discussions for gathering information about students' learning performance. Generally, students will get marks according to the following basis (Ministry of Education, 2007):

- Students who mastered all the core skills will get score 1.
- Students who mastered 66% of the core skills will get score 2.
- Students who mastered 33% of the core skills plus all non-core skills will get score 3.
- Students who are not able to achieve the above will get score 4, and if this is the case, then they will not be able to progress to the next grade level.

In addition, the assessment of each student's skills is conducted regularly during the term until the teacher can make a judgment about the student. In this sense, there is no limit on the number of assessments in each skill during a school term (Ministry of Education, 2007).

Challenges and implications

The focus of the implementation process of this new assessment has been based on helping teachers understand the difference between continuous assessment and traditional written tests.

It has taken a long time and effort to train teachers on how to deal with this type of assessment and persuade them of its value. One of the major challenges is the large class sizes in most Saudi schools. In fact, the average class size in Saudi Arabian schools is approximately 45. Thus, many teachers failed to observe the progress of all students and provide them with the right feedback to improve their academic achievement.

To investigate challenges that teachers face in implementing the new assessment approach, the first author of this paper interviewed six primary school mathematics teachers from different schools, within teaching experience between 5 and 14 years. The six teachers were selected randomly from about 200 mathematics teachers in 44 primary schools in Al-hofuf city, Eastern region of Saudi Arabia. We asked them the following five questions:

1. What is the aim of continuous assessment?
2. How can assessment improve students' learning?
3. How can teachers deal with this assessment?
4. Other than tests, how do teachers assess students' learning?
5. What the challenges is after implementing it?

The interviews with the teachers revealed that not all the teachers who were involved could provide adequate answers about the aims of this new assessment approach, and they do not know the purposes behind implementing it. They said the new assessment policy is not clear enough to provide teachers with the important information that will enable them to implement this assessment in classrooms.

> I would say I do not know the purpose of implementing a new approach to assess students. I think using tests is more reliable than using different tools to mark students (Nora, nine years experience).
>
> I can say assessment cannot improve students' academic achievement. We use assessment process to mark and grade our students by testing them what do they know (Maha, fourteen years experience).
>
> I thought the new policy needs to clarify some concepts like core skills and non-core skills and the curriculum skills divided into core and non-core. I say providing students with unlimited chances to pass the core skills is not a good idea, it is very difficult for teachers to implement it with 40 students or more (Maryam, eleven years experience).

Most of teachers have superficial knowledge about this assessment method and could not deal with the assessment results.

> Um yeah, I know the aim of country's assessment is using many tools to assess students, but I believe in using paper and pencil tests in mathematics because it is difficult to use other tools in mathematics (Sara, twelve years experience).
>
> Yeah. I can say that using rubrics is helpful for me as a teacher most of the time. However, some criteria are written in a very difficult way, in which case I cannot follow them (Reem, seven years experience)

Moreover, students in primary schools do not know how they are assessed and what kind of criteria have been used to assess them, which resulted in the lack of students' higher performance.

The interviews also found that dividing skills into core and non-core skills leads students to concentrate on the core skills and disregard the non-core skills. In addition, most teachers said they would rely on tests as the main tool to assess their students in mathematics, because it is very difficult to use other assessment tools with a large number of core skills, which might affect the efficiency of continuous assessment. In addition, it is also difficult for teachers to follow the new assessment instructions, and most of them reported difficulty in finding time to implement the policy, because of class size and a lesson time of just 45 minutes.

The first author of this paper also interviewed two mathematics educators in the Ministry of Education. They were asked the following two main questions:

1. What are the challenges and issues in implementing the new assessment approach?
2. What are the negatives aspects, if any?

The interviews revealed that many issues were reported or surfaced after implementing the new assessment approach in primary schools from the mathematics educators' perspectives. Some skills are complicated and there is no clear demarcation that they are core skills or non-core skills, and thus assessing students will be very difficult.

One of the most important objectives in mathematics learning in Saudi Arabia classrooms is to develop students' higher order thinking. However, the new assessment policy is mainly based on the core skills, and in order for the students to progress to the next grade level, they need to master these core skills at the lower grade level as required in the new assessment criteria. In fact, most of the core skills involve lower order thinking skills. This leads teachers and students to ignore the higher order thinking skills, which cause students' low performance in mathematics. The interview also revealed that the outcomes of classroom instruction in primary schools fell after implementing this assessment reform because of this variety of issues, so further improvement of the reform is needed.

Moreover, the mathematics educators interviewed highlighted the issue of ambiguity in the new assessment policy; in some places, the assessment policy did not help teachers to understand this assessment accurately. Some teachers cannot set good questions to assess students' skills or they just set simple questions in order to record a high pass for their students. In addition, using numbers from 1 to 4 to record or label students' academic achievements does not really show the individual differences among students or reveal their abilities.

Concluding remarks

Meeting students' needs is a both challenging and complex work for teachers and schools. Students' performance and academic achievement depend on many factors; one of them is how students are assessed. Moreover, the assessment of what students know and can do is an essential process, which can help improve teachers' teaching and students' learning.

According to Nitko and Brookhart (2007), teachers need to change assessment tools regularly because using only tests will make it difficult to implement continuous assessment. From our discussion above, we can see that, on the one hand, there have been positive influences of the new assessment that can been observed in the Saudi Arabian classrooms; but on the other hand, the implementation of the new assessment approach also faced many challenges that need to be addressed.

More specifically, the assessment policy makers need to clarify this policy so it is easier for teachers to understand and to implement in classrooms. The educational system in Saudi Arabia really needs this new approach towards assessment but at the same time, there is a need to be aware of the challenges and issues in implementing it, which affect students' learning. This assessment needs to be more reliable so parents and students will trust its results.

One of the important needs for both researchers in mathematics education and policy makers is to look for solutions to improve the whole process, and determine the skills based on the curriculum goals rather than relying on the objectives that are listed in the textbooks.

Finally, the success in introducing and implementing continuous assessment depends on many aspects. As reported earlier, teachers need to realise the aim of continuous assessment, and we believe that relevant professional training is provided for teachers so they are able to use the assessment results for improving education in schools.

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Flemish mathematics teaching: Bourbaki meets RME?

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The Programme of International Student Assessment (PISA) and Trends in International Mathematics and Science Study (TIMSS) create much international interest in those countries perceived as high achieving. One such system, rarely acknowledged, is Flanders, the Dutch-speaking region of Belgium. In this paper I present the results of focused analyses of four sequences of video-taped mathematics lessons taught to students aged 10 to 14 years. These confirmed a mathematics education tradition drawing on two well-known curricular movements. The first presents mathematics as a Bourbakian set of interconnected concepts. The second exploits realistic problems in its presentation of mathematics.

Keywords: PISA, TIMSS, Flemish mathematics education, Bourbaki, RME

Introducing TIMSS, PISA and their impact

For nearly two decades educational policy makers, teacher educators, and researchers have been scrutinising the results of international assessments of students' mathematics achievements. Alongside the Trends in International Mathematics and Science Study (TIMSS), repeated every four years since 1995, has been the Programme of International Student Assessment (PISA), repeated every three years since 2000. Each report provokes a flurry of activity, frequently misplaced, focused on understanding how other systems, not always culturally and demographically similar to one's own, have appeared to achieve better results.

One European system whose students have shown higher than typical European achievement on both forms of test has been Flanders, the autonomous Dutch-speaking region of Belgium. It has participated in three of the five reported TIMSS and all reported PISAs. Its rankings on mathematics, not always well-known because PISA reports Belgium as a whole[1], are unparalleled in Europe, as shown in table 1.

TIMSS (Grade 8) Mathematics			PISA (Age 15) Mathematics				
1995	1999	2003	2000	2003	2006	2009	2012
1	1	1	2	2	1	1	2

Table 1: Flemish Europe-related rankings on TIMSS and PISA

On Flanders and Flemish education

Despite its unique successes Flanders remains largely unexplored as a research site, prompting the question, what characteristics of Flemish education in general and

[1] Until the reporting of PISA 2012, the Flemish authorities have published their own PISA summaries. These have been produced at the University of Gent by Inge de Meyer and her colleagues (De Meyer, 2008; De Meyer et al., 2002, 2005; De Meyer and Warlop, 2010).

Flemish mathematics teaching in particular have precipitated these successes? In this paper I examine the second half of the question and, as a way of framing my analysis, draw on anecdotal accounts describing Flemish mathematics teaching as a juxtaposition of the formalism of Bourbaki and the informalism of Realistic Mathematics Education (RME). However, before summarising these seemingly incompatible conceptions of mathematics education, I present a case as to why English policy-makers - and here I focus deliberately on England - should be interested in Flemish mathematics education.

Firstly, not only is Flanders economically comparable to England, but it has experienced similar socio-economic and ethnic segregation, due mostly to post-war immigration from Southern Europe, Turkey and North Africa (Agirdag et al., 2012a). Such communities typically congregate in working class districts, leading not only to higher proportions of immigrant children in particular schools but lower achievement in comparison with that of native Europeans (Dronkers and van der Velden, 2013) and increasing feelings of isolation and helplessness (Agirdag et al., 2012b).

Secondly, like England, Flanders operates a complex system of school types, although, unlike England, these are typically fully funded by the state (Cherchye et al., 2010). Significantly, the core secondary curriculum is the same for all. However, with guidance students choose either a vocationally oriented, a humanities oriented or a classically oriented track (Op ’t Eynde et al., 2006). This right to choose initially puts more students in higher tracks than in selective systems, leading to high expectations and achievement (Prokic-Breuer and Dronkers, 2012). Interestingly, school type and track influence significantly the mathematical learning of Flemish students (Pugh and Telhaj, 2007), with Catholic schools outperforming municipal schools, which outperform national schools (Cherchye et al., 2010).

Thus, in the light of Flemish mathematical success and cultural resonance with an English context, I argue that an analysis of Flemish mathematics classrooms may yield insights relevant to curricular and pedagogical developments in England. Also, as indicated above, there is anecdotal evidence that Flemish mathematics teaching is an unlikely juxtaposition of Bourbakian mathematics and realistic mathematics education (RME) (Dyckstra, 2006), traditions summarised briefly in the following, making it a particularly interesting research site.

Bourbaki and Realistic Mathematics Education

The realistic mathematics education (RME) movement was an alternative to the deductive mathematics experienced by Dutch students, whereby learning entailed the acquisition of isolated and decontextualised knowledge and skills (Wubbels et al., 1997). Such knowledge, divorced from children’s experiences, is rapidly forgotten and “children will not be able to apply it” (Van den Heuvel-Panhuizen, 2005a: 2). Based on a belief that “what humans have to learn is not mathematics as a closed system, but rather as an activity” (Freudenthal, 1968: 7), RME comprised three key elements; learning mathematics necessitates doing mathematics; the subject matter of mathematics should draw on solutions to problems derived from reality (Van den Heuvel-Panhuizen, 2005b); and through the processes of mathematising, students come to reinvent mathematics (Gravemeijer, 2004).

Nicolas Bourbaki, a pseudonym adopted by group of French mathematicians, published material intended to remedy a lack of structural integrity and intellectual rigour in university mathematics (Munson, 2010; Weintraub and Moroski, 1994). The group promoted axiomatic mathematics, based on set theory, from which all topics are derived (Guedj 1985; Landry 2007). Such perspectives, on mathematics as the study of “objects as sets-with-structure” (Awodey, 1996: 211), not only influenced how Piaget came to view mathematics and mathematical learning (Munson, 2010) but provided the intellectual underpinning of the *New Math* movement, which, in an attempt to bridge the gap between school and university mathematics, privileged principles over procedures, highlighted structures, sets and patterns, and emphasised experiential over rote learning (Klein, 2003).

Method

This paper draws on data derived from a video study of mathematics teaching in five European countries. Focusing on how four Flemish teachers, each considered locally as effective, present mathematics to their students, data are sequences of five lessons taught on standard and previously agreed topics. Filming sequences reduced the possibility of showpiece lessons and agreed topics facilitated comparative analyses. All four teachers were involved in initial teacher education, while one had been filmed as part of a national project aimed at improving mathematics teaching quality.

Tripod-mounted cameras were placed discretely at the side or rear of project classrooms and videographers instructed to capture all teacher utterances and as much board-work as possible. Teachers wore wireless microphones, while static microphones captured most public student talk. The first two videos in each sequence were transcribed and translated into English by English-speaking colleagues. This enabled the production of subtitled videos that colleagues from all countries could view and analyse. The accuracy of the transcripts was verified by a Dutch-speaking graduate student at my former university.

Analysis was undertaken in three phases. Each subtitled lesson was viewed several times to obtain a *feel* for how it played out; each of these subtitled lessons was viewed again, usually several times, to identify episodes characteristic of either Bourbakian mathematics or RME; the remaining lessons - three in each sequence - were viewed several times for additional illustrative evidence.

Results

The analyses yielded several themes, two of which are reported here. The first concerns teachers′ exploitation of realistic problems, while the second teachers′ attention to the rigorous development of those conceptual structures that underpin mathematics typically associated with Bourbakian mathematics.

Realistic mathematics education

While not a common occurrence, the analyses indicated that realistic problems were not only conceived very broadly but were used strategically by all four teachers to provide pedagogical and affective motivation for their teaching. For example, as they arrived for their first lesson on linear equations, Pauline informed her students that the lesson would be based around a problem involving the cartoon family, the *Simpsons*. This brought forth discernible gasps of excitement and many smiles. Pauline then questioned the class to establish the names and ages of both the children and their mother before posing, very slowly and animatedly, the following question.

> Bart is 7 years old and Lisa is 5 years old when their little sister Maggie is born. Their mother, Marge, 34 years old, wonders if there will be ever a year in which she will have exactly the same age as her three children together.

Having posed the problem it then provided the impetus for three different episodes of the lesson. Firstly, Pauline asked her students to copy a table she had prepared on the board (see figure 1), which comprised three rows - Marge´s age, the sum of the children´s ages and the difference between the two. The first three columns were completed publicly before she invited the class to complete the rest. She allowed several minutes before initiating a public discussion of the results. However, before attending to the answer she focused on the ways in which the figures in each row changed, highlighting the growth in the two rows relating to age and a decline in the difference between them, which she annotated in red. Next, after eliciting the answer of 11 years, she asked what figures would be in the final column and received the answers 45, 45 and 0, which she added to the table. Further questioning found that zero was a consequence of subtracting 2 eleven times from 22. Finally, she amended the table, as shown below, to highlight what the public conversation had yielded.

+1 +1

Marge's age	34	35	36	 45
Children's total age	12	15	18	 45
Difference	22	20	18	 0

+3 +3

-2 -2

(-2) . 11

Figure 1: The table of values publicly constructed for the solution of *the Simpsons* problem

Secondly, although students were not asked to do anything similar for themselves, Pauline switched on the overhead projector and, by means of three prepared transparencies - axes and two straight lines - demonstrated how the straight lines representing the two growth patterns intersected after 11 years. This episode of the lesson also involved much questioning.

Thirdly, Pauline initiated a public discussion on an unknown as a means of representing the number of years that would pass before the two sums would be equal. This extensive bout of questioning led to her writing on the board that if x represented the number of years necessary for the two sums to be equal then Marge would reach 34 + x years, while the children would reach 7 + x, 5 + x and 0 + x respectively, which was written very slowly and neatly. Finally, 7 + x + 5 + x + x = 34 + x was written on the board, at which point Pauline announced that the topic under scrutiny was equations, which was then written in red and underlined. However, she did not simplify the equation to the 12 + 3x = 34 + x or attempt, at this point, to solve it.

A second example occurred during Emke´s introduction to her sequence of lessons on percentages. Percentages is one of a small number of topics to have little meaning when taught independently of its role as a mathematical tool for understanding and managing many aspects of the real world. In this respect, her introduction demonstrated well this understanding. She had, prior to the lesson, asked students to bring in artefacts relating to percentages before spending the first ten minutes of the lesson discussing them. The first part of this public discussion went as follows:

Emke Who can give me an example? A large one, so that everybody is able to see it. Yes, Katrijn?
Katrijn A yoghurt pot.
Emke A yoghurt pot, yes. Please show it. ... And?
Katrijn With nine percent fruit.

Emke Nine percent fruit. What would that mean? Katrijn has a yoghurt pot containing nine percent fruit. Sofie?
Sofie There are nine percent of one hundred fruit in it.
Emke Nine percent of one hundred fruit in it. I do not understand this very well. Who can explain it? Nine percent fruit are in it. Does that mean there are nine pieces of fruit in it?
Several No.
Emke Maybe nine grams?
Several No.
Emke Would it make a difference if it is a large or a small pot? Or would it remain nine percent?
Joost It depends. Yes, I think it would stay the same.
Emke That those nine percents will always remain, yes. The amount will change. Afterwards, we will learn what those nine percent actually mean.
Emke Does anyone have still a similar example concerning a part in respect of a whole? So the whole is the pot of yoghurt and there is a part in that pot of yoghurt. Is there anyone with a similar example?

Following this, Emke spent much of the next two lessons posing a series of problems that exposed the multiplicative structure of percentage calculations. However, at no point did she undertake any calculations, but, through the use of number base blocks, focused attention solely on the conceptual underpinning.

Towards the end of a sequence of lessons on plane shapes, in which tangrams had been used to highlight the fact that polygons are typically irregular and frequently concave, Peter wheeled his bicycle into the classroom and asked how students might predict how far he would travel if his wheels underwent one full turn. This particular problem then motivated an investigation into the constancy of the relationship between diameter and circumference, which Peter managed as a collective activity.

Bourbaki

Throughout their lessons project teachers were observed not only to focus extensively, and frequently to the exclusion of procedural skills, on the conceptual underpinnings of mathematics but also to make explicit the links both within and between topics. Moreover, a key element of this tradition seemed to lie in the encouragement of students to acquire and use the formal vocabulary of mathematics. For example, as indicated above, following her introduction, Emke invited students to use base blocks to model the conceptual underpinning of percentages. The first task, publicly posed as were all such tasks, went as follows:

Emke Everybody should place four hundred squares in front of him. Not stacked, but next to one another. There should be some distance between them, so that you can see very well that you have four times one hundred, right?... Now put five in front of each hundred (she waits while students follow her instructions)
Emke So, what did you do? What have you done? Yes, Elke?
Elke I've put five unit cubes in front of each hundred square.
Emke And which value has a hundred cube?
Elke One hundred.
Emke One hundred. So what have you done?
Joost I've put five unit cubes in front of each hundred.
Emke You can also say, I've put five per 100, or five at 100. Or you can also say, I've put five on 100. These are all different ways of saying the same thing.
Emke How should we write it down? You have put five units for every hundred. Of how many? How many do I have in total?

Frederick Of four hundred.

Following this she wrote on the board, including brackets; (five for every hundred) of four hundred. In so doing, she exposed, something she later came to exploit, the multiplicative basis of percentages. Moreover, she ensured that students were able to follow the procedure correctly, wrote what she later came to call a formulation and, finally, highlighted the linguistic arbitrariness of positional prepositions.

A more prosaic example was seen in Heleen´s teaching of grade seven polygons. During her second lesson she invited three girls and a boy to come to the board and, respectively, draw a right-angled trapezium, a rectangle, a parallelogram and a rhombus. As they worked, using set squares and lengths of string as compasses, she drew two intersecting but arbitrary line segments and began questioning the remainder of the class about the ways in which various configurations of diagonals define different quadrilaterals. Her questioning yielded a set of relationships pertaining to parallelograms, rectangles, kites, rhombi and squares. Finally, turning her attention to the now completed diagrams, Heleen initiated a discussion of the properties of each of the four quadrilaterals, asking about their angles, sides and diagonals. In so doing she focused attention on how different ways of analysing quadrilaterals´ properties created different hierarchies, each of which started with the most general form of quadrilateral but, through ever more tightly defined structures, led to different conclusions. In so doing, she made her students aware not only of the role of classification in mathematical structures but also how they may or may not be equivalent according to, essentially, arbitrary decisions as to the focus of attention.

Shortly before she began her exposition on the solution of linear equations, Pauline initiated a discussion on some of the fundamental relationships of arithmetic. She wrote the statement, a = b, on the board and questioned her students as to what could be inferred if, for example, c was added to both sides. Her students responded appropriately, telling her that the equality would remain. After two or three minutes, the following was completed on the board.

$$a = b \quad \Rightarrow \quad \begin{aligned} a + c &= b + c \\ a - c &= b - c \\ a : c &= b : c \\ a \,.\, c &= b \,.\, c \end{aligned}$$

Such insights, written very formally but always discussed publicly, permeated Pauline´s work. Later the same lesson, after she had begun to discuss the solution of 6(x - 5) - 8 = x - 3, which was the focus of her exposition, students introduced words like associativity, commutativity and distributivity to justify the various actions they proposed. In such vocabulary lies a deep-seated and conceptually sound, awareness of mathematical structures and their interconnectedness.

Discussion

Acknowledging that the data are limited to analyses of just twenty lessons, the juxtaposition of two substantially different traditions is interesting. Many of the teachers' approaches were consistent with the traditions of Bourbaki. For example, they encouraged high levels of mathematical rigour consistent with Bourbakian expectations that "mathematics itself should rest on a bedrock of unshakable fundamental principles" (Munson, 2010: 18). Moreover, teachers´ structural emphases accord with the Bourbakian belief that pedagogy should "elucidate the fundamental structures" to the extent that "underlying ideas must be elevated above the examples that illustrate them" (ibid: 19). Also, their ambitions were clearly commensurate with Bourbakian perspectives on school mathematics, in that what they did could be construed as attempting to bridge the gap between the mathematics of school and university, privileging principles above calculations, emphasising structures and autonomous experimentation (Corry, 2007).

Secondly, if realistic mathematics entails "taking students' initial understanding as a starting point, providing them with problem situations which they can imagine, scaffolding the learning process via models, and evoking reflection by offering the students opportunities to share their experiences" (Van den Heuvel-Panhuizen, 2005b:.36), then Flemish project teachers clearly appeared to work within such frameworks. However, typical RME problems encourage multiple solution strategies allowing students to compare approaches and gain new insights (Treffers, 1987). Moreover, typically in the early stages of a topic´s development, RME exploits students´ informal solution processes as an important first step in the process of mathematisation (Van den Heuvel-Panhuizen, 2005b). Indeed, project teachers, despite extended periods of public discourse, encouraged neither the sharing of multiple solutions nor informal methods; the tasks they exploited were always directed towards well defined goals closer to the aims of Bourbaki than RME.

In conclusion, it is probably more accurate to describe Flemish mathematics as Bourbaki moderated by RME than an equal partnership of the two, although there is clearly more to Flemish teaching than such descriptions suggest. For example, the analyses highlighted two other characteristics in need of further research. Firstly, all lessons were largely teacher-centred with relatively little individual work. Secondly, the pace of lessons was slow, in that teachers seemed content to take several lessons to develop the conceptual basis for any procedures they introduced later. However, this is only an initial analysis and much further research will be needed before we understand the construction of Flemish students´ mathematical achievements.

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Progression towards understanding functions: What does spatial generalisation contribute?

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We focus on a 'typical' task in which students have to give a functional generalisation in algebraic form of a growing sequence of spatial structures. We analyse the contribution of this task to a coherent knowledge of functions. Despite a plethora of research about misconceptions and the teaching of functions, little is known about the overall growth of students' understanding of functions throughout schooling. We aim to map the development of students' understanding of concepts which contribute to understanding functions in two different curriculum systems: the UK and Israel. The research uses a survey instrument that was developed in collaboration with a group of teachers and the task for this paper is one of six that span several routes to understanding functions. Our data appears to contradict some other studies as well as to suggest conjectures about how students' willingness to use covariational reasoning depending to some extent on task features.

Introduction

The function concept is both an explicit and implicit foundation for advanced study in mathematics itself and as a tool in other subjects. The roots of function understanding do not consist of a single hierarchical pathway (Schwindgendorf, Hawks and Beineke, 1992). This paper examines one small part of a project to construct a description of progression towards functions based on probing students' understanding. The research has several stages and we are currently analysing the implementation of a survey instrument that is being used across two countries.

Learning does not only depend on the written curriculum, it also depends on school and classroom context, teaching, and possibly on the level to which teachers are 'functions aware' (Watson and Harel, 2013) and on national expectations through assessment regimes. In order to juxtapose such national expectations we are working in two countries: UK and Israel. The curriculum in the UK has an informal approach to functions, not requiring a formal treatment until year 12[2] for those who continue to advanced study, but younger students will, for example, generalise sequences and meet input-output models as 'function machines'. In the Israeli curriculum approaches to functions are more explicit for younger students and the word and the notation are introduced in year 7. All project teachers are 'functions aware' due to their self-identified levels of mathematical knowledge. In this paper we outline our research approach and demonstrate its application to one task (out of six) in one national context (UK). Due to issues of gaining ethical approval the data collection from Israeli schools will take place in 2014.

We developed a survey instrument over several design cycles working closely with teachers to adapt existing tasks and develop new ones (Wilmot et al., 2011; Swan 1980). We then selected an optimal set of questions that addressed distinct routes to understanding functions that we had identified from the literature. The questions had

[2] We are using UK years in this paper.

to be accessible for students in years 7 to 13, and had to be completed in one lesson. This survey was implemented in two schools in the UK to provide data for analysis to learn about progression towards functions in secondary school, while being aware of grouping, teaching, curriculum, prior attainment, and other variables. The schools and teachers were similar in many ways (size, socio economic factors, ethnicity, stability, qualifications) but differences in spread of prior attainment in constructing teaching groups are likely to have an impact on learning.

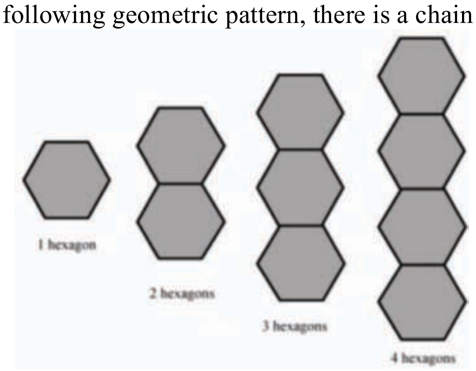
Spatial sequence generalisation tasks

The findings we present in this paper are from a spatial growth pattern generalisation task. There is a considerable body of literature about such tasks, investigating either the processes of generalisation or the effects on algebraic understanding more generally (e.g., Carraher, Martinez and Schliemann, 2008; Dörfler, 1991; Radford, 2006, 2008; Stacey, 1989). Many studies have been conducted to identify processes in building generalisations from spatial sequences. These studies vary in the types of patterns, the population studied and their perspectives and accordingly in the categories of generalisations they present. For example, Dörfler (1991) defined two types of generalisations: empirical, referring to the recognition of common features, and theoretical ‘systems of action’, identifying variables and expressing prototypical relations between objects. Radford (2006) identified three generalisation strategies used with patterns: algebraic generalisation: ‘grasping’ a commonality, generalising for all terms and forming a rule; arithmetic generalisation; and inferences based on local guesses. Rivera and Becker (2008) distinguish between constructive and deconstructive forms of generalisation. The constructive form results from perceiving figures as consisting of non-overlapping parts, exhibiting the standard linear form $y = mx + b$. The deconstructive form is based on initially seeing overlapping sub-configurations in the structure and would lead to separately counting each sub-configuration and taking away parts (sides or vertices) that overlap. Constructive generalisation seemed easier for middle school children to establish while deconstructive was more difficult to achieve. Stacey (1989) defined four main generalisation approaches: recursive; counting from drawing; searching for a functional relationship from a figure; and making an incorrect proportional assumption, using the ratio $f(x) = nx$, when the relation is $f(x) = ax + b$ $(b = 0)$. Most studies agree that students find it difficult to reach theoretical generalisations and that they tend to begin with a recursive approach to sequential data. In England recursion is often referred to as ‘term-to-term’ and a functional relationship as ‘position-to-term’, where ‘position’ refers to sequential position. Several researchers claim that presentation influences approaches since presenting data in order can encourage a term-to-term approach (e.g. Stacey, 1989; Steele, 2008).

The aim of these studies was to focus on obstacles to generalising the underlying function. It cannot be claimed, however, that students who succeed have any sense of functions, even though some authors describe this as a ‘functional approach’. As Dörfler points out (2008), these tasks only model particular kinds of function, those that are expressible as strings of arithmetic operations that relate to continuing patterns with integer inputs. To have a sense of ‘function’ would require variation and comparison of functions and their properties (Carraher et al., 2008).

Our aim was not the same as these studies. We used a spatial sequence generalisation task as one of six tasks, all of which address components of the function concept. We expected this task to provide information about how students try to identify relations when prompted to state different kinds of generalisation. By seeing what they use and how they try to generalise, whether successful or not, we can identify ways of attending to data and spatial information that might form a basis for future knowledge of functions. Spatial tasks can provide: early experience in modelling relations between two variables; opportunities to explore co-variation, relating to gradients and early calculus; experience in analysing simple functions in given domains; and experience in expressing input-output relations algebraically. Our analysis, therefore, does not imply preference for any approach. Dörfler (2008: 147) says that different approaches "... shed different light on the common underlying functional relationship." For example, term-to-term perception is a plausible pre-concept towards gradient if variation in the output variable is related to variation in the input variable to show some sense of co-variation (Carlson et al., 2002). As well as relating to gradient, co-variation fits well with modelling natural phenomena, where data typically consists of changes in a phenomenon. Position-to-term generalisation is a plausible pre-concept towards understanding that relations between two sets of numbers might (sometimes) be expressed as general algebraic 'rules'. However, in spatial sequence tasks the position number may not be understood as a variable but merely as a label, so we do not assume that such tasks provide information about students' understanding of variables.

For the following geometric pattern, there is a chain of regular hexagons (meaning all 6 sides are equal):



1.
For **1** hexagon the perimeter is **6**.
For **3** hexagons the perimeter is **14**.
For **2** hexagons the perimeter is _____________
For **5** hexagons the perimeter is _____________
Note: perimeter is the number of outside edges.

2. Describe the process for determining the perimeter for 100 hexagons, without knowing the perimeter for 99 hexagons.

3. Write a formula to describe the perimeter for any number of hexagons in the chain (it does not need to be simplified).

You can use: $p(n) =$

4. Explain why you think your formula in question 3 is correct.

Figure 1. The task

The task we used (see Figure 1) is a 'typical' spatial sequence task. The task was constructed to allow the students to achieve full generalisation gradually through different kinds of generalisation (Stacey, 1989). The language used was agreed with the teachers to ensure access and familiarity (for example you do not need to remember what perimeter means). With the above considerations in mind we tried to disturb the normal outcomes of such tasks by not providing an ordered data table, thus preventing a quick response of spotting number patterns. We did not expect, therefore, to replicate the findings in which students try to construct from recursive reasoning. We assumed some familiarity with this task type which is ubiquitous in English schools and which might lead some to assume they need to find a position-to-term relation. It is important to note that the previous survey task involved contextual sequential data and questions about rates of change, so students might be starting the task with sequential strategies in mind.

Population

The survey was given to year 7 to 13 classes from two schools with each school providing data from alternate years. We wanted data from a suitable spread of students in terms of their past attainment and asked each school to use their highest achieving class (called A) and a middle achieving class (called B) in each of years 7 to 11 plus their advanced mathematics classes. The teachers provided random anonymised samples of 10 scripts from each class. In this way we received 20 scripts from each UK year 7 to 11 inclusive, and we also had 10 scripts from years 12 and 13 (total of 120 scripts).

Data analysis

We looked for evidence of all attempts to make relationships between data items, since in these tasks such relations could contribute to understanding functions. The method of analysis was to code each student's responses according to pre-concepts related to functions, whether used correctly or not. We then classified them according to approaches to functional reasoning evidenced across the whole sample. We also coded generalisation types. The analysis process was iterative and comparative.

Student awareness of functionality through generalisation

Analysis led to five categories of functional reasoning:

a.	No answer, often accompanied by "I don't know".
b.	No conceptualisation of functional relationship: Empirical methods involving counting.
c.	A *correspondence approach* to develop a general rule for the relation between the number of hexagons and the perimeter.
d.	A *covariation approach* to coordinate the two varying quantities – number of hexagons and the perimeter – while attending to the ways in which they change in relation to each other.
e.	A *correspondence approach* followed by a *covariation approach*: Expressing a *correspondence approach* when addressing the question of finding the perimeter of 100 hexagons, and then, when asked to generate the formula for any number of hexagons, moving to *covariation approach*.

Examples:

Category		Response to Q2	Response to Q3&4	Year/ group
1	b	You can count up in 6 until you get to 100	No response	7 B
2	b	You count how many edges in the chain	No response	9 A
3	c	Because the edges join up they all link together but the 2 end hexagons have 5 sides so you do 5*2 then the hexagons in the middle have 4 sides so you do 98*4 then add the two together	2*5, then how many hexagons in the middle of the end ones * 4. Because if you had 3 hexagons together the end ones have five sides 5*2 then the middle one has 4 so 1*4 then add the answers together = 14.	8 B
4	c	You would have to multiply 6 (the number of sides) by 100 (number of hexagons) = 600, because some sides are joined you have to take them away	6*(number of hexagons) = [space] – number of joint sides. Because if you multiply the number of sides by number of hexagons and then subtract the number of joint sides it will be correct.	10 A
5	c	Each edge hexagon has 5 outside edges, and each hexagon between has 4. Therefore out of 100, 2 would contain 5 and 98 would have 4. If p = perimeter and x = number of hexagon, p=4x+2.	p = 4x+2. I can check it using values which I already know, like 1 and 2. P = 4 * (1) + 2 = 6 for 1. P = 4 * (2) +2 = 10 for 2.	13
6	d	For the amount of hexagons, if you go up by one the perimeter goes up by four.	4n+2. If you do 1 * 4 + 2 that = 6 and that is your answer	8 A
7	d	You are adding 4 sides every time you add a hexagon, so you would multiply 4 by 100 then minus 4 and add 6 for the first hexagon	((4n) - 4) + 6. Because it works for all the numbers used in 2.1	10 A
8	d	4n + 2. You can see the pattern of the perimeter value. They increase by 4 every time for every 1 hexagon! but start at 6 (for 1 hexagon)	4n+2. The jump is 4 every time but it doesn't start at 0 it starts on 6 which is 2 more than 4	13
9	e	1 hexagon is six so if you times 6 by 100 it is 600 that is how much the perimeter would be	Every 1 hexagon you add the perimeter goes up 4. Because one hexagon is 6 and two hexagons is 10 but then 3 hexagons is 14 these numbers are going up by 4	8 B
10	e	Get one hexagon and times it by a hundred	p(n) = n*4.Because the perimeter increases by 4 for	9 B

			every 1 hexagon so it is number of hexagons multiplied by 4	
11	e	To find the perimeter of 100 hexagons you can multiply the perimeter for 1 hexagon by 100. This works as it's in proportion. Example: 1 hexagon – perimeter of 6. 100 hexagons = perimeter of 600	p(n)=(n)-1.For every shape added, one side of the hexagon is lost	11 A

Type of generalisation

Each student's response was categorised according to its generalisations: (1) no correct generalisation of any kind; (2) generalisation expressed correctly in verbal terms only, or (3) generalisation expressed correctly verbally as well as algebraically. Thus the response in example #1 was coded as (1); the response in example #3 was coded (2); example #5 was coded as (3).

Results

Table 1 presents the distribution of the approaches within the A classes. We quantified outcomes across years and groups in order to see if there is any evidence of progression towards successful generalisation, or variation in approaches, bearing in mind that our sample is too small to make generalisations and what we are looking for are conjectures about development. As shown in Table 1, the *correspondence approach* to conceptualising the functional relationships was the most common within the A classes (57%) with many younger students making the proportional assumption referred to by Stacey in an effort to state a position-to-term rule (1989). The *covariation approach* was less widespread (27%). The *correspondence approach* followed by a *covariation approach* constituted 14% of the responses. The categories are distributed among years in a rather 'messy' form, with no specific pattern or order, with the exception of the first two categories of absence (1) and pre-functional approach (2) which are marginal and are expressed in early years only. The *correspondence approach* was the most common with the B classes as well (46%) with the *covariation approach* and *correspondence* + *covariation approaches* both at 14% (B classes not shown in Table).

Table 1: distribution of the approaches within the A classes

A classes	UK07A	UK08A	UK09A	UK10A	UK11A	UK12	UK13	Total
No answer	0	0	1 (1,0,0)	0	0	0	0	**1 (1%) (1,0,0)**
No conceptualization	0	0	1 (1,0,0)	0	0	0	0	**1(1%) (1,0,0)**
Correspondence approach	6 (6,0,0)	3 (2,0,1)	5 (3,0,2)	4 (2,0,2)	7 (2,2,3)	8 (1,1,6)	6 (0,0,6)	**39 (57%) (16,3,20)**
Covariation approach	1 (1,0,0)	5 (3,0,2)	1 (0,0,1)	4 (0,0,4)	2 (0,0,2)	2 (0,0,2)	4 (0,0,4)	**19 (27%) (4,0,15)**

Correspondence then covariation	3 (3,0,0)	2 (2,0,0)	2 (2,0,0)	2 (2,0,0)	1 (1,0,0)	0	0	**10 (14%) (10,0,0)**
Total	**10 (10,0,0)**	**10 (7,0,3)**	**10 (7,0,3)**	**10 (4,0,6)**	**10 (3,2,5)**	**10 (1,1,8)**	**10 (0,0,10)**	**70 (100%) (32,3,35)**

The triples in the cells of Table 1 show the distribution between generalisation types (1), (2) and (3). As with another task (Ayalon, Lerman and Watson, 2013) results from the A groups have indications of progression towards full generalisation (see bottom row). B group results are dominated by no correct generalisation across years, suggesting perhaps grouping, school or teaching effects that require further probing.

We relate approaches to success in achieving a correct generalisation, since expressing relations symbolically is also a precursor to functional understanding. On the basis of this small data set we can conjecture about the strongest connection between method and success being with the co-variation approach, and this does not accord with most other studies. 15 out of the 19 who tried it in A classes were successful (in contrast to 20 out of 39 within the correspondence approach). In B classes only 2 students reached successful generalisation at all, and they used correspondence. Of both groups, those who did not succeed with co-variation failed because they did not take starting values into account, as Dörfler (2008) and Radford (2008) point out. Those who did not succeed with the correspondence approach either assumed proportionality or took a deconstructive approach but did not deal adequately with the subtraction required (see example 4).

Discussion

Our data appears to contradict to some extent other studies in two main ways. The non-sequential presentation of data appears to have prevented some over-dependence on a recursive approach that focuses solely on differences in output terms. Instead, it is plausible that when a sequential approach requires some deliberate reorganisation of data, it is more likely to be associated with successful identification of co-variation and its role in constructing full generalisation. A correspondence approach, which is reported as not being usually the first resort, was indeed the first resort of about two-thirds of our students. This suggests that correspondence is a concept available for students so long as they are not immediately distracted by sequential features, and also that past experience has an effect although we presented the data in an unusual form. Our approach to analysis therefore offers three features of students' pre-functional approach to data: willingness to look for correspondence; willingness to explore co-variation; and flexibility in combining those approaches. Our initial analysis of the previous task which was numerical, contextual and with sequential variations (not yet published) showed similar propensities. When correct symbolisation is added to the mix, we note that co-variation had a higher success rate than correspondence among the relatively high attaining students.

Although this task and its analysis are only a small part of our whole project, and progression and success is seen clearly only in the A groups, we conjecture that further research about students' search for co-variation and their understanding of non-sequential data might reveal more pre-functional strengths than are shown in these typical generalisation tasks. The decisions we made about presentation have, we believe, enabled students to show co-variational understanding. We also need to find out more about how generalisation in these tasks was emphasised in the schools in order to understand the difficulties experienced by the B classes, and there is a need to consider the role of sequential generalisation tasks in the curriculum as our data suggests they can be used as a pre-conceptual pathway towards gradient in contrast to the usual curriculum function of expressing generality. The next stage of our research will be to compare the UK findings with the Israeli findings.

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Using a digital tool to improve students' algebraic expertise in the Netherlands: crises, feedback and fading

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Enhancing ways of developing students' algebraic expertise remains an important focus for research. This paper reports on a design research study which involved a digital intervention for 17-18 year old students, implemented in nine schools in the Netherlands (N=324). For the intervention, algebra tasks for the conceptual and procedural components of algebraic expertise were placed in a sequence based on three design principles: (i) 'crisis' items that intentionally questioned the use of standard algorithms, (ii) feedback provided by the digital system, and (iii) the 'fading' of feedback during the sequence to increase transfer. Data collected included results from student pre- and post-tests, questionnaires, and scores and log files of their digital work. Results from the study show that the intervention was effective in improving algebraic expertise, and that the aforementioned design principles have merit. This paper reports on the effects and illustrates the design principles through a case example. The intervention shows a significant effect in improving algebraic expertise. It shows that well-thought-out design principles augment learning. The paper fits in a broader discussion on how to integrate algebraic expertise and ICT use in the classroom through the use of educational design.

Keywords: algebraic expertise, digital intervention, Netherlands, design principles.

Introduction

The distinction between procedural skills and conceptual understanding is a highly researched field of interest. In *Adding it up* (Kilpatrick, Swafford and Findell, 2001) synthesised the research in this area with the concept of *mathematical proficiency*. Mathematical proficiency comprises five strands: *conceptual understanding, procedural fluency, strategic competence, adaptive reasoning* and *productive disposition.* Conceptual understanding is defined as "the comprehension of mathematical concepts, operations, and relations" (p. 116), and procedural fluency as the "skill in carrying out procedures flexibly, accurately, efficiently, and appropriately" (ibid.). Furthermore, "the five strands are interwoven and interdependent in the development of proficiency in mathematics" (ibid.).

Both conceptual understanding and procedural fluency have been discussed extensively in research. For example, Arcavi (1994) introduced the notion of symbol sense, which includes "an intuitive feel for when to call on symbols in the process of solving a problem, and conversely, when to abandon a symbolic treatment for better tools" (p. 25). Arcavi (1994) exemplifies eight behaviours that describe symbol sense, and posits that these behaviours show the intertwinement between procedural skills and conceptual understanding, both being complementary aspects. In line with this work, Drijvers, Goddijn and Kindt (2010) define algebraic expertise as a dimension ranging from basic skills to symbol sense. Basic skills involve procedural work with a local focus and emphasis on algebraic calculation, while symbol sense involves strategic work with a global focus and emphasis on algebraic reasoning.

Acknowledging the potential of ICT for mathematics education (e.g. Heid and Blume, 2008a, 2008b; Pierce and Stacey, 2010), this study uses an ICT intervention for acquiring, practising and assessing aforementioned algebraic expertise. This paper reports on the study by first describing the design research approach and the algebra content of the intervention, then elaborating the three guiding principles behind the intervention, and its implementation in Dutch classrooms. Finally it presents the results, draws conclusions and presents some challenges for discussion. The focus of this paper is on the question whether three carefully chosen design principles for feedback have an effect on the acquisition of algebraic expertise. To further exemplify these principles, apart from quantitative data, a case example of one student is provided.

Methodology

The complete study 'Algebra with Insight' followed a design research approach with four phases, one preliminary phase and three intervention cycles (Van den Akker, Gravemeijer, McKenney and Nieveen, 2006). The preliminary research phase concerned the choice of the digital tool (Bokhove and Drijvers, 2010a). The first intervention cycle focused on whether it was possible to design a tool in such a way that it would allow symbol sense activities (Bokhove and Drijvers, 2010b). The second cycle consisted of a small-scale field experiment with two teachers in one school. The third and final cycle involved a large-scale classroom experiment. This research set-up shows a progress from small-scale to large-scale in 'layers of formative evaluation' (Tessmer, 1993). The last cycle, aimed at the intervention effects, is the focus of this paper, and in particular the effects of the three design principles for the intervention.

Algebra content of the intervention

In the design of the study intervention, we want to address algebraic expertise, both basic algebraic skills and symbol sense. To do so, the digital intervention needs to offer symbol sense opportunities (Bokhove and Drijvers, 2010b). Tasks were sourced from exit and entry examinations, textbooks, journals and remedial courses. Several suitable 'symbol sense type items' were identified and selected, with the main criterion being that items covered both basic skills and symbol sense. With this content, an intervention called 'Algebra met Inzicht' [Algebra with Insight] was designed in the Digital Mathematical Environment (http://www.fi.uu.nl/dwo/en) of the Freudenthal Institute.

The complete cycle consisted of a pen-and-paper pre-test (eight items), a digital practice module (sections d1-d4, 45 items, excluding randomisation), a digital diagnostic test (section d5, 23 items, excluding randomisation), a digital summative test (section d6, 23 items, excluding randomisation) and, finally, a pen-and-paper post-test (10 items). The time needed to complete the intervention was estimated at six hours, excluding pre- and post-tests.

The intervention was used in fifteen 12th grade classes from nine Dutch secondary schools (N=324), involving eleven mathematics teachers. The design set-up did not include control groups. The schools were spread across the country and showed a variation in school size, pedagogical and religious backgrounds. The participating students were pre-university level ‘wiskunde B’ students, of whom 43% were female and 57% were male. The participating schools subscribed after an open invitation in several bulletins for mathematics education. Schools received an example course plan and some suggestions on using the intervention, they were however free to adapt the intervention to their own requirements. Schools deployed the intervention in the last three months of 2010, just before preparations for the final national exams would start. Teachers received mailings on a regular basis, and could visit a project website with support materials like screencast instructions.

Data collection for the intervention included results from a pre- and post-test, and the scores and log files of the digital activities. The log files record information on students’ individual item scores, feedback, answers, and number of attempts per step. Apart from marking for correct and incorrect, pre- and post-test were also marked with regard to symbol sense behaviour. Using a second marker, inter-rater reliability was good with an alpha=.91 for all items of 5% of the students’ pre-tests and 5% of the post-tests.

Three main design principles

The three underlying principles for the intervention are answers to three challenges that arose from the three design cycles prior to the last cycle: (i) students learn a lot from what goes wrong, (ii) but students will not always overcome these difficulties if no feedback is provided, and (iii) that too much of a dependency on feedback needs to be avoided, as summative assessment typically does not provide feedback. These three challenges are addressed by principles for crises, feedback and fading, respectively.

The principle of using crises is based on, as the poet John Keats so eloquently described in the early 19th century, failure being ‘the highway to success’. The same idea has had many forms during the years. Piaget (1964) used the concept of disequilibrium and equilibrium. Tall (1977) refers to ‘cognitive conflicts’. Van Hiele (1985) distinguishes a ‘crisis of thinking’ with a need for challenge. More recently, Kapur (2010) coined the term ‘productive failure’. Most sources, however, see crises as an inherent part of learning when solving problems; in this case the idea was to embed tasks that could intentionally cause a crisis (Bokhove and Drijvers, 2012b). In this intervention intentional crisis tasks are added to sequences of near-similar tasks, as depicted in Table I. The general structure of a sequence is: pre-crisis items, crisis item, post-crisis items. First, students are confronted with familiar equations (pre-crisis items).

1.1	Tasks: “Solve the following equation:”	**Pre-crisis items** In the initial items students are confronted with equations they have experience with. Students may choose their own strategy. Many students choose to expand brackets as that is the strategy that they have used often: work towards the form $ax^2 + bx + c = 0$ and use the Quadratic Formula. There is some limited feedback on the task.
1.2	$(4x-3)\cdot(4x-1)=(4x-3)\cdot 2$	
1.3	$\sqrt{3x+2}\cdot(3x+3)=\sqrt{3x+2}\cdot(6x-2)$	
1.4	$(x-4)\cdot(2x-5)=(x-4)\cdot(-3x+3)$	

1.5	**Opgave 1.5** Los de volgende vergelijking op:	$(5x-13)\cdot(4x-3)-(5x-13)\cdot(-2x+3)=0$
1.6	$(x^2+3x-3)\cdot(8x-6)=(x^2+3x-3)\cdot(4x+1$ $8x^3+18x^2-42x+18=4x^3+24x^2+24x-3$ $4x^3-6x^2-66x=-54$ $4x\left(x^2-1\frac{1}{2}x-16\frac{1}{2}\right)=-54$ ✓	**Crisis item** Students are then confronted with an intentional crisis: if a student uses his/her conventional strategy of expanding the expression. The yellow tick at the bottom of the screen denotes that the equation is algebraically equivalent to the initial one, but that it is not the final answer (half-correct). This is accompanied by a partial score for an item and some feedback in Dutch: ‘You are rewriting correctly’. Although these students showed good rearranging skills, in the end they were not able to continue, as they did not master the skill to solve a third order equation. There is some limited feedback on the task.
1.7	**Opgave 1.7** Los de volgende vergelijking op:	tip stap losop $(2x^2-3x-2)\cdot(7x-3)=(2x^2-3x-2)\cdot(3x+12)$
1.8	$(x^2-3x-2)\cdot(6x-3)=(x^2-3x-2)\cdot(4x+12)$	**Post-crisis items** After the crisis item students are offered help by providing a ‘voorbeeldfilm’, an instructional screencast, and buttons to get hints (‘tip’), the next step in the solution (‘stap’) or a worked solution (‘losop’). These features have in common that they provide feed-forward information at the task level and self-regulation.
1.9	$\sqrt{3x+3}\cdot(2x+4)=\sqrt{3x+3}\cdot(6x-5)$	
1.10	$(4x+4)\cdot\sqrt{-2+3x}=\sqrt{3x-2}\cdot(7x-5)$	
1.11	$(-5+{}^{2}\log(x-2))\cdot(6x-6)=(-5+{}^{2}\log(x-2))\cdot(3x+14)$	
1.12	$(4x-13)\cdot(3x-3)=(4x-13)\cdot(-3x+2)$	
1.13	$(-4x+5)\cdot(8x-5)=(-4x+6)\cdot(3x+14)$	

Table 1 Sequence of items illustrating crises and feedback. The sequence starts with conventional pre-crisis items, then a crisis item that cannot be solved with the ‘standard’ procedure and ends with feedback to overcome the crisis and further practice items (post-crisis items).

Then the student encounters a carefully designed ‘crisis item’: this item intentionally confronts conventional strategies head on, meaning that the ‘standard procedure’ will not work. Finally, having experienced a ‘crisis’ students are offered help to overcome the crisis by providing feedback; the second design principle. As Hattie and Timperley (2007) pointed out, one effective action for learning would be to provide hints and corrective feedback. Feedback would then very much have the role of aiding assessment for learning, formative assessment. Black and Wiliam (1998) define assessment as being ‘formative’ only when feedback from learning activities is actually used to modify teaching to meet the learner’s needs. Feedback in this intervention is provided at different levels: corrective through green, yellow and red symbols, and supportive by providing screencast movies, hints, next steps and worked solutions.

However, in my personal experience as a teacher I have seen there can be an over-reliance on feedback that is provided: when students take an exam there is no feedback present, so can students still solve tasks correctly, without any feedback? The third principle of fading addresses this. The digital intervention initially provides a lot of feedback, but the amount is decreased towards the end (Renkl, Atkinson and Große, 2004; Bokhove, 2008). Figure 1 shows how this principle was implemented in the intervention. At the beginning of the intervention, in sections d1 to d4, feedback is provided for all intermediate steps of a solution. The next part of the intervention, section d5, concerns self-assessment and diagnostics: the student performs the steps without any feedback and chooses when to check his or her solution by clicking a 'check' button. Feedback is then given for the whole of the exercise.

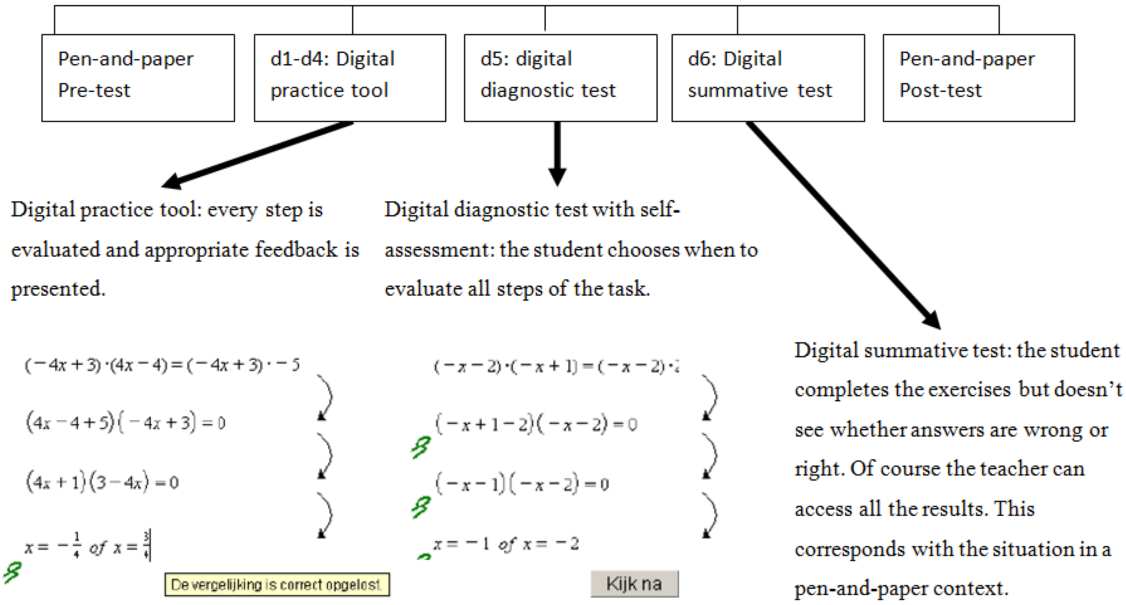

Figure 1 Outline of fading feedback in formative scenarios. The boxes at the bottom say 'The equation has been solved correctly' and 'Check', respectively.

In section d6, students get a final summative test with no means to see how they performed. Just as is the case with a pen-and-paper test, the teacher will check and grade the exam (in this case automatically) and give students feedback on their performance. In sum, the complete narrative behind the three design principles is that (i) *intentional crises* are provoked in students, (ii) enable students to overcome these crises by *providing feedback*, and (iii) to avoid a dependency on feedback *fade the feedback* in the course of the digital intervention.

Results: a case example

The following sequence of events during the intervention, concerns a student named Paula. She starts with a pre-test. Apart from the calculation error on the right hand side of the equation, Figure 2 shows that Paula's strategy here is to expand the expressions, similar to students in earlier phases of the study (Bokhove and Drijvers, 2010b).

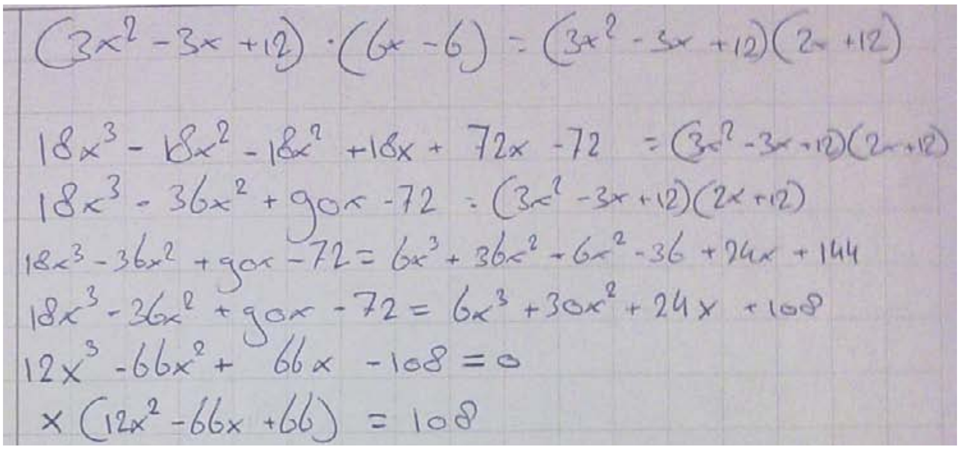

Figure 2 Example of Paula's pre-test pen-and-paper work.

Not surprisingly this strategy fails in the case of this equation. Paula scores low during the whole pre-test, only 14 out of 100. With regard to symbol sense, Paula scores poorly as well. Paula then starts with the sequence of digital tasks. In the first task she has to get acquainted with the digital environment. The pre-crisis items pose no problem for most students, including Paula. On arriving at the crisis item students exhibit three behaviours, roughly corresponding with the ones already observed in the pre-test: (i) students solve the equation correctly, (ii) students recognise the pattern of the equation but subsequently make mistakes (for example by losing solutions in the process), and (iii) students expand the expressions and get stuck with an equation of the third power. Figure 3 shows that Paula exhibits the third type of behaviour, quite similar to what she did in the pre-test.

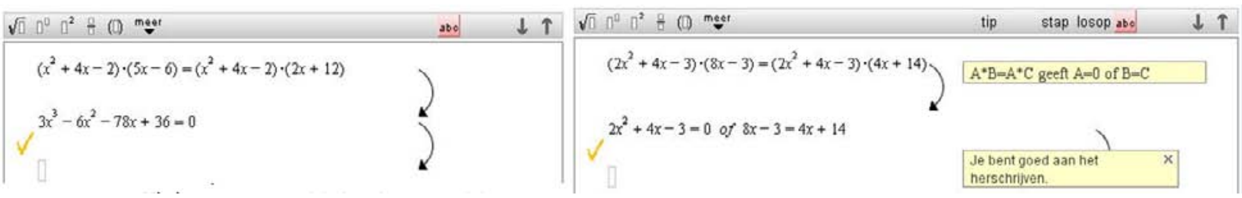

Figure 3 Paula's digital work. Left: crisis item. Right: post-crisis item. The boxes to the right say 'A*B=A*C yields A=0 or B=C' and 'You are rewriting correctly', respectively.

At this item feedback is still restricted to correct/incorrect. In addition, students are allowed to choose their own strategies, even when they aren't efficient or would lead to problems. In the post-crisis items, as well as feedback correct/incorrect, Paula is provided with buttons for hints and worked solutions, and the option to watch a screencast demonstrating the solution. The log-files from the online environment show that Paula fails at the crisis item (0 out of 10 points), but is successful at the post-crisis items with feedback (10 out of 10 points). Looking at the attempts made, intermediate steps for the equation that were sent to the system, Paula attempts the crisis-item 73 times, and the post-crisis items, aided by feedback, only three times. Finally, in the post-test Paula shows a significant increase in the total score (70 out of 100) and symbol sense behaviour. Even though mistakes are made they were not caused by a lack of symbol sense any more but errors in calculations. Focusing only on similar types of equations it becomes clear that Paula manages to solve these equations correctly. As the general results have shown, Paula is not a unique case in this school.

Overall results

Overall, dependent t-tests with pairwise exclusion if data was missing, show that students in participating schools improved on their scores and symbol sense behaviour. The score on the post-test (M=78.71, SE=15.175) is significantly higher than the pre-test score (M=51.55, SE=21.094), $t(286)=-22.589$, $p<.001$, $r=.801$, $d=-1.34$. For symbol sense behaviour scores on the post-test (M=1.462, SE=1.504) also is significantly higher than the pre-test score (M=-1.493, SE=2.339), $t(285)=-20.602$, $p<.001$, $r=.773$, $d=-1.22$. According to Cohen's benchmark (1992) this is a large effect. Both specific case examples and more quantitative analyses show that crises together with feedback decrease the number of step attempts needed, while fading feedback does not prevent a large effect (Bokhove and Drijvers, 2012b).

Conclusion and discussion

In this article I focused on the question whether three carefully chosen design principles for feedback have an effect on the acquisition of algebraic expertise. Overall, the use of the intervention for an average of six hours has a large effect on improving algebraic expertise. This effect did not only entail an improvement in score, but also an improvement in recognising patterns and having a sense for symbols. The question whether the three main design principles were the cause of this is much harder to answer with a 'yes' or 'no'. The principles seem to have merit: the *crises* together with *feedback* decrease the number of step attempts needed, and even with less and less feedback through *fading,* the effect on higher scores remains strong (see also Bokhove and Drijvers, 2012a; 2012b).

To conclude I want to address two points that led to discussions within and about this study. Firstly, I noticed that some educators were concerned that students were 'set up to fail'. A lot of this seemed to correspond with negative perceptions towards words like 'crisis' and 'failing'. This is understandable, as the words have a negative connotation in society. I would like to emphasise that students naturally are not told about an imminent crisis item. The whole intent is that students *could* fail if they exhibit mathematical behaviours we don't want. It would be quite unethical if this was not followed up by a solution as well: detailed feedback to overcome the crisis. In fact, I would contend that the whole combination of crises and feedback strengthens the learning, as set out in the section on design principles. A second point concerns the limitation of the study that the design set-up does not include a control group. As a more philosophical and final comment, I often wonder what the control group should be when one is introducing a new approach in the classroom. If I did not have the opportunity to provide feedback automatically it would be completely unfeasible to do the same thing as a teacher, without ICT. With a collaborative approach students could help each other, but I set out to look at the potential of ICT for acquiring algebra. Is it really so useful to have a control group in studies where the discerning factor, for example use of ICT, has inherent and obvious advantages?

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How-many-ness and rank order—towards the deconstruction of 'natural number'

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This paper concerns a philosophical, but highly practical, issue arising at the interface between mathematics and education, and I claim that mathematics education offers insights where mathematical philosophy has ground to a halt. More specifically it concerns the two related but distinct concepts of how-many-ness (alias *quotity*) and rank order, whose separate identities are traditionally obscured by the language of 'number'. (Sometimes they are called 'cardinal number' and 'ordinal number'. More often they are wrapped up together as 'natural number'.) The teacher of young children has the advantage over the philosopher that she works with people before they have acquired all the prejudices of their native language, and we shall build on the analysis of *counting* by Gelman and Gallistel (1978), concluding that 'natural number' as normally conceived is something of an illusion, for only quotity has the properties expected of 'number', while rank order is a mere *quality*.

Keywords: natural number, cardinal/ordinal, quotity

Introduction 1: the mystery of 'natural number'

Nothing does more to unite the mathematician with the person in the street than their shared belief in 'number'. Yet few things have proved more elusive under close investigation or more difficult to pin down in a definition.

Mathematicians speak on the issue with many voices. Dedekind declared enthusiastically:

> Of all the aids which the human mind has yet created to simplify its life… none is so momentous… as the concept of number. (Ewalt, 1996: 837)

In this, however, he raises eyebrows at the claim of human authorship, for Plato had assigned all mathematical objects to a domain of external objective super-reality. Kronecker struck a famous compromise:

> The dear God has made the whole numbers, all else is man's work. (Grattan-Guinness, 2000: 122)

But this raises the enormous issue of the higher 'types of number'. Aristotle, with a wisdom that may yet return to favour, divided mathematics into the study of multitude (*plethos*), which is discrete, and magnitude (*megathon*), which is continuous, and this distinction was still retained at the time of Newton, who identified continuous number (real number) with the ratio of continuous quantities. But in the development know as the *arithmetisation of analysis* it came to be believed that the study of continuous quantities could be reduced to that of 'real number', which in turn could be defined in terms of other numbers and ultimately natural number. Whereas Aristotle made mathematics stand on two legs, the modern mathematician believes it can balance on one. To challenge this thesis is to undo the work of the last two hundred years and clearly requires more space than a single article. I will therefore say no more about it here. But it does bear on our immediate issue to this extent, that the modern number system requires a foundation, and that foundation is looked for in 'natural number'.

Alas 'natural number' has proved to be less straightforward than expected. Peirce put his finger on one critical issue when he asked

> whether the cardinal or the ordinal numbers are the pure and primitive mathematical numbers (Peirce, 1933: para. 658).

In other words does the essence of natural number lie in *how-many-ness* or in *rank order*? Broadly speaking, Dedekind, Peano and Peirce himself (after an initial flirtation with cardinality) favoured rank order ('ordinal number'), while Frege, Husserl and Russell went for how-many-ness ('cardinal number'). Only Cantor provides a more balanced view, suggesting that cardinal number (which he sometimes called the 'power' of a set) and ordinal number (alias 'ordinal type') might be two independent, if related, concepts. But he applied this analysis only to transfinite numbers, supposing that ordinary numbers somehow contain the potentiality for both concepts within a single entity. (See Cantor, 1900)

One modern view—sometimes promulgated even to teachers—is that natural numbers are defined by Peano's axioms. Yet, as Russell at least realised at the time, these axioms are satisfied by any rank-ordered sequence (in Russell's terms any *progression*), and from the available candidates he wanted to select "such as can be used for counting common objects" (Russell, 1920: 12). And there is now serious doubt as to whether the axiomatic method is suitable at all for pinning down a specific entity like natural number—see Wilder (1967).

The whole situation is admirably summarised by Goodstein:

> It is surely a very remarkable thing that despite the range, power and success of modern mathematics, the concept of natural number, on which the whole edifice rests, is still something of a mystery. (Goodstein, 1965: 68)

But in this summary Goodstein unwittingly perpetuates the two implicit assumptions that render the issue so intractable: the assumption that 'the whole edifice' must have a single foundation, and the assumption that that foundation is provided by 'natural number'.

Introduction 2: towards a solution and the role of mathematics education

The way to a solution, I believe, is sign-posted by three insights, all of them more than half a century old. The first is from Frege

> A frequent fault of mathematicians is their mistaking symbols for the objects of their investigation (Frege and Gabriel, 1984: 229)

Although Frege was perhaps thinking of higher mathematics, his stricture applies very well to arithmetic. In geometry, when the vertices of a triangle are labelled *A, B, C*, no-one would for a moment think that the 'objects of investigation' were letters of the alphabet, because the idea of a geometrical point is well enough understood and the letters are mere labels. But in arithmetic, where the labels are the numerals '1', '2', '3', it is far from clear what objects are being labelled, so that one is tempted to cling to the labels as if they were the real thing.

The way to track down the elusive 'objects of investigation' in such cases is indicated in the second insight, from Wittgenstein, who notes

> … the question, 'What do we actually use this word or this proposition for?' repeatedly leads to valuable insights. (Wittgenstein, 1961: 65)

Again Wittgenstein was presumably thinking more generally—and we need not here concern ourselves with the further views on language that he developed later—but what he says applies with special relevance to arithmetic, covering specialist symbols like numerals as well as number-words. There is a corollary to this insight: if it turns out that a symbol is used in two or more ways, then it has two or more meanings.

And this leads to the third insight, from Ayer, who refers to

> the superstition that to every name a single real entity must correspond (Ayer, 1946: 42)

If it turns out that a name (like, for example, 'natural number') refers to two or more distinct concepts (as it might be, say, how-many-ness and rank order), then it is erroneous to suppose that there exists some third thing (perhaps called natural number) over and above the concepts already identified, which somehow subsumes these into a whole. To this also we may add a corollary: once it is established that a word or other symbol is used to denote two or more separate concepts, then it may be helpful, at least in serious analysis, to given the concepts proper names of their own (as, for example, 'how-many-ness' and 'rank order' or perhaps 'cardinality' and 'ordinality') and drop the original name altogether because it is merely a source of confusion. For example, if one wishes to delineate the relation between how-many-ness and rank order, the task is made unnecessarily difficult if the two things are both called 'number'. (Indeed this common name may disguise the fact that there is any relationship to be investigated anyway.)

It is not perhaps obvious that mathematics education has any special contribution to make to the resolution of these fundamental issues. Teachers employ the vocabulary of number as freely as anybody. Nor are teachers especially scrupulous in distinguishing number-symbols from the things they denote. (And in any case such delicacy would be unhelpful if the things denoted were simply called 'numbers' without any finer discrimination.) In consequence the literature of mathematics education, like that of mathematical philosophy, contains much that is open to criticism in the light of the Frege/Wittgenstein/Ayer insights.

I believe nonetheless that teachers of young children (and those who teach them, advise them, watch them and do research in the classroom) do enjoy some advantages when approaching the foundations of mathematics. Children (some of them all the time and all of them some of the time) find mathematics difficult, and work has to be done to analyse the sources of their difficulties. And those who work with young children meet some of their fellow human beings before they have acquired all the prejudices of their native language. Anyway, for whatever reasons, the literature of mathematics education contains insights into mathematics as well as education, and we shall draw on some of these in the present paper.

Counting

Although mathematicians and philosophers often refer to counting as if it were the bedrock of arithmetic (if not of all mathematics), they also seem to regard it as too elementary and familiar to require further analysis. That task has therefore been left to mathematics education.

Counting in the simplest sense is the mere recitation of the number-names in order, but the counting *of objects* requires a one-to-one coordination between the spoken words and the objects in a given set. It thus provides a prime example of the process of *symbolisation* or *throwing together*. It is for this reason, of course, that number-words and numerals are often called 'symbols'. However, that term could with some justice be applied to the things that are thrown from the other side, and in their masterly analysis Gelman and Gallistel (1978) achieve a little extra clarity by avoiding the actual word 'symbol' and calling the number-words simply *tags*.

These authors observe that counting is governed by several principles. The tags themselves must have a fixed order (the *stable-order principle*). The objects being counted need not have any pre-existing order, but, even if they do, you can ignore it and count them in any order you like, so long as each object gets a unique tag (the *one-to-one principle*). Whatever the order of the counting, the last-used tag will always be the same (the *order-irrelevance principle*). Finally, the last-used tag must be re-interpreted as saying how many objects in the set (the *cardinality principle*).

By far the weightiest of these principles is the last. The order-irrelevance principle has its own importance because, whereas the tags are at first attached to individual objects, the last-used tag, being invariant, can now be thrown also against the complete set-of-objects. But the cardinality principle goes further, because that tag is not just a label for the set-of-objects, but states *how many* objects are in the set. It is a label for the *degree of how-many-ness* of the set, or (in one sense of the term) the *size* of the set, as witnessed by the fact that the same tag would be attached to any other set of the same size. In order to use the tag in this new sense a person must already have some notion of how-many-ness or size-of-set, and it is important to realise that, whereas the process of counting provides a *technique* for finding how many, the *concept* of how-many-ness has its origins elsewhere.

Approaching the matter from the direction of cognitive development, Schaeffer, Eggleston and Scott (1984) note that the degree of how-many-ness of a small set (say, up to five elements) can be seen at a glance without the need for counting, and they suggest that, if counting is now applied to such a set, the child may learn that, in these cases at least, counting gives the same answer. From there she will—according to the *skills-integration model* they employ—go on to make the required generalisation.

Approaching from a different perspective, it may be argued that the process to which how-many-ness is most intimately related is not counting at all but the direct matching of sets-of-objects by one-to-one correspondence. When this is done, it will be found that in most cases one set is larger than the other (contains more objects), but sometimes they are the same size (in Frege's term *equi-numerous*). This equivalence relation of equi-numerosity underlies the abstract concept of how-many-ness, and the order relation in which it is embedded (has-more-than) underlies the abstract order relation on degrees of how-many-ness (three-ness is greater than two-ness).

Whatever the precise analysis, however, the conclusion must be that the use of counting to find the degree of how-many-ness of a set of objects requires a grasp of how-many-ness as well as a grasp of counting, otherwise the purpose of the exercise will be missing. That is the essence of the cardinality principle.

We may note that, when counting is mentioned in academic mathematics or mathematical philosophy, there seems to be no appreciation of the cardinality principle at all. Thus Benacerraf (1965: 50) distinguishes *intransitive* counting (the mere recitation of number-names) from *transitive* counting (the counting of objects), but the latter he takes to be all the rest of the process, making light of the difference between throwing a tag at an object and throwing it at the whole set, let alone throwing it at the degree of how-many-ness of the set. Peirce (1933: para 156) identifies what he calls The Fundamental Theorem of Arithmetic, but this title he bestows on the order-irrelevance principle, not the cardinality principle. He seems to think it sufficient that a numerical tag should be associated with a (finite) set, and ignores the further step of associating it with an *attribute* of the set.

Rank order and the use of letters as numerals

Although the size of a set is unaffected by the order in which the elements are counted, the process of counting does nevertheless have the incidental effect of placing the objects in order, at least temporarily. But, of course, some sets of objects—for example the pages of a book or the houses in a street—are ordered permanently, and here numerical tags can be used permanently to indicate position in the sequence.

It is worth noting, however, that alternative notations are available for this ordinal purpose. There are, firstly, the dedicated adjectives, ‘first’, ‘second’, ‘third’ etc. Whether a street is called ‘Fifth Avenue’ or ‘Avenue Five’, whether a house is called ‘Number 6’ or ‘Sixth House’, whether a person is called ‘Second-in-Command’ or ‘Number Two’—these are matters of style not mathematics, for in either case the concept being indicated is rank order. Only the symbols are different.

Another alternative is provided by letters of the alphabet. The sections of a report can be labelled Section *A*, Section *B*, Section *C* just as well as Section 1, Section 2, Section 3 (or Section I, Section II, Section III). Letters are suitable for this purpose because, like numerals, they themselves possess rank order. They also provide, like numerals, a comprehensive system because, after the first twenty-six places, they can be combined into compound tags by the principle of place value. (After *Z* you take *AA*, *AB* etc, and after *AZ* you take *BA*.) In theory letters of the alphabet could replace numerals in all ordinal applications. For example, the pages of a book could be lettered instead of numbered. I think that the use of letters has a special role in conceptual analysis, where it can avoid the ambiguity of using numerals simultaneously in both an ordinal and a cardinal role.

Note in passing that in this alphabetic system there is no sign corresponding to zero. It is sometimes said that the principle of place value depends on the introduction of zero. But this is an over-simplification. Zero would not be needed with numerals either if these were used only for rank order. (It would then be acceptable for 9 to be followed by 11, 19 by 21, and 99 by 111.) Zero is only needed for the denoting of how-many-ness.

Addition

The degrees of how-many-ness—which we are calling ‘one-ness, ‘two-ness’ etc—possess a rich structure. In the first place, they form a rank-ordered sequence. Normally one-ness is the first, two-ness the second and so on. If you start with zero-ness, then zero-ness is the first and one-ness the second. But either way they possess *rank order*. For this reason they can be represented by the points of a (discrete) number-line—see further below.

The degrees of how-many-ness also possess the normal operations of arithmetic, starting with *addition*. In order to add four-ness and five-ness, for example, you take any set of four objects, combine it with any (disjoint) set of five objects, and find how many objects in the union. The mathematician will note that this operation is *well-defined* because the result is independent of the choice of objects. The teacher will note that the usual symbolic statement ‘4+5=9’ can be interpreted either at the concrete level, meaning that a set of 4 objects combined with a set of 5 objects gives a set of 9 objects, or at the abstract level, meaning that the sum of four-ness and five-ness is nine-ness. In any matter to do with addition of degrees of how-many-ness, however abstract the setting, if one gets stuck or fails to remember a result, one can always appeal to sets of concrete objects.

At this point it is necessary to stress a very negative fact: **rank-ordered sequences in general do not possess an operation of addition**. Up to a point this is obvious. You do not 'add' the fourth person in a queue to the sixth person to get the tenth person. Nor is there any sense in which pages 7 and 8 of a book can be combined to give page 15. Unfortunately neither mathematicians nor teachers have fully faced up to this natural deficiency, and they have tried to make up for it in various ways, particularly when they have in mind the sequence of 'numbers'.

One way, popular in mathematics, is to *impose* a kind of addition on rank-ordered sequences. Take, for example, the letters of the alphabet—and you can take them either as objects in their own right or as tags for other objects. You can then decide that $A+A$ will be, say, B (or any other letter), and complete an 'addition' table in the obvious way to get $A+B=C$, $A+C=D$ and so on, making use of the successor relation. This is often called 'definition by induction'. I think a better name is *definition by imposition*. The trouble is, of course, that the result does not correspond to any natural feature of the domain on which it is imposed. In this way you can 'add' the kings and queens of England so that, for example, Edward VI, the successor of Henry VIII, becomes the 'sum' of Henry VIII and William the Conqueror!

Another way of bringing in addition, popular in teaching, is to involve *displacements* or jumps along a sequence: a jump of 3 steps, combined with a jump of 4 steps in the same direction, gives a jump of 7 steps. This is a genuine addition, being an example of the *vector addition* that can be found in spaces of any dimension, both discrete and continuous. It may be remarked, however, that, if one were setting out to investigate jumps along a dotted line, and anticipated that the sizes of the jumps would be denoted by numerals, it would be asking for trouble to use numerals to tag the points of the line at the same time—letters would be better. (Any alternative would be better.) But this is what happens in the primary classroom. And trouble does ensue. If, using letters, you start at point C and move 4 steps along, you will finish at point G, an operation that might with a good notation be symbolised as $C \xrightarrow{+4} G$. However, if the points (as well as the displacements) are labelled with numerals, the formula that springs to mind is $3+4=7$, and this gives a kind of hybrid addition or *pseudo-addition* in which the first and last numerals are tags for points and the '4' says how many steps in the jump.

The attempt can be made to justify pseudo-addition in terms of vector addition, but this is more tricky than it looks, partly because it means abandoning the original assumption that some of the numerals are simply tags for points, and partly because it requires the introduction of a zero-point or origin. One can also try to justify pseudo-addition by counting suitable subsets of points, but this raises doubts as to whether it is points or steps that are to be counted.

Fuson (1984: 219), taking a purist view, argues that the number line is a 'measure model', not a 'count model', and insists that numbers are represented by lengths and not by points at all. This is very close to an endorsement of what I am calling vector addition, and certainly rules out pseudo-addition. But I have one serious quarrel with this analysis. In my view the number line is not a 'representation of number' at all because 'number' does not exist.

Among the things that do exist are degrees of how-many-ness, numerical tags, and various geometrical models including the *continuous half-line* and the *dotted half-line* or *stepping stone model* (both of which are sometimes called number lines). The stepping stone model may certainly be used to represent the degrees of how-many-ness, demonstrating their rank order, so long as it is realised that in this model there is no representation of addition. (The continuous half-line can similarly be used to represent degrees of any continuous quantity, with the same proviso regarding addition.) But the geometrical models may also be studied for their own sakes, and in the study of jumps along a dotted line number-symbols will naturally be used to indicate how many steps in a jump. Note, however, that the objects here are steps, jumps, the sizes of jumps, and numerical tags. It only confuses the situation to talk about ‘numbers’ as well.

Conclusions

In our analysis the cardinality principle is important because it brings together counting and how-many-ness (the ordinal and the cardinal). The mathematician’s lack of interest in it can perhaps be blamed on the division of that community into two camps, neither looking for rapprochement because each believes that its own position provides a complete account of the entire subject. In mathematics education such a reductionist stance is less common. It is true that Padberg and Benz (2011) detect “two very different tendencies” (*zwei sehr unterschiedliche Ansätze*), one associated with Piaget, stressing things like invariance and one-to-one correspondence, and favouring the cardinal, and the other, based on the skills-integration model, laying more stress on counting, and favouring the ordinal. But it is hard to believe that practising teachers have ever entirely neglected either how-many-ness or rank order, and Padberg and Benz themselves propose that primary teaching should be based on the concurrent development of several different “aspects of number”, including a cardinal aspect and an ordinal aspect.

This plan is broadly attractive on both practical and theoretical grounds. My objection is that, while it follows the Wittgenstein insight by looking for usage, it is less attentive to the insight of Ayer on the avoidance of spurious objects. In fact the very phrase ‘aspects of number’ implies that the various concepts identified by studying usage, notably how-many-ness and rank order, are merely ‘aspects’ of some further thing called ‘number’. This is Ayer’s superstition without apology or mitigation. It can be avoided if one says, more simply, that

> Children should study the elementary uses of number-symbols, especially in the denoting of how-many-ness and rank order.

There is no need for the word ‘number’ at all.

I am not the only person to deny the existence of ‘natural number’. Benacerraf (1965: 73) also reached the conclusion that “there are no such things as numbers”. Benacerraf, however, was too pessimistic. He was looking for number in rank ordered sequences generally and was concerned (rightly) at their lack of uniqueness. He gave no credence to *how-many-ness*, which is the hero of my own analysis, and in fact possesses all the properties that are normally looked for in ‘natural number’.

What’s in a name? you may ask, and that is a good question. I have tried to argue that the term ‘natural number’ is too soiled by misuse to merit any continuing place in serious analysis. It also carries overtones of the arithmetisation of analysis—and that is seriously prejudicial to further developments. However, ‘degree of how-many-ness’, which I have used in the present paper is cumbersome and inelegant. It could perhaps be replaced by ‘cardinality’, which emphasises the contrast with rank order. For further work, however, my preferred term will be ‘quotity’ from the Latin *quot*, meaning how many. This term has been introduced into mathematics by Lucas

(2000: 98), who points out the contrast with ‘quantity’ from *quantum*, meaning how much. The parallels and contrasts between quantity and quotity provide a further rich field of enquiry. (And both can be compared with mere *qualities* such as shape and rank order, which lack addition.) But, of course, there is still much to be done to establish the credentials of quotity itself.

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Assessment Practices in Secondary-school Mathematics Teaching in Brazil

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Classroom-based assessment has been a matter of concern and discussion in academia, especially in recent years. Many studies have been conducted, particularly about the implementation of formative assessment. Although it has been heralded as an important practice, there is still little research about this subject related to mathematics education, particularly in Brazil. Aiming to seek information about the types of approach that secondary-school mathematics teachers in Brazil have been taking in their classrooms, survey research was conducted via an e-questionnaire. The teachers were asked, among other aspects, about the frequency with which they apply and the importance they give to specific assessment methods or procedures. The results from the quantitative analysis show that tests and homework assignments are the methods most commonly used by mathematics teachers, whereas self- and peer-assessment are still not common practice.

Keywords: Assessment, mathematics, secondary school, Brazil

Introduction

The issue of assessment has been a subject of intense research. Discussions concerning the importance and purposes of assessment have played a central role among researchers, particularly about its formative and summative functions, with evidence that the former improves learning (Black and Wiliam, 1998).

Even so, as argued by some researchers (James and Lewis, 2012; Shepard, 2002), it seems that the re-shaping of assessment is not occurring at the same pace as the re-shaping of instruction.

Recognising the relevance of teachers' conduct to the development of assessment practices, the present study aimed to analyse which types of approach have been adopted by secondary-school mathematics teachers in Brazil to assess their students. This study also aimed to analyse if there is any evidence that they are implementing assessment with a formative purpose.

Teachers' Assessment Practices

The expression *assessment practices* does not simply refer to techniques, procedures or instruments. It has a broader meaning, covering events that occur in the assessment of daily schoolwork. Both formal procedures, i.e. those that are planned and which inform students that they are being evaluated (e.g. tests and homework); and informal procedures, that occur through the interaction of teachers with students and the students themselves (e.g. observations of students' responses in class), can be included in these criteria.

Black and Wiliam (1998), discuss the results of studies conducted by Crooks (1988) and Black (1993), which revealed many weaknesses in assessment practices:

- The practices generally encouraged superficial and mechanical learning;
- Teachers, in general, did not review the assessment tasks and procedures. Moreover, they were not critically discussed with the students, which indicated little reflection on what was being assessed;
- The attribution of marks was the primary purpose rather than the promotion of learning;
- There was a tendency to conduct norm-referenced rather than criterion-referenced assessment.

Susuwele-Banda (2005), using a questionnaire, interviews and observations concluded that teachers perceive classroom assessment as tests that they give to their students at specific time intervals. Moreover, as they perceive classroom assessment as tests, they showed a limited ability to use different methods and tools to assess their students.

In contrast, Pacheco (2007) investigated primary teachers' assessment conceptions in Brazil and found that, although the participants are still implementing assessment for summative purposes, they recognise the importance of formative assessment and the use of diverse instruments and procedures to assess their students.

The same characteristics were found in the study conducted by Albuquerque (2012), who concluded that, although teachers recognised the necessity of using different methods and instruments to assess their students, the two methods that they used widely are homework assignments and tests. Indeed, some of them still use tests as the only method based on the justification that it is the most practical and objective method, and also because of the time constraints and number of students per class. Homework was mentioned as being used only to provide marks related to the fulfilment of the task or otherwise.

In many cases, as reported by Johnston and McClune (2000), teachers adjust their teaching style in ways they perceive as necessary because of the tests. They spent most of the time on direct instruction and less on providing opportunities for their students to learn. Moreover, Harlen (2004) shows that the teachers' assessment practices are inevitably influenced by the external assessment and that teachers often use these assessments as models for their own, even if they do not use them directly.

However, there is a major thrust of new research and professional thinking about assessment that has been seeking to distinguish the formative and summative functions and suitability of various types of assessment methods for these different purposes.

Frohbieter et al. (2011), for example, analysed three mathematics formative assessment programmes in the US, which showed many different approaches to assessment. Several teachers reported using some form of warm-up exercises and/or pre-assessment at the beginning of an instructional unit in order to determine whether or not their students had specific skills or had mastered the topics that would be covered. Some teachers also reported using these assessments to understand some of the more common mistakes made by their pupils as well as exploring where a solution process had broken down and why, or describing a partially-correct conception that produced the right answers only in certain cases. Many of them pointed out that they preferred using tests with open-ended questions (constructed-answers) because this allows them to see what their students are thinking a lot more clearly. In addition to tests, some teachers reported using informal observations, quizzes and homework assignments.

While Frohbieter et al. (2011) analysed the current practices of the teachers, Black et al. (2003) focused their attention in the King's-Medway-Oxfordshire

Formative Assessment Project (KMOFAP) on innovative strategies that emerged from teachers' participation in the project. Teachers made changes in relation to classroom questioning, feedback through marking, peer- and self-assessment, and the formative use of summative tests. In order to take a formative approach in preparing their students for summative tests, the teachers encouraged the pupils to use *traffic lights* to improve their review schemes. The traffic lights were also used as a means for self-assessment. Teachers also encouraged pupils to generate and then answer their own questions.

Concerning classroom questioning, the teachers wanted to give more time to students to think about their responses; and this made them realise that it would also be necessary to spend more time designing those questions, so they would indeed evoke student understanding. In relation to *comments-only feedback*, some teachers simply stopped marking; others allocated marks only for their own records without showing them to the students, while others only gave marks after receiving answers from the students following the teachers' feedback. In this way, they started thinking about how they should write the feedback so the students could realise what they had already achieved and which specific areas they should improve and also how to engender attitudes that would make the students act upon the feedback given.

Based on the pieces of research reviewed and the fact that almost no research relating to Brazil was found, four specific research questions were addressed:

1. What types of approach have been adopted by mathematics teachers in Brazil to assess their students?
2. How are teachers using the information gathered from assessments?
3. Is there any evidence that professional development courses have influenced teachers' approach to assessment?
4. Is there any connection between teachers' conceptions and their approach to assessment?

In this paper, only results from research question 1 will be discussed.

Methodology

As the main goal of this research was to explore what types of approaches have been adopted by secondary-school mathematics teachers in Brazil for assessing their students, we decided to implement an exploratory questionnaire survey (Cohen et al., 2011).

Sample

The questionnaire was delivered to mathematics teachers in Brazil. However, Brazil is a huge country with approximately 70 000 secondary mathematics teachers. Due to the time and distance constraints and the large number of teachers in the population, we decided that the questionnaire would be delivered only to those who participated in the Gestar II Programme, which was a teacher-training course offered from 2008 to 2011 to those teaching in secondary schools.

The choice of this sample was made in order to facilitate the data collection, since one of the researchers was one of the trainers for the programme, and still had access to the participants. Moreover, having access to all the regions in Brazil would produce more accurate results about the assessments being used throughout the country. Thus, in this particular study, the sample can be characterised as an opportunitistic sample that, conveniently for purpose, is already geographically clustered.

However, the choice of this sample carries some biases that were taken into account in the analysis and in the intention of generalising the results. The teachers who were nominated by the local government to take part in the Gestar II programme were chosen from amongst the more knowledgeable and/or professionally active teachers in order to guarantee that the training would be successful in its third level. Therefore, it might be expected that their assessment practices are more effective, with the students and the teaching-learning process playing a central role.

Instrument

For the present study, a structured e-questionnaire was adopted. It was designed using the web-application Qualtrics® and was divided into five sections: one section required the teachers to provide personal profile information, with the intention of defining the context and classifying the data. The second section was related to the frequency in which the teachers applied specific types of assessment and the importance they ascribe to each of them. Two sections were dedicated to better understand the use of tests and homework assignments. In the last section, the teachers were asked about the actions taken after the implementation of an assessment.

The response options provided for all questions were taken from other studies (Albuquerque, 2012; Black et al., 2003; Black and Wiliam, 1998b; Brookhart et al., 2004; Hodgen and Wiliam, 2006) as well as from our experience as mathematics teachers. The Portuguese version was developed using terms that are widely used in the Brazilian teachers' daily practices, and reviewed by a specialist in assessment in mathematics education (from Brazil). When judged necessary, some specific items had further explanation (e.g. long tests (taking more than one hour to complete)).

Procedure and Data Analysis

The questionnaire was sent out at the beginning of May 2012 and replies were accepted for approximately a month. The first step was to use descriptive statistics to determine the overall characteristics of the data. The relevant findings were organised in the form of tables and figures. After that, we divided the questionnaire according to the research questions, and cross-tabulation tables were generated to analyse the degree of association and homogeneity amongst all of the questions that were related.

With nominal variables, Chi-squared tests were applied and p-values were obtained through Monte-Carlo estimation to evaluate the statistical significance of the relationship between the questions, taking into account the degrees of freedom (df). The same steps were followed with regard to the ordinal variables, through a Kendall's tau-b test, which measures the relationship between two ordinal or ranked variables.

Findings

The questionnaire was answered by 332 mathematics teachers, which can be considered an acceptable sample size since this study was intended to be exploratory, where the findings were not intended to be generalised to the entire target population. The majority of the respondents were experienced teachers with more than 16 years of teaching experience, having at least a post-graduate degree. The Northeast was the region with most respondents (38.8%) and those from the North comprised the minority (3.9%). The remainder was almost equally distributed among the Southeast, South and Midwest (18.7%, 18.7% and 19.9% respectively).

The types of assessment that had the highest number of responses in each category of frequency are presented in table 1.

Question 11: In assessing the work of the students in a typical class of yours, roughly how often do you use each of the following types of assessment?	Never	Annually	Termly	Monthly	Weekly	Daily
Q11.13: Portfolios of student work	**54.8%**	18.1%	14.2%	7.8%	2.7%	2.4%
Q11.12: Peer-assessment between students	**53.0%**	6.6%	12.0%	16.9%	9.9%	1.5%
Q11.1: Standardised tests produced outside the school	24.7%	**26.5%**	23.2%	16.3%	7.5%	1.8%
Q11.5: Long tests (taking more than one hour to complete)	24.4%	4.2%	**44.9%**	25.0%	0.6%	0.9%
Q11.2: Teacher-made tests involving open-ended questions	5.7%	2.1%	19.0%	**54.5%**	16.3%	2.4%
Q11.6: Homework assignments	2.7%	0.3%	1.2%	3.6%	**36.1%**	**56.0%**
Q11.10: Attention to responses of students in class	0.3%	0.0%	1.2%	2.1%	10.2%	**86.1%**

Table 1: Frequency of assessment types.

The result indicate that *portfolios of student work* (Q11.13) and *peer-assessment between students* (Q11.12) are the types of assessment that are least widely used among the respondents, since 54.8% and 53.0% of the total respectively, stated that they *never* use these methods to assess their students.

However, the same logic cannot be applied to conclude that *attention to responses of students in class* (Q11.10) is the method most used solely because the majority of teachers stated that they use it in their day-to-day classes. Different methods are expected to be used with different frequencies. That is to say, it is not expected, for example, to apply *long tests* on a daily basis, since the available time also needs to be used for teaching and learning activities, but it is perfectly reasonable to implement it on a *termly* basis.

Following this argument, it can be inferred that *attention to responses of students in class* (Q11.10) is the method most used on a *daily* basis (86.1% of the respondents), which is unsurprising since the teachers are considering their own actions, i.e., their observations of what the students are doing and saying in the classroom, as an important means of assessment. *Homework assignments* (Q11.6) were also indicated as being largely used on a *daily* (56.0%) and *weekly* basis (36.1%).

Different types of test were indicated as most commonly used on a *monthly*, *termly* or *annual* basis. However, it is important to notice that the number of teachers who affirm using *standardised tests produced outside the school* on an *annual* basis (26.5%) is almost the same as that of those who state that they *never* use this method to assess their students (24.7%), which shows that there is no agreement in relation to the use of this type of assessment or perhaps some teachers do not use this approach because they do not have access to these tests.

One could argue that this elevated use of tests is due to school or government rules, such as the obligation to award a final grade with marks coming from specific types of assessment. However, this argument is not supported by the results, since there is a statistically significant relationship between the frequency with which teachers apply the different types of test listed in Q11 and the weight they give to them (Q12). In other words, it was possible to observe that teachers who use tests relatively often also give *quite a lot* or *a great deal* of importance to them.

For example, 93.9% of the teachers who reported applying "teacher made tests involving open ended questions" (Q11.2) *monthly* affirm giving *quite a lot* or a *great deal* of importance to this kind of assessment (T_B = 0.230; $p < 0.0001$). The same happens to the other types of test listed in Q11. For teachers applying "teacher-made multiple choice, true-false and matching tests" *monthly*, 91.8% of them give *much* weight to this type of test (T_B = 0.271; $p < 0.0001$). For those who stated that they set "teacher-made short answer or essay test that require students to describe or explain their reasoning" *monthly*, 95.1% also give *quite a lot or a great deal* of importance to them (T_B = 0.223; $p < 0.0001$). The same was observed for those applying Q11.5 *termly*, where 81.8% affirm giving *much* weight to this type of test (T_B = 0.425; $p < 0.0001$).

Summary and discussion of the main findings

First of all, it was possible to observe that, although the teachers use different kinds of assessment and with different frequencies, tests and homework assignments are the two methods that are most commonly used by secondary-school teachers of mathematics in Brazil, as corroborated by the importance that the teachers said they give to them, which confirms the results of other studies (Albuquerque, 2012; Pacheco, 2007; Susuwele-Banda, 2005). As the questionnaire had two specific sections addressing these methods, it was possible to analyse in more detail how teachers are using them.

Referring to the tests, before setting them, the teachers reported frequently giving a review lesson, in which they include the contents that were covered in previous lessons, as well as practise in basic skills and in tasks similar to those contained on the test. This action could be considered summative in spirit, if the intention is simply to give students practice in taking the test in order to get better results. On the other hand, if the review also has the intention of promoting learning and helping the students to understand their strengths and weaknesses in order to use the content covered to see what is the next step in their learning process (Harlen and James, 1997), it can be considered formative.

The same can be said in relation to the type of questions that the teachers include in their tests and the kinds of skill they require from their students to answer them. The respondents reported that most of their tests comprised open-response questions, involving the application of mathematical procedures. This can be considered a good indicator, although it is impossible to affirm if they include these questions in order to try to understand the students' thoughts and make use of this to guide their teaching to suit the students' needs, or if they are just encouraging superficial, mechanical learning, focused on memorising isolated details, typical of the weak practices which Black and Wiliam (1998a) reported.

Similarly, some comments can be made about homework assignments. The majority of the teachers reported that they assign exercises and problems from the textbook to their students. After that, they record whether or not the homework was completed and give feedback to the whole class. Based on these statements, it seems that homework assignments are being used principally for accountability, where the teachers are not considering the student's individual performance, just whether they have completed the assignment or not, without taking into account what has been answered as an indication of what the students have learnt and what they still need to improve.

However, the data did not provide enough information to draw confident conclusions on this. A deeper analysis of how this feedback is being given would be needed in order to decide whether or not it is being used for the improvement of learning or if it is being used just to correct and show what is wrong or right in the assignment. It would be necessary to verify if the teachers give the students the opportunity to think about their learning and how it is possible to focus on the aspects that they still need to improve, as well as if the teachers are giving their students the opportunity "to act upon the feedback and also discuss the feedback with others" (Hodgen and Wiliam, 2006: 19).

Finally, it is important to remember that all the findings presented here are the results of an exploratory study through a self-report questionnaire, which is subject to social desirability bias and may not reflect the actual classroom practices of the respondents.

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Early Number Concepts: Key Vocabulary and Supporting Strategies

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Teacher-facilitated "math talk" in the early years significantly increases children's growth in understanding of mathematical concepts (Klibanoff et al., 2006). Although young children may have a beginning understanding of early number concepts, they often lack the language to communicate their ideas. Teacher modelling and fostering of mathematical language throughout the day and across various subject areas, allows children to articulate their ideas and communicate their understanding. Encouraging "math talk" in young children as they explain, question and discuss their strategies is important. The teacher plays a significant role in guiding children to make connections, to recognise how their thinking relates to key mathematical number concepts and to make further conjectures and generalisations. This paper will outline the theoretical perspectives underpinning the development of a resource of key vocabulary and teaching and learning strategies for teachers to support their planning and teaching in early number.

Keywords: number; language; early years

Achievement in mathematics

Achievement in mathematics is a key educational concern. Competence in mathematics is essential in meeting the demands of the workplace and in successful functioning in everyday life. However, recent research reports have indicated that many children in the North and South of Ireland are failing to reach the expected levels of achievement in mathematics (DENI, 2011; DES, 2011). Although considerable attention has been devoted to mathematics achievement at primary and secondary levels, the foundations for learning mathematics are established much earlier (Clements and Sarama, 2007).

By the time children enter preschool, they demonstrate wide individual differences in their mathematical knowledge, with children from high and middle socioeconomic status (SES) families showing higher levels of mathematics achievement (Klibanoff et al., 2006). Such early differences are a matter of some concern since levels of mathematics knowledge at the time children enter school have been shown to predict later achievement (Duncan et al., 2007). Success in mathematics in the early years is critical. If children can learn to think mathematically and to express their thoughts in mathematical terms during the preschool years, then they are better prepared to learn formal mathematics concepts upon school entry (Ginsburg, Lee and Boyd, 2008). Austin et al. (2011) argue that neglecting mathematics in the early years might hamper both mathematical development and literacy skills. Duncan et al. (2007) found that early mathematics knowledge is a more powerful predictor of later achievement than early language and reading skills. High levels of mathematical competency are also required to satisfy growing needs for a scientifically and technologically sophisticated workforce (NRC, 2009).

A socio-cultural perspective on learning

Both socio-culturalists and constructivists recognise the importance of individual activity in learning. While constructivists prioritise psychological processes, socio-cultural approaches give priority to the context for learning, placing emphasis on "the conditions for the possibilities for learning" (Cobb and Yackel, 1998: 184). According to Rogoff (1998), learning arises from both individual activity and participation in social activity. Rogoff's (1995) view is that individual learning cannot be understood outside of an activity or of the people participating in it. She views learning as the development of mind in a socio-cultural context. Children's active participation in an activity is regarded as an important element of the process by which they gain mastery. Rogoff (1990: 7) conceives of children as "apprentices in thinking, active in their efforts to learn from observing and participating with peers and more skilled members of their society." As children engage in culturally valued activities, they become more responsible participants. However, Rogoff (1995) argues that children need to be guided in that participation and she defines 'guided participation' as

> The processes and systems of involvement between people as they communicate and co-ordinate efforts while participating in culturally valued activities. This includes not only face-to-face interaction … but also the side-by-side joint participation that is frequent in everyday life and the more distal arrangements of people's activities that do not require co-presence…. The 'guidance' referred to in guided participation refers to observation, as well as hands on involvement in an activity. (Rogoff 1995: 700)

From a socio-cultural stance, learning is seen to be a consequence of collaboration in social activity.

Language and mathematics

"The ability to communicate is at the very heart of early learning and development" (NCCA, 2003: 29). For most children, language is the dominant form of communication. According to Vygotsy (1978), concepts are first introduced on an interpersonal level through social interaction and then develop, integrate and expand intrapersonally, as children work to understand and use the concept. On both levels – interpersonally and intrapersonally – language serves a primary role in understanding and mastering what is learned. Language, "the primary cultural tool … is instrumental in restructuring the mind and in forming higher-order, self-regulated thought processes" (Berk and Winsler, 1995: 5). Language also plays a crucial role in helping children to use other cultural tools, including the notational systems of writing and counting (John-Steiner and Mahn, 1996), and is necessary to understand (Jordan et al., 2007) and express (Ginsburg et al., 2008) other kinds of mathematical thinking. Although the notational system for numbers is governed by different rules than those for writing, Austin et al. (2011) argue that the process of developing facility with one cultural tool enables the child to gain better facility with another. Further, it appears that proficiency in language is a key factor in predicting proficiency in mathematics (Austin et al., 2011).

Language is fundamental to education because it is the major form of representation of cultural knowledge and the principal medium of teaching. The nature of the relationship between language and mathematical cognition is currently the subject of much debate (Donlan et al., 2007). While some argue that increasing the time spent on mathematics activities could decrease time available to spend on language activities, thus impeding children's development of language, Sarama et al.

(2012) argue that this is based on the assumption that mathematics activities have little or no positive effects on language. However, evidence from both educational and psychological research suggests that language and mathematics have co-mutual beneficial influences. Development in both domains appears to follow similar pathways (Sarama et al., 2012). Moreover, Duncan et al. (2007) suggest that mathematics learning has the potential to make a unique contribution to children's emerging literacy due to its emphasis on reasoning, problem solving and communication (NCTM, 2006; Senk and Thompson, 2003).

A number of studies show that children's language acquisition is related to the overall amount of language input they receive (Weizman and Snow, 2001). Furthermore, the specific lexical terms acquired appear to be sensitive to variations in the amount of input. It therefore seems reasonable to suggest that children's acquisition of mathematical language is also related to the amount of "math talk" they are exposed to. Klibanoff et al. (2006) contend that the amount of teachers' mathematics-related talk is significantly related to the growth of young children's mathematical knowledge. In other words, teacher input that helps children to learn the language of mathematics will have a positive impact on the development of their mathematics skills. Although acquiring the language of conventional mathematics is only a part of developing understanding in mathematics, it is an important tool for fostering mathematical thinking.

Number sense

The importance of number sense in school mathematics has been highlighted by many national reports (Cockroft, 1982; NCTM, 2000; NRC, 2009). However, there is no consensus on a precise definition of the term. Over thirty years ago, Cockcroft (1982) established that a "feeling for number" is an important mathematical requirement of adult life and used the word 'numerate' to imply the possession of two attributes:

> an 'at-homeness' with numbers and an ability to cope with the practical mathematical demands of everyday life …. an ability to have some appreciation and understanding of information which is presented in mathematical terms,... (Cockcroft, 1982: 11).

More recently, 'numeracy' is highlighted in national strategies north and south of Ireland (DES, 2011; DENI, 2011) and is defined as "the ability to use mathematics to solve problems and meet the demands of day-to-day living" (DES, 2011: 8) or "the ability to apply appropriate mathematical skills and knowledge in familiar and unfamiliar contexts and in a range of settings throughout life, including the workplace" (DENI, 2011: 3).

The introduction of the term 'number sense' was aimed at embracing a range of real-life applications of number as well as balancing the traditional skills-based curricula with approaches which included other aspects of number (Dunphy, 2007). Number sense, in curriculum documents worldwide, refers to "flexibility" and "inventiveness" in calculation and is a reaction to an "overemphasis on computational procedures devoid of thinking" (Anghileri, 2000: 2). Not only does it relate to the development of understanding but also to the "nurturing of a positive attitude and confidence" (Anghileri, 2000: 2). Consistent with a socio-cultural perspective on learning where children's number sense is viewed as developing in collaboration in activity with others, Dunphy (2007) considers that number sense in very young children will look different from that of older children. Her framework reflecting key aspects of number sense as it relates to four year olds includes: pleasure and interest in number; understandings of the purposes of number; ability to think quantitatively; and awareness/understanding of numerals (Dunphy, 2006).

The role of the teacher

The mathematical knowledge teachers possess has a profound impact on what and how they teach (Bobis, 2004). Teachers play a key role in helping children develop number sense through creating a learning environment that encourages children to freely explore numbers, operations, and their relationships in meaningful contexts (McIntosh, 2004; Siegler and Booth, 2005). Similarly, Dunphy (2006) highlights the importance of mathematical language in the provision of a quality early years' mathematics curriculum and acknowledges the pivotal role of the teacher:

> Responding to children's curiosity and interest about numbers, encouraging children to use number and number language as a means of organising and communicating their experiences, modelling of skills related to quantification, and drawing children's attention to the use of numerals in different contexts are also essential pedagogical tasks for the early years teacher' (Dunphy, 2006: 72-73).

Yang et al. (2009) suggest that teachers' lack of number sense as well as their lack of knowledge on how to help children develop number sense may account for weak performance in number sense. They argue that teachers empowered with knowledge and appreciation for number sense will be more likely to attend to number sense when working with learners. Greeno (1991) explicitly acknowledges the role of adults in relation to the development of number sense and recognises that "someone who already lives in the environment is an important resource for a newcomer" (Greeno, 1991: 197). Consistent with Rogoff (1990; 1995), this acknowledges that the development of children's number sense needs to be guided by more experienced others and is intrinsically bound up in everyday experiences. Through guided participation in a range of meaningful mathematical experiences, young children become more skilled in understanding and using number.

The theoretical perspectives underpinning the development of a resource of key vocabulary and teaching and learning strategies for teachers to support their planning and teaching in early number have been outlined with regard to achievement in mathematics, the socio-cultural perspective on learning, language and mathematics, number sense and the role of the teacher.

Methodology

The NRC (2009) recommends that number should be emphasised in the development of young children's early mathematics. In light of this, it was decided to focus on the development of young children's early number concepts with a particular emphasis on the key associated vocabulary. The proposed research questions included: (1) What is the core vocabulary children require to understand, communicate and apply early number concepts? and (2) What approaches/strategies could assist teachers in their planning and teaching of the language of early number? Cooper's (2007) model of research synthesis was adapted for the project, namely: step 1, formulating the problem; step 2, searching the literature; step 3, gathering information from literature sources; step 4, evaluating, analysing and integrating the studies; step 5, interpreting the evidence, and step 6, developing the resource.

The research methodology utilised in the project was documentary analysis. During this review, books, papers, research reports and policy documents using library and internet sources were consulted and reviewed. The areas of focus emphasised children's development of number, mathematical language and intervention techniques/strategies used to support the development of number and language. The principal focus of the research was on recent national and international research from an Irish, UK and international perspective. The researchers completed a rigorous literature search examining the role of mathematical vocabulary and language in the acquisition of early number. In addition, evidence-based research was reviewed to identify strategies supporting the teaching and learning of early number concepts.

Major Education and Social Science Databases (for example, Australian Education Index; British Education Index; Education Research Abstract; PsychINFO; International Bibliography of the Social Sciences; and the Mathematics Didactics Database) were searched using search terms such as mathematical language, language development, development of mathematical language, analysis of number, early number concept, number sense, and so on. These sources provided an extensive basis of documentary evidence and information. Emphasis was given to peer-reviewed sources. Documents were evaluated and critiqued on four criteria, namely; authenticity, credibility, representativeness, and meaning (Denscombe, 2004). The analysis of documentary evidence was the central and exclusive research method. Content analysis was considered the most appropriate approach in analysing the documents. It was important that appropriate categories and units of analysis, both of which reflect the nature of the documents being analysed and the purpose of the research were identified (Cohen et al., 2004). The studies were reviewed and compared and conclusions drawn concerning the nature of early number concepts and language.

As already stated, this project gathered data from the analysis of secondary sources, namely document analyses. In this sense, no defined research sample was involved in the project. As this research project centred on the development of a resource, there was need for independent review by teachers. The resource was piloted and reviewed by teachers of infant classes (Republic of Ireland) and the foundation stage (Northern Ireland). This process involved teachers familiarising themselves with the resource, implementing the activities and strategies in their classrooms with a focus on facilitating "math talk", and subsequently critiquing the resource by completing an evaluation form. The review process was completed at three different stages of the project to reflect the three core areas highlighted above. The involvement of key stakeholders facilitated the socio-cultural perspective.

Outcomes

The principal outcome of the project was the production of a teaching and learning resource for teachers in the area of early number concepts with an emphasis on developing associated language. As a result of the documentary analysis detailed above, it was decided to organise the resource into three core areas: Number and counting; Number relationships; and Number operations. The number core considers the different uses of number and draws attention to the use of number symbols. The five key principles that underlie counting (Gelman and Gallistel, 1978) are also highlighted. The number relationships core addresses comparing, ordering and structuring numbers (with particular emphasis on the use of spatial and finger patterns), and partitioning and combining numbers. Finally, the number operations core focuses mainly on early addition and subtraction. With a view to empowering teachers (Yang et al., 2009), each section begins with an overview explaining the underpinning mathematical concepts and principles; a table setting out the key vocabulary and examples of learning experiences associated with these important mathematical ideas; and a sample of activities for use in the classroom. Each activity is presented according to the following subheadings: mathematical focus, key vocabulary, resources required, activity and possible interactions, taking ideas further, and assessment opportunities. A socio-cultural stance was adopted in relation to the development of the resource which also includes introductory guidance material on the provision of a number rich environment, ideas for developing number across the setting, and suggestions for promoting home-school links.

The title of the resource is 'Number Talk'. The resource was designed specifically to be a practical support for early years' teachers in developing early number concepts and the associated language. The resource may be useful to teachers in planning their teaching of early number, and, thereby, to children in aiding their understanding and use of language with regard to early number concepts both in school and in their day-to-day lives. It is important to acknowledge that this resource builds on materials already developed for teachers.

Conclusion

The aim of this research project was to develop a resource of key vocabulary and teaching and learning strategies to support teachers in their planning and teaching in early number. "Improvements in early childhood mathematics education can provide young children with the foundational educational resources that are critical for school success" (NRC, 2009: 331). Yang et al. (2009) believe that if children are to develop number sense, then teachers must first be empowered with knowledge and appreciation for number sense. The resource acknowledges the critical role that teachers play in developing young children's number sense through the environment created, the language and behaviour modelled, and the involvement of children as they communicate with them in worthwhile number activities.

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Mathematics curriculum reform in Uganda – what works in the classroom?

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The Ugandan secondary school mathematics curriculum was established in colonial times to serve a small, select minority of academic high-achievers, and it is delivered with a 'dominant pattern of expository, whole-class teaching' (Centre for Global Development through Education 2011). But following the introduction, in 1997 and 2007, of Universal Primary and Secondary Education Policies the curriculum has become increasingly irrelevant and inaccessible to the majority of learners. How can mathematics education be reformed to make it more appropriate to the modern Ugandan context, with large, mixed-ability classes and very limited resources?

In an effort to increase access and improve learners' performance in the first four years of secondary education, the Ugandan National Curriculum Development Centre is developing a new mathematics curriculum with a range of materials that encourage alternative teaching and learning strategies using low-cost, locally-available resources. These have been trialled in urban, peri-urban and rural secondary schools, where lessons have been observed and learners' work has been collected and analysed. The trials indicate that learners may be more willing to adopt a new approach when the tasks are novel and unfamiliar. When they are associated with established mathematical knowledge and techniques, pedagogical change may be more difficult to establish.

Keywords: curriculum development; practical activities; group work; universal secondary education; pedagogy; Uganda

Changing pedagogy

The established practice of chalk-and-talk, teacher-dependent pedagogy in Uganda is well documented (CGDE, 2011; Opolot-Okurut et al., 2008; World Bank, 2008). The teacher stands at the front of the class, textbook in hand, and writes notes and examples on the board. The 'silent learners' copy it all down (Clegg et al., 2007). Why?

Many Ugandan mathematics teachers would rather use a more active, learner-centred approach, but they feel they have little choice given the pressure to 'cover' the syllabus with large classes and very limited resources (CGDE, 2011; Sikoyo, 2010). Learners normally sit on 'forms' – benches attached to narrow desks – with three or four learners sharing a form. The school may have class sets of textbooks that the teacher can borrow for the lesson, but even the process of distributing the books – and getting them all back – is challenging in a tightly packed classroom with no room to move between the forms (Clausen-May and Baale in press). Furthermore, the ratio of textbooks provided to learners is one between three or four (Government of Uganda 2010; Japan International Cooperation Agency and International Development Centre Japan 2012). If three (or more often four) learners are sharing one textbook then, in reality, at least one of them will not be able to see it. By writing everything on the board for the learners to copy into their exercise books the teacher is ensuring equality of access, in the most fundamental sense, for all learners. This practice also gives each learner a copy of the material that they can use later for independent study.

However, this teaching approach creates a highly teacher-dependent ethos, with the teacher serving as the fount of all knowledge. It reinforces the academic approach, as the textbooks used are designed to be copied onto the board with the requisite 'facts' presented as concisely as possible (Namukasa et al., 2010). It also takes an inordinate amount of time as learners copy everything down (Hannon, 2009). And finally, it is increasingly irrelevant to a school population that is expanding rapidly following the successful introduction of the Universal Primary and Secondary Education Policies (Baguma and Oketcho, 2010; Clegg et al., 2007; Nakabugo et al., 2008; World Bank, 2007). Most learners may be able to copy down what the teacher writes on the blackboard and learn it off by heart but, in all but a few highly selective schools, it will make very little sense to many of them. This being the case, one of the aims of Uganda's National Curriculum Development Centre (NCDC) in developing the new Lower Secondary mathematics curriculum is to reduce the learners' heavy dependence on the teacher, with more active, practical activities and an increase in group work. However, this must be done within very tight budget constraints, often in classes of over a hundred learners with varying levels of mathematical attainment.

Independent learning through practical work

Although it is recommended in the current Lower Secondary syllabus, particularly in relation to work on three-dimensional shapes, little practical work is done in Ugandan mathematics classrooms at present. For example, according to the syllabus for Senior 2 (the ninth year of formal education), in Topic 16, *Nets and Solids*,

> The emphasis should be on practical work – construct nets from wires, sticks manila card using tacking pins, sellotape or adhesives. Properties should be discovered from the practical work. (Ministry of Education and Sports, 2008)

But even such materials as wire and manila card are in scarce supply in many schools. Learners have little experience of handling three-dimensional shapes, and they often struggle to visualise them. Teachers have difficulty drawing them free-hand on the blackboard, and learners cannot interpret the drawings effectively.

As part of the wider reform of the lower secondary mathematics curriculum NCDC is seeking alternative readily available, low-cost materials that can be used to make geometric shapes. Bananas are grown in many (although not all) parts of Uganda, and the dried fibre from the central stem of the plant is a waste product that is already used as a cheap resource for arts and crafts both in schools and for products made for the tourist market. Learners can use it to make a range of mathematical shapes, whose geometric properties they can explore for themselves.

This idea was trialled with four Senior 1, 2 and 3 classes in two private and one Government Universal Secondary Education (USE) school, in Wakiso and Mbarara Districts. Learners' ages in any year group vary greatly, but ideally Senior 1 is the eighth consecutive year for learners who started their schooling at the age of 6 years, so some Senior 1 learners could be about 14 years old. Class sizes varied from 75 in a private school to 120 in the USE school. In each trial the lesson was presented by the mathematics teacher while the NCDC researchers acted as participant observers, talking to the learners and supporting the teacher. Learners' models and written work were collected and scrutinised, and the researchers' observations were recorded.

Learners worked in groups of about 6 with printed sheets (one or two copies per group) which gave step-by-step instructions to make banana fibre 'sticks' by rolling the fibre around lengths of sisal string. These 'sticks' could then be tied together to make frames of three-dimensional shapes (see Figure 1).

Figure 1. Banana fibre and sisal tetrahedron

In all four of the trials learners tended to look first to the teacher for guidance, and they had to be encouraged to read the instructions on the sheets. Once they got started, however, very little further input was needed from either the teacher or the researchers. In the context of this very practical activity learners in all classes were able to follow the printed instructions and to work in their groups largely independently of the teacher. Each group made a tetrahedron out of banana fibre, and this was taken as a significant measure of success for this previously untrialled (in any country) activity. Having made their tetrahedron, learners were asked to observe and sketch it and record its properties – the number and shape of its faces, and the number of vertices and edges. This, too, the learners were able to do with little assistance.

The work on three-dimensional shapes was taken further with the Senior 3 classes. Each group first made a tetrahedron, and then went on to make one of a selection of different polyhedra including prisms, pyramids and an octahedron. Again, the process of making the shapes was effective: in one class of ninety-odd learners, seventeen banana fibre tetrahedra and thirteen other polyhedra were made in the course of an eighty-minute lesson.

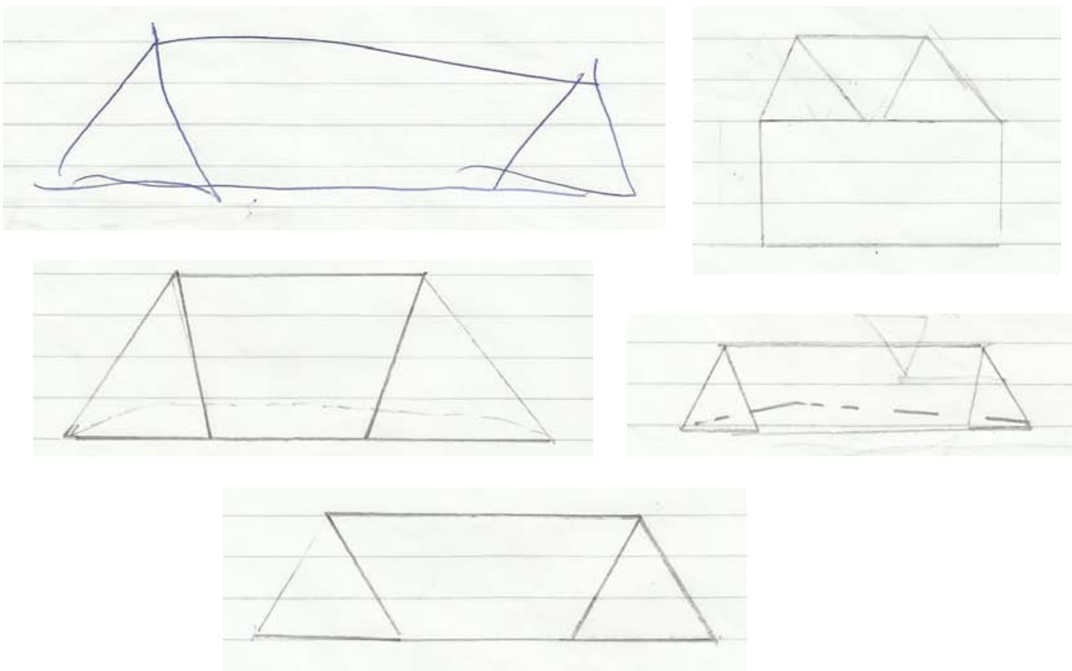
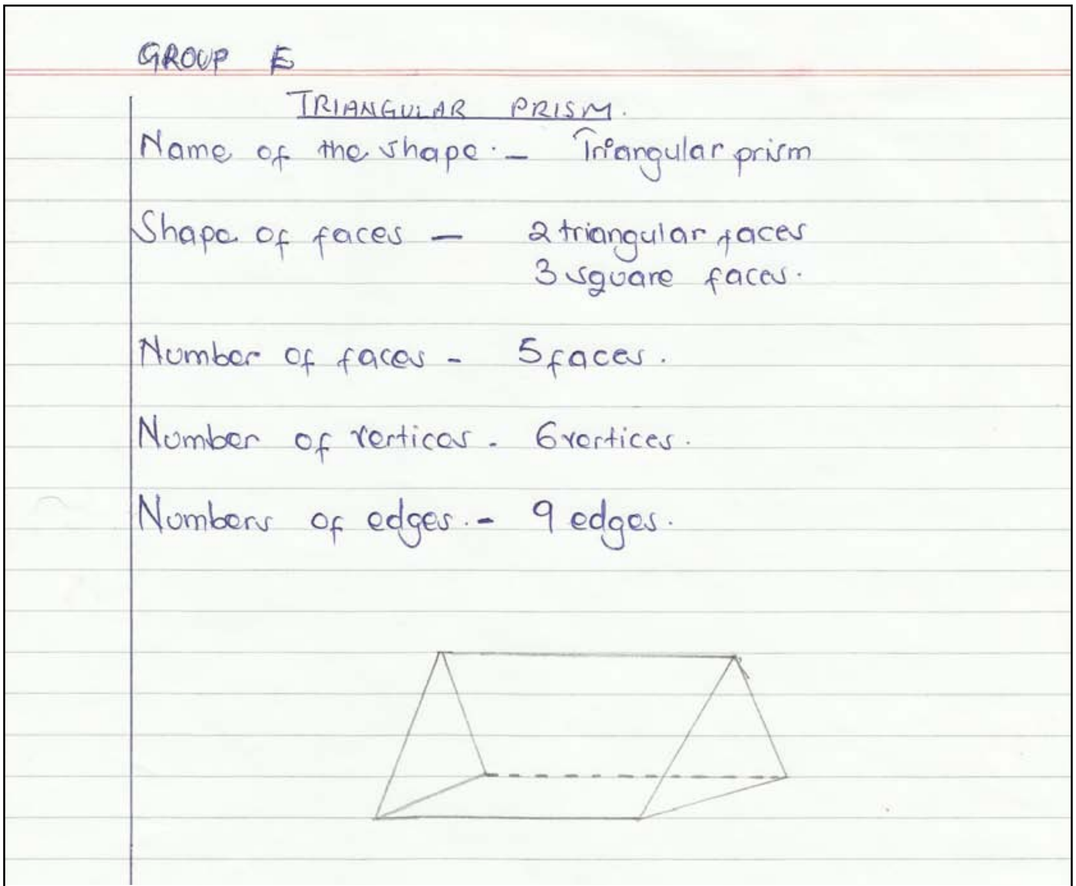


Figure 2: Learners' individual attempts to draw a tetrahedron

In the following session each group was asked to observe, discuss and record the properties of the shape it had made, and to produce a group report. This was clearly an unfamiliar concept and some groups split up, with each member working independently of the others. However, the learner's written work showed that at least some groups worked cooperatively as intended. For example, different members of a group that had made a triangular prism sketched it with varying degrees of accuracy (see Figure 2). They then used their various efforts to produce a report with a more accurate diagram of their triangular prism and a list of its properties (see Figure 3).


GROUP E

TRIANGULAR PRISM.

Name of the shape :- Triangular prism

Shape of faces — 2 triangular faces
3 square faces.

Number of faces - 5 faces.

Number of vertices - 6 vertices.

Numbers of edges :- 9 edges.

Figure 3: Group report on a triangular prism

In the final part of the lesson the teacher led the whole class in counting the numbers of faces, vertices and edges of each shape in turn, and recording these in a table on the blackboard. The teacher clearly had not previously handled three-dimensional shapes like these himself, and he sometimes struggled to count efficiently and accurately.

Once they had constructed their table the learners worked in pairs, following the instruction: *For each 3-D shape in the table, add the number of faces to the number of vertices (corners), and compare it to the number of edges. Write what you notice.* They explored the relationships between the numbers and expressed their observations in their own words.

Three pairs started by noticing that for some of the shapes – the tetrahedron, the square-based pyramid and the cube – both the sum of the faces and vertices and the number of edges were even. For example, Jovitah and Sylvia wrote:

Terahedro → 8, 6

Squ based pyramid → 10, 8   They are all even numbers

The girls then crossed out their observation when they realised that their hypothesis, that the number of edges and the sum of the faces and vertices would always be even, was refuted by the figures for the triangular prism (11, 9). However, their work was encouraging as it indicates that these learners had the confidence to put forward their own hypotheses, and to test them with further data.

Nearly all the learners discovered Euler's Theorem ($V - E + F = 2$) with no further guidance from the teacher or the researchers. They expressed their findings in different ways. Some focused on the *difference* between the number of edges and the sum of the numbers of faces and vertices. Kenny and Paskazia, for example, wrote:

> When you add the number of faces with the number of vertices you get 2 more than the number of edges.

Margaret and Suzan expressed this as the *addition* of 2 to the number of edges:

> There is the addition of 2 to the number of edges to get the sum of the vertices and faces.

Others, such as Sonko and Jose, focused on the *subtraction* of 2 from the sum of the numbers of faces and vertices:

> The sum of the number of vertices and faces minus two gives the number of edges

while Yvonne and Constacia saw it as the subtraction of the number of edges from the sum of the numbers of faces and vertices:

> When you get the sum of number of faces and vertices subtract it with number of edges you get 2.

The great majority of responses were of one of these forms, using words to express the relationship. However, a few of the learners were more succinct:

> The number of faces + Number of vertices = Number of edges + 2
>
> Number of faces + Number of vertices – 2 = Number of edges

These responses indicate that, with a bit of encouragement, these learners might be ready to move on to a formula expressed in symbols:

$$F + V = E + 2, \text{ or } F + V - 2 = E$$

Teacher dependency

The ability of the learners to work independently of the teacher when they were making and exploring three dimensional shapes was very encouraging. However, an attempt to adopt a similar approach to another, more familiar, area of mathematics was less successful. A lesson on subtraction was developed after an observation in a previous trial that many learners had only a limited range of highly formalised algorithms for straightforward calculations. So, for example, to subtract 272 from 400 nearly all learners used a standard 'borrowing' structure – whether accurately or not (see Figure 4). An attempt was made to increase the learners' flexibility by introducing an alternative strategy based on a model of *Subtracting by adding on* (Department for Education and Employment 1999). This approach to subtraction involves seeing the difference between two numbers as a 'distance' on a number line. For example, the subtraction *400 – 272* may be seen as the distance of

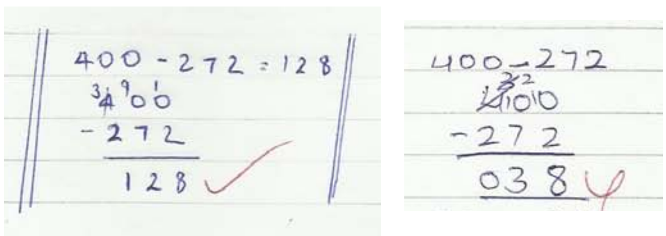


Figure 4. Algorithmic subtraction

the ‘jump’ from 272 to 280, plus the distance of the ‘jumps’ from 280 to 300 and from 300 to 400 (see Figure 5).

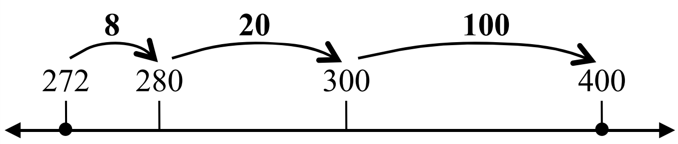


Figure 5. Subtraction by adding on

A worksheet designed to introduce this concept was trialled in one of the schools which trialled the lesson on three-dimensional shapes. It was distributed to learners in a parallel Senior 1 class, with one copy between two to ensure access. The learners were encouraged to read and work through the explanation, examples and exercises without any further input from the teacher.

However, it quickly became evident that the learners found this a very challenging task. Most of them copied the diagrams in the examples and questions provided, but many did not fill in the gaps for the missing numbers in the exercises. Some then went on to do the calculations in the conventional way (sometimes scribbling on the cover of their exercise books or their hands rather than on their answer sheets), while others left these blank (see Figure 6).

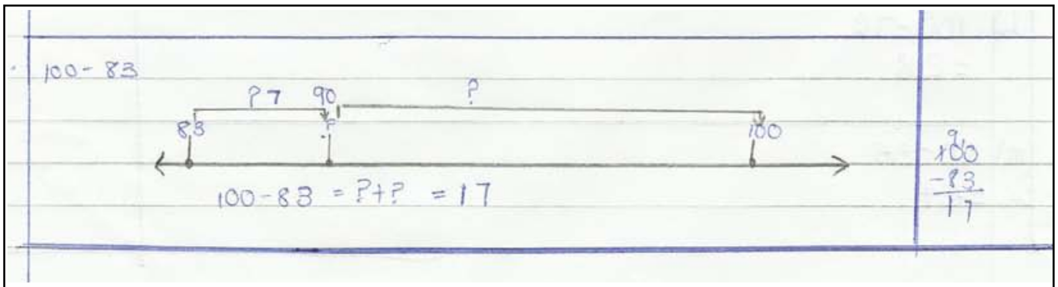


Figure 6. Continuing to use the familiar method to find 100 – 83

Some 25 minutes into the lesson the researchers agreed with the teacher that the learners needed more guidance. The teacher therefore instructed the learners to stop work, copied the two examples from the sheet onto the board, and talked them through. After this the learners got back to the task with rather more enthusiasm and some success. However, there was no evidence in their subsequent work that they had retained or could apply the new method. Further trials of the same activity in a high-achieving private school and a USE school showed similar problems when learners were asked to work independently to explore this new idea relating to a topic in which they were already very well drilled.

So while the trials indicated that the learners were able and, indeed, eager, to work independently on the unfamiliar practical task of making and exploring three-dimensional shapes, where the mathematics seemed familiar old habits were more likely to reassert themselves as learners looked to the teacher as the sole source of all knowledge. This observation reflects similar findings elsewhere. For example, university students in the UK sometimes ‘feel it is more convenient to engage in familiar mathematical tasks than [to] engage with new practices’ (Radu, 2012), so any attempt to introduce a new approach may meet with ‘limited appreciation from the students’ (Alcock and Simpson, 2002). On the other hand, Ainley and Goldstein (1988) found that introducing Logo, a new area of mathematics, encouraged learners to ‘take control over the learning situation’. However, teachers who worked with Logo felt the same pressure to cover the syllabus as they do now in Uganda (Ansell, 1987). The contexts are very different, but the issues may be similar.

Conclusions

The lesson trials described briefly here were selected because they bring out a number of key factors that need to be borne in mind as the new Lower Secondary mathematics curriculum is developed and introduced.

There is a strong habit of dependence on the teacher in Ugandan Lower Secondary mathematics classrooms, fostered by a heavy diet of didactic, teacher-focused classroom practice and a lack of independent learner access to textbooks and resources. However, the lesson trials indicate that the practical activity of making and observing mathematical models can offer an effective introduction to independent group work. The process of writing a group report is unfamiliar, but there is some evidence of group members sharing their efforts as they fine-tuned their report. But this change in the learners' behaviour may be easier to establish in the context of activities that the learners regard as new. When they were asked to explore a familiar topic, subtraction, in a new way, the learners were much more hesitant.

If group work and independent learning are to be encouraged in the mathematics classroom under the new curriculum then focusing on practical activities in contexts that are unfamiliar to both learners and teachers may offer a useful strategy. However, this will take up more time in the classroom. It takes much longer to make a tetrahedron and to observe, discuss and record its properties than to copy a list of these from the blackboard and learn them off by heart. The coverage of the reformed curriculum will have to be reduced to allow time for learners to understand, not just to memorise, mathematics.

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University Schools: A Collaborative Approach to ITT in Secondary Mathematics

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In this paper we discuss the difficulties and tensions currently facing our initial teacher training (ITT) programme in secondary mathematics at Manchester Metropolitan University, and outline the pilot we have trialled to try to address some of these issues. We have called the schools we have worked with for this pilot 'University schools'. This model of teacher training is currently in its third year of development in 10 schools; it has been evaluated throughout this period via interviews with student teachers, weekly feedback from student teachers in the form of reflections, lesson observations, discussion with MMU staff involved in the programme, and discussions with teachers in the pilot schools. The paper outlines our findings and then draws conclusions about the success of this training model.

Keywords: collaboration, teacher training, practice, theory, pedagogy

Background: the problems of initial teacher training (ITT)

The difficulties we have identified at Manchester Metropolitan University which present barriers to high quality teacher training can be conveniently categorised into three broad areas: placement, integration and quality. Although by no means limited to the training of teachers of mathematics, the problems are especially pressing on such courses of study. We summarise the issues briefly here, prior to explaining our model in the next section.

Placement

Traditional routes into teaching via a college- or university-based course require that student teachers be 'placed' in appropriate school(s) for practical experience. In subjects such as mathematics, as well as other subjects that have shortages, it is often difficult to find placements, not least because of a corresponding shortage of existing teachers to support the training of new ones. A further consideration affecting mathematics is the heavy emphasis government places on examination league tables and success in English and mathematics, creating potential reluctance on the part of schools to allow inexperienced student teachers access to these crucial subjects.

The lack of placements for mathematics student teachers has a further negative aspect in that university tutors may have to accept placements that may not be entirely suitable. Even if it was possible to find sufficient places, the great variation in the effectiveness of serving teachers means that not all those who agree to mentor student teachers are necessarily well equipped for the task. Universities provide training for the teachers who volunteer to mentor student teachers, but uptake is reported to be variable, and universities cannot adequately monitor mentors, nor compel their attendance at training. Consequently some student teachers may receive a less than ideal training experience with respect to support.

Integration

The integration of subject knowledge with sound principles of pedagogical practice to produce what is commonly known as pedagogical subject knowledge (Shulman, 1998) is a key concern of teacher training. The student teacher is keen to pass on the principles of mathematical thought and reasoning but must be alert to the need to provide support for learners to construct mathematical knowledge for themselves, rather than 'absorb' it in the mode implied by the traditional transmission style which they might have experienced in their own education. Additionally, they must meet the challenge of engaging often reluctant and occasionally recalcitrant learners. In such circumstances the student teacher must find a way to adapt pedagogy to serve their subject. Wideen et al. (1998) note that the implicit theory underlying traditional teacher education is that the university provides the theory, methods and skills, while schools provide the setting in which that knowledge is practised; the student teacher then provides the effort to apply the knowledge. However, as Barone et al. (1996) report, the theory presented in teaching programmes often has little connection to the practice that is seen in schools, echoing Zeichner and Tabachnick's (1981) comment that many notions and concepts developed during pre-service training were 'washed out during field experience'. Comparable findings were also reported by Cole and Knowles (1993).

Once on placement and away from the advice and sounding board provided by university tutors and fellow student teachers, the novice teacher has to negotiate a tortuous path that may dent the enthusiasm and idealism which drew them into teaching in the first place. Too often, pedagogy is sacrificed to ensure compliance. Inventive and imaginative practice enthused over at university becomes reductionist process-driven teaching. In some cases school-based mentors and other staff in school encourage student teachers to adopt an unduly cynical approach to teaching (Nolan, 2012). Brouwer (1989) and more recently Allen (2009) report on the dominating influence of school placement on pre-service teachers; for Brouwer (1989), the most important factor promoting transfer from teacher education into practice was the degree to which there was alternation and integration of the theory and the practice.

Anecdotally, university tutors on their periodic but infrequent visits to student teachers on placement are frequently surprised and dismayed to see student teachers previously noted for their innovation and creativity relapsing into didactic teaching. One of the problems faced by university tutors concerned with the gap between theory and practice is that "teachers often feel threatened by theory" (Elliot, 1991: 45): many feel that they are unable to use the theory presented to them by experts and are therefore 'falling short of living up to expectations'. Student teachers in turn see that they are not the only ones struggling to embed these ideals into their classroom and the whole idea of applying theory is discarded. "The only way out of the feeling of always falling short is to adapt to the common habit of teachers to consider teacher education too theoretical and useless" (Korthagen, 2001: 5). Elliot (1991: 47) similarly concludes that "The perceived gap between theory and practice originates not so much from demonstrable mismatches between ideal and practice but from the experience of being held accountable for them".

Quality

Initial teacher training providers who find themselves with a cohort of high calibre students, sufficient placements and committed school-based colleagues to support the development of effective pedagogy will still face the hurdle of monitoring the outcome of their efforts. Ensuring a common understanding of 'quality' in ITT presents difficulties despite the existence, in England, of a set of Teachers' Standards (DfE, 2013). Adding in the criteria used by Ofsted to judge lessons as either 'outstanding', 'good', 'requires improvement' or 'inadequate' adds further confusion. Informally, what constitutes an 'outstanding' lesson may vary depending on the school context, so that an 'outstanding' lesson in one school may be judged as 'requiring improvement' in another: mentors supporting student teachers in 'challenging' schools may deem simply surviving a Year 9 lesson an outstanding achievement. Consequently university tutors' infrequent visits are inadequate to monitor and support a shared sense of quality across multiple schools in a university partnership. The consequence of this can be novice teachers released into teaching wrongly graded and lacking understanding of their true capacity and potential for development.

Moving towards a new model of ITT

Finland's teacher training focuses heavily on the integration of theory and practice in schools, and has strongly influenced the development of our model of teacher training. Teacher training in Finland is completed in specialist teacher training schools, where theory and practice are considered to be conceptually inseparable. Integration of theory with practice is seen as fundamental to promoting teacher autonomy and professionalism (Heikennen et al., 2011). Furthermore, the relationship between the university teacher and the training school teacher is considered crucial to teacher development, and both class teacher and university tutor are equally responsible for supervision, but from different, overlapping, perspectives. The university tutor observes lessons to ensure that the student teacher is building their practice upon theory and can identify the theory that arises out of their practice in the classroom. The classroom teacher focuses on subject knowledge. Similar practices have also been developed in the United States, in 'Professional Development (or practice) Schools' (Bullough and Kauchak, 1997). This collaboration between the school and the university is an area we wanted to develop.

The second aspect that we wanted to develop in our teacher training course was the use of collaborative teaching to develop innovative lessons as well as encouraging our student teachers to become more reflective practitioners. Northfield and Gunstone (1997: 49) state that "Learning about teaching is a collaborative activity" and that it is most productive when conducted in small groups where ideas and experiences can be shared and discussed.

The University School Model at MMU

The secondary mathematics department at MMU has placed mathematics student teachers into selected schools in multiples of three; a university tutor works in the mathematics department of the school each week that the students are on placement there. There are as many as 15 mathematics student teachers in each school, down to a minimum of six, and the tutor works between one and two full days a week in the school. Schools are paired and work together to create a cohesive training model for the students they share.

The students in the university school model are enrolled on a PGCE course and have the same entitlement as those on the traditional PGCE route. The difference in the university school model is the way they work in their placement schools. The role of the university tutor is similar to that identified in the Finnish system, that is, as support for the class teacher's role by observing student teachers' lessons, giving feedback, and helping to plan lessons, with a focus on developing pedagogy throughout this process. There is an opportunity for lessons to be modelled and delivered by the university tutor during the block placements in school. This is extremely powerful in linking theory to practice and showing how it is relevant for the classes being taught by the student teachers.

Block A

During the first teaching placement the student teachers teach in threes (or triads). Each triad has approximately three classes that they are responsible for, and the remainder of their timetable is used for intervention that the school feels is beneficial (usually year 11 exam preparation classes to support revision). They are encouraged to ensure that all three take part in both planning and delivery of shared lessons. They are equally responsible for the learning that takes place in these lessons regardless of who is delivering, since the exchange of ideas and the discussion around why certain tasks have been chosen is crucial to their development. After each lesson they teach, they give and receive feedback to each other. They are encouraged to be precise in this feedback. The model has developed this year to trialling the use of two shared classes and one individual class.

Block B

During the second teaching placement there is a greater emphasis placed on individual teaching. Student teachers continue to teach one class on their timetable as a shared class, and the other two classes are taught individually. This usually also includes some year 11 intervention. There continues to be at least six student teachers in each school with an emphasis on sharing ideas, collaborating and supporting each other. They are encouraged to observe each other teach and give feedback even in their individual classes. They continue to receive feedback from each other in each lesson taught as a triad.

Methodology

The aim of this research is to assess the impact that the university school model of training has on the placement schools in terms of staff development and pupil progress; the quality of lessons the trainees are delivering by working collaboratively; the impact on pedagogy through bridging the gap between theory and practice; and to ensure a consistency in the training experience. However, this paper reports on only part of the action research cycle as discussed by Cohen et al. (2011) as there is further research to be developed. The university schools model has been piloted in 10 schools over three years involving a total of 148 student teachers. The schools were chosen based on previous experience working with Manchester Metropolitan University and established expert practice. The location of the schools has determined which trainees are involved in the model based on their travel time. Over the three year period, we have collected a range of data, including: lesson observations by university tutors; students' weekly reflections on their teaching practice; further reflections at key points of the year on their involvement in the university schools model; year-end group interviews with both university school and traditional route students focusing on their experience in the PGCE and how they have developed in preparation for their induction year as a newly qualified teacher (NQT). Additionally, school mentors have given feedback during the pilot in the form of an open discussion where they were asked to comment on this model of teacher training. The data reported in this paper is primarily concerned with participants' perceptions at particular stages along with impressions from the university tutors and school based mentors.

Findings and discussion

Placement

Being able to place multiple student teachers into our selected university schools affords us the opportunity to work with colleagues who see training teachers as a fundamental role of their school, and who recognise the continuing professional development (CPD) it provides. However, the issue of having numerous student teachers working in a mathematics department and the effect this has on results for government league tables is a crucial consideration. Each university school has used the numerous student teachers in their mathematics departments for some form of intervention, in most cases year 11 GCSE revision. This has varied in its nature from in-class support in small key groups to working with targeted pupils outside the classroom. This has been perceived as extremely beneficial in raising school standards. An e-mail from one university school reported:

> Involvement in this model has directly contributed to raising standards in maths. Since becoming involved in the multiple placement model in 2010 we have seen a distinct rise in maths results. Prior to involvement in the multiple placement model results were around 62-64% A*-C for maths. In the first year of the model (2010-11) it rose to 66% A*-C. In the second year of the model (2011-12) the results were 76% A*-C. (School A)

Due to the collaborative nature of their first teaching placement, student teachers have had experience of teaching classes that in the traditional model would have been considered unsuitable for them to teach. These have been described as the lower attaining classes where pupils exhibit more challenging behaviour and many pupils are on the special educational needs register. Teaching in threes means that pupil understanding can be aided by the availability in the room of additional mathematics specialists. These classes have been frequently used for both Block A and Block B placements as schools recognise the benefit of a high teacher-to-pupil ratio:

> In addition to the benefits for the trainees, the model has had particular benefits for our pupils … Many of the lessons benefited from having three additional maths specialists in the room. (School A)

A student in a different school also commented on this issue in their reflection:

> I was very anxious about taking over year 9 as I had no idea how I was going to work with and contend with the behaviour that I had seen them portray to their class teachers…all of them seem to be working and I think it's due to the large amounts of adults that are in the room…all pupils get the opportunity to have their own one to one time throughout a lesson and this works for them well. (01-02-13)

Integration

One of the main benefits of the university tutor working alongside the teachers in the university schools is the ability to focus on pedagogy and theory. They are not constrained by the same pressures experienced by class teachers and subject mentors and so can spend more time developing these aspects of the student teachers' teaching. This is very relevant in schools where students are guided to teach procedurally with little emphasis on underlying concepts. A student who had made excellent progress on Block A commented in a Block B reflection that she was not enjoying teaching because of this issue:

> The trouble seems to have come with the class teacher in that he has a specific way of teaching the class, and that's the way that he wants me to teach. His way seems to be very textbook led, whereby an example is gone through on the board with the correct method and then the pupils answer questions similar to this, and because of this reason he seems to be stripping my lessons back to the minimum. This isn't the way that I would like to teach and so have found it a challenge to plan these kinds of lessons. After planning these lessons because of the lack of enthusiasm I have about them this is making me less confident in the classroom, which in turn is affecting my teaching style. (08-02-13)

A related issue is that beliefs about effective teaching styles in mathematics go beyond individual class teachers and are noticeable in pupil perceptions as well (Swan et al., 2000). Pupils play an important part in the learning process and the importance of this in the planning stage cannot be dismissed. Whilst reflecting on this lesson the student teacher said:

> … then when we got into the class I was faced with resistance – not from the teacher, from the pupils. "You only learn when you copy things down and answer loads of questions, there's no point in doing stupid card sorts – it's not beneficial". (15-02-13)

The university tutor has a unique opportunity to work alongside teachers and pupils in the school as well as students. By the end of the placement the teacher mentioned above was actively encouraging the student to do card sorts and mathematics trails around the room.

A unique benefit of the university schools model is the opportunity for student teachers to witness the university tutor delivering a lesson to a class that the students are teaching which is based on the theory discussed at university. This is extremely powerful in bridging the gap between theory and practice and showing that theory is relevant for the school they are placed in. After planning a lesson with their university tutor, one trio delivered the lesson but failed to recognise the importance of the use of the context and decided to not include this in the lesson and to revert to procedural methods. The lesson was not successful and the pupils had not made any progress when assessed at the end of the lesson. The lesson was then delivered by the university tutor, with emphasis placed on using the context to secure understanding. The students reflected on this experience in their weekly reflection:

> We've learnt a lot in university about conceptualising maths in the classroom and how that is a really good foundation for understanding….that is one lesson we had planned to deliver in terms of concept but we all during the lesson fell back to procedure. For me this was highlighted when [the university tutor] started with a concept and at no point during the lesson did they revert to procedure, they kept the theme throughout and used questions to build up their understanding. It was good to see all that theory in practice, from conceptualising, to planning for questioning, to dealing with misconceptions. (24-10-13)

> …the lesson was a success and it demonstrated to me that a contextual lesson, focussed on developing pupils' understanding, can show progress and it can be done in a similar amount of time as traditional lessons, based on methods. (22-10-13)

The collaboration between the student teachers in these lessons encourages a wealth of ideas to be shared before delivery. This often results in the students taking more risks in their lessons. In the first week of teaching in one tutor group the university lecturer observed lessons involving play-doh to conceptualise fractions, ‘people maths’ to explain bearings, and one group taking a class outside onto the field. The second lesson that one trio taught in their first week on teaching practice involved a mixture of ‘people maths’ to explain concepts, mathematics trails around the room and Tarsia puzzles to consolidate understanding. This exciting and innovative approach to learning has been witnessed in numerous lessons in the university school model and one subject mentor compares this to her experience with student teachers on a traditional route of training:

> The lessons I have observed have certainly been of a higher quality than the majority of lessons on the previous PGCE model. The trainees are taking risks, being innovative and working on developing strengths that would usually not be considered until Block B. The lessons are focusing on understanding rather than procedural algorithms, and their ability to discuss pedagogy and critically reflect on their own (and each other’s) lessons is excellent. (School B Mentor)

Quality

After each lesson has been delivered, the student triad meet and give each other feedback on the lesson. They complete a ‘feedback sandwich form’ for each lesson, which focuses them to discuss positive aspects and targets arising from the lesson. This not only aids the development of the student receiving the feedback, but also the critical analysis techniques of those giving the feedback. They are encouraged to be precise in their feedback and give clear targets to develop. This ensures that students are receiving valuable feedback every time they teach and helps them to develop effectively. Furthermore, the presence of the university tutor working in the school throughout the placement means that the grading of students’ practice becomes more consistent. The students value this as they are being observed throughout the placement not just on pre-planned visits. As one student observed in an interview:

> Because you’re [the tutor] in school every single week it’s not as big a deal. Because you have to make sure all your lessons are good. Whereas people that are in a school on their own, if they’ve got an observation, ‘Oh my God I’ve got to do this really outstanding amazing lesson’ but it might not be consistently good which is something you have to try to make sure you keep on top of. (04-2013)

This regular contact with the students, and the discussions with the mentors, ensures a secure understanding from all parties as to the progress the students are making. It also helps to ensure that targets for the induction year are more consistent throughout the university schools.

Conclusions

We have discussed only a few of our findings in relation to concerns about the integration of theory and practice in the training of pre-service teachers, the placement of these student teachers in schools that would support their training, the quality of the training they receive and the quality of the lessons they deliver. The research suggests that delivering lessons in collaborating groups of three and the university tutor working within and alongside the school has definite benefits in these areas.

Are there any emerging issues with the university school model? One concern is the quantity of solo teaching versus the quality of their teaching. This is the subject of on-going research. The development of university schools is still in its early stages and we are currently collating more substantial evidence and comparing this to the traditional PGCE route, in order to evaluate the impact of both routes on the development of the teachers in their NQT and RQT years.

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The case of the square root: Ambiguous treatment and pedagogical implications for prospective mathematics teachers

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I report on a small-scale study rooted in the UK context that was conducted with eight volunteers from a cohort of PGCE secondary mathematics students (participants). The participants' own understanding of the square root concept and use of the associated symbol were explored and the findings revealed that they may not possess adequate subject knowledge about and for teaching this concept. Access to instructional materials, mainly textbooks and discussions with other more experienced teachers were identified as the main external sources consulted by the participants in order to refresh their knowledge of the square root concept. During this study, those participants who became aware of the shortcomings of their conceptual understanding of the square root felt, at first, uncomfortable with modifying their personal knowledge and their long held beliefs about this concept. Group discussion helped most of the participants become aware of connections between their more advanced knowledge of mathematics and the square root concept. Such awareness empowered the participants to clarify this concept for themselves and critically scrutinise the (re)sources available. A tension between employing their modified knowledge about the square root and adherence to the widely accepted view about this topic in school mathematics has also been identified.

Keywords: square root, radical symbol, Advanced Mathematics Knowledge

Background to the study

This paper originates in an informal conversation between myself and another mathematics teacher educator colleague who debated whether teachers should ask pupils the question What is the square root of 16? or What are the square roots of 16?

This conversation motivated me to carry out a review of the use of the radical symbol $\sqrt{}$ amongst students, undergraduates, mathematics teachers and most of the authors of school textbooks. The review reported in Crisan (2012) identified a widespread misuse of the radical symbol, as well as a lack of consistency in treating the square root concept in a large number of textbooks and other instructional materials such as GCSE and A-level examination papers together with their mark schemes, workbooks, instructional software and/or web-based content. Some materials introduced a new symbol notation $\sqrt[\pm]{}$, according to which the notation $\sqrt[\pm]{16}$ stands for the positive and negative square root of 16, without modelling explicitly the use of this new notation. Other materials introduced the symbol $\sqrt{}$, exemplifying its use such as $\sqrt{16} = 4$ to be read as 'the square root of 16, which equals 4 *or* - 4' or as ‘the square root of 16, which equals 4 *and* - 4’, while a handful of textbooks suggested that $\sqrt{16}$ equals 4 only.

This review highlighted the fact that there is no consistency in the way the square root concept is presented in school mathematics textbooks, not to mention the misuse of the radical symbol itself. This is worrying, as it misleads users of these textbooks such as pupils and parents/carers, but also teachers. It is known that instructional materials remain the major source used by teachers in presenting topics to their students when selecting methods of teaching (TIMSS, 1995). This raises the issue of the quality of teachers’ subject knowledge that would empower them to scrutinise the authority of the (re)sources they consult and it is this issue which I will explore in this study.

Rationale of the study

Ball and Phelps (2008) argue that teachers need to be able to make judgments about the mathematical quality of instructional materials and modify them as necessary. But what knowledge is needed to make such judgments?

It is widely recognised that knowledge of the subject matter is an essential component of teachers’ professional knowledge base for teaching. It includes knowledge of the subject itself, extent, depth, structure, concepts, procedures and strategies (e.g. Shulman, 1986; Grossman et al., 1989). While subject matter knowledge is necessary in teaching, there is no consensus as to what depth or breadth of knowledge is essential. There is disagreement about the extra knowledge of mathematics needed; some argue that teachers also need some additional number of years of further study of the subject at undergraduate level, while others argue that teachers need to know the curriculum but ‘deeper’, and so the mathematics education literature has seen a widening of the definition of subject matter knowledge over the years. More recently, Zazkis and Leikin (2009) put forward Advanced Mathematical Knowledge (AMK) as “systematic formal mathematical knowledge beyond secondary mathematics curriculum, likely acquired during undergraduate studies” (p. 2368). The authors looked at teachers’ ideas of how AMK is implemented into their teaching practice. Their study called for further research to determine whether teachers’ ability to identify explicit connections between AMK and mathematics taught in school is a rare gift of only a few teachers or whether specific prompting is needed to bring this ability to the surface. Related to Zazskis and Leikin’s call, the focus of the study I report in this paper is to explore how an awareness of the connection between more advanced knowledge of mathematics and the square root leads to a modification of the participants’ personal knowledge of this secondary school mathematics topic. Implications for their pedagogical practices are also considered.

The study

In this study the eight secondary mathematics PGCE volunteers were engaged in a number of mathematics and pedagogically specific tasks with the aim of gaining access to their conceptions (knowledge, views and beliefs) of the square root concept.

The main goals of this research were:

1. *to gain access to prospective teachers’ conceptions of the square root*

by asking participants to go through a mathematics task consisting of a number of questions related to the concept of square root. It was envisaged that participants might hold competing conceptions about the square root and thus group discussion was carried out in order:

2. *to identify some of the prospective teachers' sources of conceptions about the square root* and
3. *to find out what triggers changes (if any) in their conceptions of the square root.*

Methodology

The participants

The participants were selected from a cohort of PGCE secondary mathematics students that I was teaching. The purpose of the research study was explained to the whole cohort, but the specific mathematical topic was not mentioned to the students at this stage. It was made clear to them that the study was not part of the course requirements and that it was not going to be linked with any student assessment processes. An information sheet explaining the aims of the study, what the participants were expected to do and the methods of data collection was made available to the whole cohort and confidentiality issues were discussed. The students were then invited to think about participation with this small study and interested parties were asked to email me to volunteer and to return the consent form.

The Mathematics Task

The participants were asked to take home a mathematics task, complete it and return it to me the following week, on a particular day. An excerpt of which is included in the data analysis section (Figure 1).

The task consisted of a number of questions related to the concept of square root. The aim of this task was to encourage the participants to refresh their subject knowledge and revisit some of the topics where the concept of square root comes into play. Some of the questions were designed to elicit the participants' understanding of the subtleties of the concept. The questions were designed so that they would bring to the surface the implications of the widespread misuse of the radical symbol and of the inconsistent way in which the square root concept is treated in school mathematics. At this stage, the task was situated in the mathematical space (Stylianides and Stylianides, 2010) with no pedagogical constraints, at least not explicitly at this stage.

The decision to set the task as a piece of homework and not as a test was deliberate. I wanted the participants to solve the mathematics questions using their pre-existing knowledge, at the same time, if they needed or wished to do so, being able to consult other sources such as a textbook or any other instructional materials to remind themselves of the concept (definitions, facts, examples, related mathematical topics, etc.) or, even consult their colleagues or more experienced teachers.

The group discussion

The participants were invited to work in groups of four (Group I - pseudonyms: Jan, Jemma, Jack and Joan; Group II - pseudonyms: Billy, Barry, Ben and Bea) to talk to each other about how they solved/answered the questions set in the mathematics task. The group discussions were video-taped, while at the same time I took written notes of some of their explanations and comments. I probed further any issues that arose during the group discussions.

As the participants discussed their solutions and answers to the mathematics task questions, the need to clarify/defend/justify a definition of the square root of a positive real number surfaced. During the discussion, implications for teaching about square roots arose naturally, either through the participants' reflection on how they were taught the topic or on how they would teach the topic themselves. Indeed, Stylianides and Stylianides (2010) argue that when working with prospective teachers the answers to the questions posed in the mathematics tasks cannot be sought in a purely mathematical space, but rather in a space that intertwines content and pedagogy. Immersion of the participants' mathematical work in the pedagogical space was taken further through a further task using fictional pupils' scenarios.

The fictional pupils' scenarios

The participants were asked to give written feedback to three fictional pupils' responses (Emma-KS3, Peter-KS4 and Lucy-KS5) characterised by a subtle mathematical error to a question involving the square root, throwing further light on the choices the participants made about treating this concept. This is a well known approach proposed by researchers such as Biza, Nardi and Zachariades (2007) to be employed in a teacher education context "as tools for the identification and exploration of mathematically, didactically and pedagogically specific issues regarding teacher knowledge" (p. 308).

The intention with this task was to encourage the participants to reflect further on their own conceptions of the concept of square roots in the light of having done the mathematics task and having discussed the mathematics tasks questions as a group. Excerpts of the fictional pupils' scenarios are incorporated in the data analysis section (Figures 2 and 3).

Data analysis

I present the participants' approaches to solving some of the mathematics task questions, supporting their written and oral explanations with data collected during the group discussions and some of their written feedback to the fictional pupils' scenarios.

The participants' conceptions of the square root

Overall, the participants provided a variety of answers to the mathematics task questions.

Q1. Find the square roots of the following numbers. Write the answers in the boxes provided and where necessary, show your workings out.

36, 225, 0.64

Q2. Answer the following questions:

$i)\sqrt{4}=?\ \ ii)\sqrt{89-25}=?\ \ iii)\sqrt{9^2}=?\ \ iv)\sqrt{a^2}=?\ \ v)\sqrt{25y^2}=?\ vi)\sqrt{(8c+1)^2}=?$

Q3. Solve the following equations to find x. Explain how you work it out.

$i)\,x^2=16,\ ii)\,x^2=a^2$

Figure 1. (Part of) The Mathematics Task

All the participants were happy with their answers to Q1 ($\pm 6, \pm 5, \pm 0.8$) and Q2 i) ± 2 and ii) ± 8. "I'll always put $\pm$ because I am so used to it, but I did get in a muddle with some of questions in the HW," explained Jack and the others agreed with him. They all encountered some difficulties with the rest of Q2. The following answers and alternative explanations to Q2 iii) $\sqrt{9^2}=?$ were provided by both groups:

1) $\sqrt{9^2} = \sqrt{81} = \pm 9$, 2) $\sqrt{9^2} = (9^2)^{\frac{1}{2}} = (81)^{\frac{1}{2}} = \sqrt{81} = \pm 9$, 3) $\sqrt{9^2} = (9^2)^{\frac{1}{2}} = 9^{2\times\frac{1}{2}} = 9^1 = 9$ and 4) $\sqrt{9^2} = 9$ (as "the square and square root cancel each other"). All four explanations were regarded as being valid and the participants did not seem to be able to find a 'fault' in their reasoning, which seems to contradict the obvious equality $\sqrt{9^2} = \sqrt{81}$. "This is how we were taught since very little" (i.e., $\sqrt{81} = \pm 9$), said Jan and so when encountering disagreements or ambiguities in their solutions, the participants worked on the premise that their knowledge is correct, hence looking elsewhere for resolving the issue.

The discussion moved on the Q3 *i*) solving $x^2 = 16$. All the participants were in agreement that the solutions were $x = \pm 4$. The solutions were reached either by solving the equation by factorisation or by using the graphical approach or by 'taking the square root' of both sides. Jan explained that taking the square roots of both sides of the equation gives $\sqrt{x^2} = \sqrt{16}$, hence $x = \pm 4$ since $\sqrt{16}$ equals ± 4 and the rest of her group seemed happy with this explanation. The same approach to finding the solution of the same equation was put forward by Billy in Group II. However, he changed his mind very soon after offering his explanation,

> Actually, strictly speaking that is not right, is it? Looking at it now, I would amend it to say that $x = \pm\sqrt{16}$ since $\sqrt{x^2} = \pm x$ and $\sqrt{16}$ equals 4.

Nobody responded to Billy's comment, who himself did not look convinced at what he had just said.

The next question Q3 *ii)* in the mathematics task asked for the solution of $x^2 = a^2$. The use of the parameter a instead of a specific real number on the right hand side of the equals sign made things slightly more problematic. For example, the participants in Group I were not sure on which side of the equals sign to include the ± sign after taking the square root of both sides. While all the participants seemed content with $\sqrt{a^2} = \pm a$, some doubts were raised by Jack (Group I) and Billy (Group II) about whether $\sqrt{x^2}$ should also equal $\pm x$. In the end, however one wanted 'to look at' the square root, the matter was quickly settled when the participants realised that either $\pm x = \pm a$ or $x = \pm a$ yields the same solutions to the equation.

The participants in Group II had a similar debate when comparing each other's answers to Q2*v)* asking them to simplify $\sqrt{25y^2}$. The participants soon realised that they did not need to resolve their disagreement about whether $\sqrt{y^2} = y$ or $\sqrt{y^2} = \pm y$, as multiplication by $\sqrt{25} = \pm 5$ gave the same answer, namely $\pm 5y$.

Almost one hour into their group discussion, the participants started to become less concerned with agreement over the answers and more interested with the square root concept itself and settling the ± issue.

External sources of conviction

Most of the participants' sources of conviction, which they used in order to justify their answers, were external in nature. The participants relied on what they remembered from school or what they learned from the instructional materials they brought along to the group discussion. The participants became aware of the inconsistencies of how the square root was presented in textbooks. "They [authors of textbooks] don't care. I'm disappointed about this lack of agreement," said Billy, while Ben said, "I'd like to go to the National Curriculum exam board because I would feel secure if I knew what people will go for, who makes the decisions? Who wears the hats?" On a frustrated note, Jack summarised his Group II's desire, "We want an answer!"

Internal sources of conviction

The lack of agreement between the resources browsed led to an interesting turn in the discussion. Group I set out to bring clarity to the concept. For example, Jan talked about the possibility of the more mathematically rigorous origin of the square root as the output of the inverse of the square function. Drawing on her first year undergraduate analysis course she concluded that only the positive value should be accepted as a correct answer and explained this to the whole group in great detail. While agreeing with Jan, Jemma, who "only studied an applied mathematics course as an undergraduate", enquired about why the square root needs to be a function, and as such have only one output given the input; she was not clear about why one should not consider the square root as just a relationship, or a mapping. Jan concluded, "It seems to me that as a function and inverse, we can only accept one answer, while as a process it is acceptable to have two answers." After some discussion, while Joan pointed out to the $\sqrt[\pm]{81}$ notation she came across in a textbook, Jan said, not very convincingly, that maybe the symbol usage needs to be addressed, namely that the radical symbol should perhaps only be used for the positive square root of a number. Her point was acknowledged by the whole group, who sighed with relief for finally reaching a conclusion.

In Group II, the participants also expressed their frustration with the polemic surrounded the + or - . Bea remembered that she was taught at college that $\sqrt{x^2} = |x|$ for any x real number, and so $\sqrt{a}$ can only be a positive real number. She went on to explain how based on this fact $\sqrt{16} = 4$ and that taking the square root of both sides would then yield $|x| = \sqrt{16}$, hence $\pm x = 4$ resulting in $x = \pm 4$. This clarified the presence of $\pm$ for the other participants, some of whom needed help to remind themselves about the modulus function and why the relationship $\sqrt{x^2} = |x|$ holds true in the first place. The participants in Bea's group realised that using this definition of the square root the ambiguities encountered before would be eliminated and so they happily adopted it for solving the other mathematics task questions.

Pedagogical decisions

The participants were asked to take home the fictional pupils' scenarios.

Give the answer to the following questions:

a) $\sqrt{4} = ?$ b) $\sqrt{36} - \sqrt{9} = ?$ c) $\sqrt{7^2} = ?$ d) $\sqrt{25 - 9} = ?$

Emma responded as follows:

a) $\sqrt{4} = 2$ *and* -2 b) $\sqrt{36} - \sqrt{9} = \pm 6 - \pm 3 = 3, 9$, *got stuck here, sorry...*

c) $\sqrt{7^2} = 7$ *because square root and square cancel each other*

d) $\sqrt{25 - 9} = \pm 4$

What comments would you make to this pupil with regard to her answers?

Figure 2: Emma's (Year 8 - KS3) scenario

Jan, in Group I, who so eloquently talked about the ambiguity of the radical symbol notation, concludes her feedback to Emma, the KS3 fictional pupil that " $\sqrt{7^2} = \sqrt{49} = \pm 7$ so when you see $\sqrt{\ }$ you must consider both the positive and the negative roots", a position in contrast with the one she reached during the group discussion. However, in her feedback to Lucy's (KS5 fictional pupil) solution to the same question, she explains that "$\sqrt{7^2}$ can only equal to 7, as the square and the square root cancel out each other".

Bea in Group II was consistent in her feedback that the radical symbol is to be used for positive answers only, as $\sqrt{x^2} = |x|$. Ben, too employed the modulus, providing a detailed feedback to Lucy's solution seen below.

Solve the following equation: $3x^2 = a$

Lucy responded as follows:

$3x^2 = a$ *divide both sides by 3*

$x^2 = \frac{a}{3}$, *square both sides and so* $\sqrt{x^2} = \sqrt{\frac{a}{3}}$, $x = \pm\sqrt{\frac{a}{3}}$ *is the solution*

What comments would you make to this pupil with regard to her answers?

Figure 3: Lucy's (Year 13 - KS5) scenario

He points out in his feedback the missing intermediate steps in Lucy's answer, "The statement $\sqrt{x^2} = \sqrt{\frac{a}{3}}$ so $x = \pm\sqrt{\frac{a}{3}}$ is fine as long as the intermediate step is $|x| = \sqrt{\frac{a}{3}}$ ".

While the whole group employed this definition when attempting the other mathematics tasks questions, from a pedagogical point of view, Billy did not think this definition would be of much use since the square root is introduced much earlier than the concept of modulus in secondary school education.

Barry in Group II writes that the radical notation is assigned to the positive square root by convention, "However, by convention, we usually take $\sqrt{4}$ to just mean the positive root, i.e. 2.", showing that he, too adhered to Bea's unambiguous way of working the square root.

Concluding remarks

The tasks in which the participants were involved in this small study provided a context in which their knowledge, preferences and choices were brought to the surface.

The mathematics tasks created some instability in what the participants knew about the square root and the discussion of how they tried to resolve the discrepancies brought out into the open their knowledge and interpretations of the subject matter and enabled the participants to question further or clarify these concepts for themselves when needed. The activities carried out during this study offered the prospective teachers opportunities to engage with the subject matter at a level deeper than simply recalling their existing knowledge. The tasks were triggers for reflection and introspection, "I thought I knew all about square roots until I worked on this homework." (Jemma) and during discussions other mathematics concepts came under consideration.

This study adds to Zazkis and Leikin's (2010) call for a more articulated relationship between AMK and mathematical knowledge for teaching. The participants in this study benefitted from recalling some of their AMK (function, relation, mapping, inverse function, modulus function) which they studied more formally and in depth in their undergraduate studies. A few of the participants (Ben, Jan and Bea) became explicitly aware of the connections between their AMK and the square root concept under scrutiny. Their articulation of such connections triggered recall of these advanced topics and the implications for the other participants' square root definition. All participants realised that with the new understanding gained about the square root, the ambiguities initially encountered were removed and so they were able to answer the rest of the questions in the mathematics task without further confusion or even disagreements.

However, their feedback to pupils' scenarios highlighted the tension between the widely accepted school definition of the square root, together with the use of the radical symbol and their own modified conceptions. Five of the eight participants in this study decided in the end to adhere to the widely accepted school practice of treating the square root and misuse of the radical symbol, i.e. $\sqrt{16} = \pm 4$.

The findings of this very small scale study suggest that more needs to be done to empower prospective teachers not only to scrutinise (re)sources through applying their AMK to ideas in the secondary school mathematics curriculum, but also to challenge the accepted, ambiguous treatment of concepts and ideas in school mathematics.

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Investigating the construction of the problem-solving citizen

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Similar to other curricula, the Swedish mathematics curriculum emphasises problem solving both as an end in itself and as a means to becoming a competent citizen. Thus, a goal for mathematics is the creation of a problem-solving citizen. In this paper, I explore how critical discourse analysis, and parts of social activity theory can be used to operationalise Bernstein's pedagogic device in relation to the construction of the problem-solving citizen. It is proposed that critical discourse analysis can be used for a linguistic analysis of official documents, like the curriculum whilst social activity theory's different domains, mainly public and esoteric, can be used to analyse the national tests. These tests assess students' problem solving both as a means and an end in two different senior secondary programmes, an academic orientated one and a vocational one. In this exploration, two examples are given to show how these methodological tools can be used.

Keywords: Pedagogic device, critical discourse analysis, social activity theory, curriculum, problem solving

Background

This paper is part of a larger project about how the goal of problem solving is recontextualised in the Swedish mathematics curriculum and in national tests for two upper secondary school programmes, vocational and academic orientated. These programmes have similar curricula and the same grading criteria but different national tests.

Sweden can be seen as following an international educational trend in the way competences are emphasised in the curriculum (Dahl and Johansson, 2013). For example, problem solving is strongly emphasised both as a means to develop other mathematical competences but also as a competence in its own right that should be assessed and graded (Skolverket, 2011c). It is also stated that students should become employable citizens and participate in democratic processes through sharing a common "civic reference framework" (Skolverket, 2011d: 13). Thus, the goal of mathematics problem solving in schools can be considered to be the creation of the *problem-solving citizen*. Although it is unclear how mathematics education contributes to the creation of this citizen, mathematics is highlighted as important in this process (Skolverket, 2011d). Giving this as a reason for including mathematics in school, therefore, seems to have become common sense without any need to question it. Consequently, my research question for the larger project is: How is the problem-solving citizen constructed and recontextualised in the curriculum and other official documents?

In this paper, I explore the research methodologies of critical discourse analysis (Fairclough, 2001) and social activity theory (Dowling, 2005) as means for questioning this common sense assumption. Both theories share similarities on an ontological and epistemological level which enables them to be related to Bernstein's pedagogical device, explained in the next section and which provides the theoretical framework for the main study. In this paper, two examples are provided as part of the exploration. The first example uses critical discourse analysis on the Swedish mathematics curriculum. The second example is an analysis of the national test.

The pedagogic device

Bernstein (2000) described the pedagogic device as illustrating how education acted as a filter for ensuring that class distinctions were reproduced. The pedagogic device consists of three sets of interdependent rules: *the distributive rules* regulate the power relationship by distributing different forms of knowledge to social groups; *the recontextualising rules* regulate the formation of pedagogic discourse; *the evaluative rules* constitute pedagogic practices that are realised in instructional and regulative texts (Bernstein, 2000). In relation to the problem-solving citizen, the distributive rules can be considered as controlling how the discussions of politicians, education bureaucrats and educational researchers are relayed to those formulating the curriculum. The recontextualising rules control how the distributed knowledge about problem solving is incorporated into the curriculum and national tests which then control the ways that educators in schools and the wider education sector come to discuss problem solving. The evaluative rules control how teachers teach problem solving in classrooms. In the wider project, it is the operation of the recontextualising rules that are in focus around the construction of the problem-solving citizen.

Bernstein identified a difference between forms of knowledge in the *vertical* and *horizontal discourse*, both of which are regulated by the distributive rules (Bernstein, 2000). The horizontal discourse includes mundane (common sense) knowledge. Abstract, conceptual (esoteric) knowledge is a form of vertical discourse and “the process of acquiring vertical discourse is through induction into that strongly classified and insulated body of knowledge” (Wheelahan, 2007: 3).

Mundane, common sense mathematics tasks that emerge from the horizontal discourse, into the institutional and regulative texts of the evaluative rules, can be used to ‘hook’ students and introduce or inspire them into mathematics (Whitty, Rowe and Aggleton, 1994). Bernstein (2000) stated “to make specialised knowledge more accessible to the young, segments of the *horizontal discourse* are recontextualised and inserted in the contents of school subjects” (Bernstein, 2000: 169). Including examples of these mundane tasks in the curriculum or national tests contributes to them being passed on through evaluative rules into teachers’ classroom practices.

However, putting mathematics in a real-world (mundane) context is, according to Bernstein to treat mathematics, which is a vertical discourse, as if it were a horizontal discourse. This is more common in vocational programmes, generally designed for low achieving students from the working class (Dowling, 2005). Therefore recontextualisation is used to “exclude[s] the working class and other disadvantaged social groups from access to powerful knowledge, because it denies students access to the structuring principles of disciplinary knowledge” (Wheelahan, 2007: 1) as it is only “esoteric knowledge [that] has the potential to challenge the social distribution of power” (ibid. p. 3). An analysis of the national tests can provide information on whether problems are from the vertical or horizontal discourse and how knowledge, and therefore power comes to be unequally distributed to students in the different courses.

Nevertheless, the pedagogic device is not about *what* is transformed, the content, but *how* the general rules control the transformation of knowledge into pedagogic communication. In order to understand how the problem-solving citizen is constructed and its effect on mathematics education in upper secondary education, there is a need for tools to analyse these transformations. *Critical discourse analysis* (Fairclough, 2001) and *social activity theory* (Dowling, 2005) are investigated as research methodologies for identifying how the problem solving citizen discourse has become a regulative discourse in distributing different forms of mathematics, mundane and esoteric, to groups of students from different social classes.

Critical Discourse Analysis

Critical discourse analysis (CDA) highlights what is not obvious, by questioning and problematising the taken-for-granted, or dominant ideology (Winther Jørgensen and Phillips, 2000), such as the belief that students should become problem-solving citizens and that school mathematics is the path to achieve this goal. It contains three dimensions: a socio-cultural practice; a discursive practice; and text (Winther Jørgensen and Phillips, 2000). The analysis on the level of the socio-cultural practice is about the relations to other discourses and non-discursive practices. The analysis on the level of the discursive practice is about how texts are produced, distributed and consumed. This division opens up the possibility to analyse the dialectic interplay between social and discursive practices:

> Unlike other forms of discourse analysis, [critical discourse analysis] also involves theorising the social processes and, in particular, the power structures, which give rise to, and are maintained by, discourse. (Oughton, 2007: 261)

At the macro level of the socio-cultural practice, the global economy, the national infrastructure or the 'neo-liberal political trend' provide an overarching structure in which there are schools and in schools there is a subject called mathematics. Within the discourse of school mathematics, a curriculum and national tests are produced. To construct the problem-solving citizen, these texts draw on different discourses, for instance the horizontal and vertical discourses of Bernstein.

The curriculum is produced on the level of discourse practice, the micro level, which is part of a socio-cultural practice. Thus these texts are produced within the recontextualising field, in which Bernstein (2000) distinguishes "between an *official recontextualising field* (ORF) created and dominated by the state and its selected agents and ministries, and a *pedagogic recontextualising field* (PRF). The latter consists of pedagogues in schools and colleges, and departments of education, specialised journals, private research foundations" (p. 33, italics in original). Between and within these fields there is a struggle over the pedagogic discourse, in regard to what should be included and emphasised. CDA is a way to analyse the connection and interplay between the micro and macro level and hence the tensions or struggle within the recontextualising field. It requires a linguistic analysis of the text and a sociological analysis of the macro level. The analysis on the micro level is then the connection between the text and the socio-cultural practice (Winther Jørgensen and Phillips, 2000).

Chouliaraki and Fairclough (1999) emphasise that Bernstein is a critical theorist "concerned to trace the embeddedness of social practice within [and between] social relations of power" (p. 98) and therefore conclude that Bernstein's theory and CDA share the same ontological and epistemological basis. Following Bernstein, Chouliaraki and Fairclough (1999) understand the pedagogic discourse to be "a recontextualising principle which removes discourses from the practices they primarily belong in and relocates them within its own practice" (p. 109). To understand *what* is relocated, they consider that something more is needed and proposed the CDA element of intertextuality, particularly interdiscursivity.

Intertextuality provides an understanding about how texts, which are in focus, have connections to and are dependent on other texts (Chouliaraki and Fairclough, 1999). Skolverket (The Swedish National Agency for Education) (2011b) as the writer of the curriculum, draws on national and international studies, to emphasise that problem solving should be viewed both as a means for learning other topics or subjects and as an end that should be assessed and graded. The distribution of the discourses about the double role of problem solving through the curriculum produces a 'blurriness' within which there are tensions about how to interpret what mathematics is and its relationship to citizenship. An intertextual analysis could highlight the differences and similarities between texts in order to place the curriculum in a context and indicate how the problem-solving citizen is constructed.

One form of intertextuality is interdiscursivity "concerned with the way in which a text appears to subscribe to one or more discourses" (Locke, 2004: 43). An interdiscursive analysis is useful in seeing whether vertical or horizontal discourses are drawn upon in representations of problem solving in the curriculum and in tasks in the national tests.

An example

As the example is used to explore the relevance of CDA, it was decided to do an initial intertextual analysis of the similarities and differences between the academic and vocational programmes. According to Dowling (2005), differences between these programmes result in different discourses (vertical/horizontal or esoteric/ mundane) being utilised. Thus, it should be possible to use an intertextual analysis see if and how vocational students come to be excluded from the esoteric domain and hence from power associated with being a problem-solving citizen.

An intertextual analysis

To exemplify how intertextual analysis can be used, the Swedish curriculum (Skolverket, 2011d) (from now on called Gy11), launched in 2011, is analysed with the old curriculum from 1994 (Skolverket, 2006) (from now on called Lpf 94). The curricula documents analysed are for Gy11: *Curriculum for upper secondary school* (Skolverket, 2011a), which has the fundamental values and overall goals and guidelines; and *Upper Secondary School 2011* (Skolverket, 2011d), which describes the purpose and structure of the upper secondary school. For Lpf 94 the comparable document is: *Curriculum for the non-compulsory school system Lpf 94* (Skolverket, 2006), about purpose and structure of the upper secondary school.

The two documents, *Curriculum for upper secondary school* (Skolverket, 2011a) and *Curriculum for the non-compulsory school system Lpf 94* (Skolverket, 2006) are similar in content and in the way ideas are stated and expressed. Consequently, I analyse *Upper secondary school 2011* as a complement to *Curriculum for upper secondary school*. This approach stresses the differences with the older curriculum and highlights the structure and purpose of upper secondary school that was not provided in Lpf 94.

There is also a point in comparing the two documents textually. For Lpf 94 there was no official document, similar to *Upper secondary school 2011*, so teachers had to rely on *Curriculum for the non-compulsory school system Lpf 94*. As Gy11 was supposed to overcome problems in Lpf 94 (Skolverket, 2011d), the construction of the problem-solving citizen in Gy11 could be considered a response to Lpf 94.

Upper secondary school 2011 is 258 pages but only discuss the same topics as *Curriculum for the non-compulsory school system Lpf 94* in the first 60. *Curriculum for the non-compulsory school system Lpf 94* is 22 pages.

'Vocational' is mentioned 15 times in Lpf 94 and 162 times in Gy11. In Lpf 94, 'vocational' is either about the difference between vocational and academic programmes (five times) or about the need for students to be prepared for a (changing) vocational life (10 times). Out of the 162 times 'vocational' is mentioned in Gy11, only one could be categorised as being about preparation for vocational life. An alternative to 'vocational', the term 'working life' in Lpf 94 is mentioned 25 times and in Gy11 14 times. This indicates that there are differences in the two documents.

Differences in meaning

Although tallying the number of times a term appears hints at differences, it is more crucial to identify how 'vocational' is expressed. In Lpf 94, the use of 'vocational' suggests similarities rather than differences between the programmes. For example:

> The school shall strive for good co-operation with working life, which is important for all upper secondary education, but of particular importance for the quality of vocationally-oriented education (Skolverket, 2006: 7)

However, this changes in Gy11, with the differences between programmes being highlighted. Pages 20-21 list the differences between the programmes. The next four pages (22-25) describe the vocational programmes and another one and a half pages describe 'Higher education preparatory programmes', suggesting that they should be considered in contrast, rather than as complements, to each other.

How vocational programmes are discussed in Gy11 changed markedly to how they were discussed in Lpf 94. In Gy11, pages 9 to 13 give 'A brief look backwards' to earlier curricula and reforms and describe how vocational programmes first became a part of the upper secondary school. It states that with Lpf 94, the programme was extended to three years and stressed 'civic competence' and "all students in vocational programmes could automatically achieve basic eligibility for higher education" (Skolverket, 2011d: 11). The following section is 'The problems of the school at the beginning of 2000' and this suggests that the emphasis on civic competence and the right of all students to have the opportunity to continue to higher education could be seen as causing the problems of school:

> Since the beginning of the current century, international studies Sweden has participated in have shown that the knowledge students leave school with is not as good as before. Many students drop out of their upper secondary studies. And Sweden has a high level of youth unemployment compared to other countries. These trends led to the reform of the upper secondary school and the compulsory school in 2011. (p. 11)

The proposed solution was to create larger differences between vocational and academic programmes so that "Vocational education should provide good preparation for working life so that students can start working immediately after upper secondary school" (p. 12). Thus, vocational and academic programmes in Gy11 are described as being more different that they were in Lpf 94.

Further, the context for Gy11 is described as:

> The upper secondary school has a broader aim than merely preparing students for working life immediately after education or for further studies in higher

> education. It should also give them a good foundation for personal development and active participation in society. (Skolverket, 2011d: 8)

This statement seems to indicate that this was not the fact in earlier curricula and seen in the light of the above analysis the ‘or’ in the quote must be interpreted as an exclusive or, suggesting the school either prepare students for working life or for further studies, but not both. Thus, there seems to be a wish that different discourses should to be drawn upon for the different programmes. Following Dowling (2005) this could result in different opportunities for ‘personal development and active participation in society’ for students.

Mathematics is highlighted as having an important role in the making of the future citizen (Skolverket, 2011d) whether or not this is a reality. To see what different discourses are drawn upon in the construction of the problem-solving citizen on the level of evaluation, an analysis of the national tests is conducted. In the next section, I add to the CDA analysis of the curriculum, by exploring Dowling’s social activity theory (SAT) as a means for analysing the national tests. Seen as a part of CDA, this analysis could be described as content analysis (Bergström and Boréus, 2005).

Social Activity Theory

Dowling (2005) draws on Bernstein to build a language of description, which he calls social activity theory (SAT), to analyse mathematics tasks from textbooks. The analysis concentrates on differences between tasks in the esoteric domain equivalent to Bernstein’s vertical discourse, and tasks in the public domain, Bernstein’s horizontal discourse.

Dowling (2005) showed how access to the esoteric domain, through textbooks was unequally distributed to low and high achieving students. Low achieving students were very rarely provided with tasks from the esoteric domain. Instead they worked with mathematical tasks, for example, about shopping, often expressed in non-mathematical language. Tasks for high achieving students could start in the public domain, but quickly moved into the esoteric domain.

However, given the differences identified between the vocational and academic programmes, it is useful to see if these differences extend to the tasks for the different programmes being constructed differently as well.

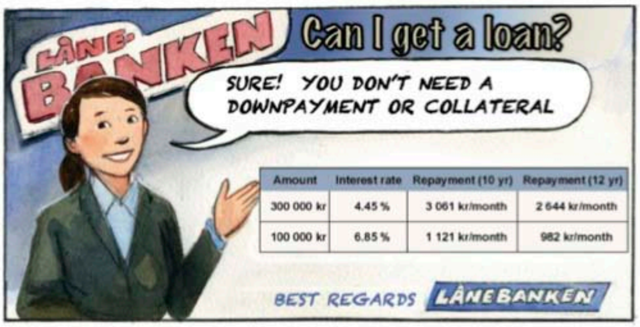
An example

With the new curricula for the first time since 1994, there were different national tests (PRIM, 2012) for vocational and academic programmes. The problem-solving tasks reinforce differences between programmes identified in the curricula documents. For the vocational programme 7% of the problem-solving tasks were in the esoteric domain, while on the test for the academic programme 33% were in the esoteric domain. This is in spite of the fact that the two programmes have the same grading criteria (Skolverket, 2011c). Figure 1 and Figure 2 are tasks from the national test (spring 2012, course 1b). Figure 1 shows a task about the smallest positive integer and so can be considered as being situated in the esoteric domain. The content is mathematical and the language draws on the mathematics register. Figure 2, a task about a loan, is a typical public domain task. Both the content and the expression are weakly classified, as they refer to the outside world, a mundane context. Although numbers are used, the language is conversational.

What is the smallest positive integer that is divisible by all i integers from 1 to 9? Explain your answer.

Figure 1 (from PRIM, 2012, p. 6)

An advertising flyer shows the following information.



Amount	Interest rate	Repayment (10 yr)	Repayment (12 yr)
300 000 kr	4.45 %	3 061 kr/month	2 644 kr/month
100 000 kr	6.85 %	1 121 kr/month	982 kr/month

The repayments include amortization, interest etc.
Renée is considering borrowing 100 000 kr and repaying over 10 years.

a) Use the information in the flyer to calculate the total amount she will have paid back to the bank when the loan is paid off.

b) What proportion of the first month's payment is interest?

Figure 2 (PRIM, 2012, p. 5)

This initial and brief analysis suggests that, in Gy11, differences between vocational and academic programmes are being promoted in regard to preparing for working life. Thus, there seems to be different possibilities for students to reach the main goal of school: to "give them a good foundation for personal development and active participation in society" (Skolverket, 2011d: 8). The analysis of task distribution in the national test supports the findings about different outcomes being promoted for different groups of students, by suggesting that access to the esoteric domain, and hence to active participation is unequally distributed.

Conclusion

Bernstein (2000) suggested that school reproduces social inequalities. In this paper I have showed how CDA and SAT can be used as tools to see how this is done in the official documents for mathematics education in upper secondary school in Sweden.

This exploration of CDA and SAT suggest that both methodologies can illustrate how the rules governing the pedagogic device operate. CDA is used to analyse the curriculum to see what happens as a result of recontextualising rules. In this initial analysis, the tension between different discourses, vertical and horizontal is illuminated by a textual analysis. This is mainly done by looking for differences in vocational and academic programmes, and especially how the distinctiveness of vocational programmes is emphasised. Thus, CDA has helped to highlight what is not obvious, and to question and problematise the taken-for-granted (Winther Jørgensen and Phillips, 2000). Dowling's domains suggest that the distinctions found in the curricula are also present in the national tests.

To see how the problem-solving citizen is constructed, as a goal of mathematics education, an examination of other parts of the curriculum is needed. A larger intertextual analysis, for instance comparing the curriculum with the PISA framework will need to be a part of this since Gy11 draws heavily on PISA (Dahl and Johansson, 2013). In this paper I have used problem-solving tasks from one test, the first national test (course 1). A further analysis of the national tests needs to be done.

In this paper I have analysed the curriculum and national tests in a Swedish context. The Swedish curriculum, especially with its emphasis on competences and the highlighting of problem solving should be seen here as a typical example of a 'modern' mathematics curriculum. These features are characteristic in many mathematics curricula, standards and frameworks, for instance the influential international PISA-tests, emphasises competences instead of content-knowledge (OECD, 2009). The rationales behind this trend are also similar, emphasising the need for citizens to be available for the labour market throughout life (Nordin, 2012).

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The relevance of mathematics: The case of functional mathematics for vocational students

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This study of recent school-leavers in Further Education explores students' opinions of relevance and how these are influenced by their experiences of different mathematics curricula in school and college. These vocational students are taught mathematics as a functional 'tool for life' rather than a discipline of rules.

Perceptions of relevance are influenced by personal goals and interests (Ernest, 2004) and may depend on whether students identify a value for the qualification, a practical usefulness or some transferable skills (Sealey and Noyes, 2010). These can provide reasons for studying mathematics but, in this study, students who encountered mathematics as a 'tool for life' engaged in learning experiences that connected with their personal life experience. This changed their conceptual view of mathematics and added a different perspective to their views of relevance.

The research is part of a larger study of the student experience of functional mathematics in colleges but this paper will focus on qualitative data from student focus groups and lesson observations. Transcriptions were coded and compared to identify common themes in student experiences. The results suggest how teaching mathematics as a 'tool for life' can influence perceptions of relevance and effect some positive changes in student attitudes.

Keywords: relevance, vocational, functional mathematics.

The concept of mathematics as a 'tool for life'

Political interest in the skills that adults need for life and work is long-standing and an intention to address these needs has been re-iterated many times. Various curricula have been developed and discarded in successive attempts to develop and assess such skills. The emphasis on using and applying mathematics in practical situations has been a common theme in Adult Numeracy, Key Skills, quantitative literacy and, more recently, in functional mathematics (QCA, 2007).

At the heart of these curricula lies the concept of mathematics as a 'tool for life', a set of knowledge and skills that, allegedly, equip students to apply mathematics and solve problems in life and work situations. This concept contrasts with the traditional knowledge-based school curriculum. Although, theoretically, functionality in mathematics has been embedded into the GCSE curriculum, there remains some doubt about whether this is a token offering or a real commitment to the meaningful application of mathematics to everyday life.

The students in this study mostly fell within the category of school-leavers at age 16 years who had failed to reach the 'gold standard' of a GCSE grade C in mathematics and had opted to take a vocational course in a Further Education college.

After following a GCSE mathematics course in school, they were then exposed to a functional mathematics curriculum in college. This involved the application of mathematics in a range of contexts, the use of 'realistic' scenarios and a problem-solving approach (QCA, 2007). This study looks beyond the difficulties implicit in implementing this curriculum, in order to understand the effects on students' perceptions and responses when taught mathematics as a functional tool rather than a knowledge-based subject.

As a result of the government's response to the Wolf report (2011), post-16 students who fail to achieve a grade C in mathematics are now expected to continue with GCSE mathematics but the suitability of repeating the same qualification is questionable. Recent research suggests that current workplace practices require applications of basic mathematics in complex situations rather than higher-level knowledge of mathematics (Hodgen and Marks, 2013). For vocational students, developing skills in applying mathematics may have more value for their intended careers than acquiring a particular academic qualification. In this research, a comparison of the impact of a different curriculum on students' perceptions of relevance provides some indications as to why an alternative to GCSE may be a preferable option for these students.

Relevance and reasons

The relevance of mathematics, as seen by educational and political figures, is linked to goals that have benefits to society but hold little meaning for students (Ernest, 2004). Their neglected perspective, Ernest (2004) argues, is an important part of the discussion about relevance and should be seen in relation to their personal interests and goals.

For students, relevance may be established for a number of reasons. Firstly, mathematics may be perceived as relevant because the qualification itself is useful to them (Sealey and Noyes, 2010). The value of mathematics in society and its 'gate-keeping' role for higher-level study make a qualification in mathematics a desirable asset in a student's portfolio. A qualification in mathematics may have a direct 'exchange value' as a prerequisite for a chosen career, or for further training related to a personal goal. In these ways, students may recognise their need for a mathematics qualification and consequently have a reason to engage with the subject.

Secondly, students may identify practical or transferable skills (Sealey and Noyes, 2010) that may be useful to them and thereby recognise the relevance of the subject. Discovering a use-value for these skills may be dependent on students' awareness of mathematics in other areas of life apart from the classroom. Similarly, appreciating that certain transferable skills have value in the future may help some students to engage with topics, but the connections are not always apparent. Their limited life experience and unfamiliarity with the demands of the workplace may make it difficult to identify how mathematics is used in different situations. Establishing links to students' lives can provide reasons to engage with the subject but a cognitive acceptance of the value of mathematics may still be insufficient to influence their views of the subject.

Attempts to make mathematics more relevant in the classroom may involve using 'realistic' contexts for mathematics questions, but the interpretation and impact of these on students' responses to questions can vary (Cooper and Harries, 2002). One of the problems may be that the context itself is often no more than a metaphor to illustrate an aspect of pure mathematics rather than authentic use of a scenario as a source of mathematics (Wiliam, 1997). Making connections between mathematics and life that appear authentic and convincing for students is not a simple task.

In the Realistic Mathematics Education (RME) approach, context is used as a base for the construction of knowledge but being 'realistic' refers to situations that students can imagine and not necessarily to authentic problems from real life (Van Den Heuvel-Panhuizen, 2003). Moving from these 'realistic' situations to symbolic representation is as an important element of the learning process. Used in this way, context is a means of developing understanding but the connections to 'reality' may also bring mathematics closer to the individual and demonstrate a value for the subject beyond the classroom.

Engagement and emotion

There is evidence that students who fail to see the relevance of mathematics also demonstrate poor motivation and disaffection, even when their attainment is high (Nardi and Steward, 2003). The term disaffection is commonly used to refer to negative attitude or emotion, suggesting that student perceptions of relevance and attitude to mathematics may be connected.

Attitudes and emotions are often seen as part of an affective structure that includes beliefs (McLeod, 1994) and values (DeBellis and Goldin, 2006). Definitions of these constructs vary and the relationships between the components are complex but there is a wide acceptance that affect is inter-linked with cognition in the learning process. Affective responses to mathematics may be seen as part of an engagement structure (Goldin, Epstein, Schorr and Warner, 2011), in which beliefs are intertwined with other aspects of affect and cognition in a personal framework that influences behaviour. Goldin et al. (2011) consider that affect permeates many of the strands within this structure, highlighting the relationships between goals, affect and behaviour. From this viewpoint, if relevance is connected to personal goals, then it involves more than a cognitive reason for engaging with the subject. Affective associations, such as interest, are influential and will have an effect on student perceptions of the relevance of mathematics.

Considerations of the affective domain have led to considering the elements, such as emotions, as having transient states or more stable traits (McLeod, 1994). Hannula (2012) takes the view that each affective component has both characteristics and suggests that attitude may be seen as the more long-lasting trait related to emotions. For the purposes of this study, the possibility of short-term fluctuations in emotions but slower shifts in attitudes means that effects may be evident over the research period that are linked to the same basic construct.

The concept of mathematics as an academic discipline, with a well-defined system of rules, has an appeal for some students but may seem unattractive to others. The role of relevance is, perhaps, an intermediary construct in the linking of student views of mathematics to their attitudes but the effects may be influential on the learning process due to the entwining of affect with cognitive processes.

Research methods

This research formed part of a wider study of vocational students' experiences of functional mathematics, which had the aim of exploring the main factors that influenced their experience. The research question of interest for this paper was to identify the ways in which functional mathematics was relevant to students. This will be examined with particular reference to findings that related to student perceptions of relevance, their views of mathematics and affective responses.

A grounded theory approach was adopted for the main research due to the exploratory nature of the study. Qualitative methods were preferred because of the emphasis on understanding 'how' and 'why' certain factors influenced the student experience. Student focus groups were used to explore their perceptions, whilst observations of functional mathematics lessons and semi-structured interviews with teachers provided additional data sources for comparison. The interviews and focus group discussions were audio-recorded and transcribed.

The student focus groups were conducted in a semi-structured manner with the researcher using a flexible framework of questions as prompts for discussion. Students also gave individual responses to sets of statements regarding their experience of mathematics in school and college. Each student group met three times during the college year, with the following topics providing the main focus:

- the transition from school to college and students' early experience of functional mathematics (Term 1)
- students' opinions of a range of contextualised materials (Term 2)
- a review of the students' experience over the year and the outcomes (Term 3)

Lesson observations were unstructured in the first term and used alongside the data from the focus groups to identify the areas to be explored further in the second phase of data collection. A series of structured observations in the second term focussed on particular aspects of teaching and learning. Written records were made rather than video-recording due to concerns regarding the possible effect on student behaviour for classes unused to being recorded.

The study involved comparisons of the student experience in three Further Education colleges on courses in Public Services, Hairdressing and Construction. These represented vocational areas with different gender biases, practical skills and training bases (salon, workshop, outdoor/classroom) that were available at all three colleges. Where possible, two student groups, with different teachers, were recruited for the research from each vocational area at each college. Within the constraints of access and consent, this resulted in a total of seventeen student groups that were studied in some depth over a period of nine months.

The analysis, based on an iterative process of coding and comparison, led to the construction of case studies of these student groups, from which the main themes were extracted and examined further. Two contrasting case studies will be used in this paper to illustrate some of the main themes that emerged in relation to the particular research question being addressed.

Research findings

Individual responses in the first term showed that many of the students in this study were disaffected by their experience of mathematics at school but there was a significant shift towards more positive attitudes in college. In particular, students indicated that in college they were less stressed, more interested and less bored in functional mathematics lessons than they were in GCSE mathematics in school. When the same individual activity was repeated in the third term the results were similar and showed that students had largely retained the attitudes they expressed in the first term.

The reasons for these changes in attitude between school and college were explored in the focus groups. Although reactions were sometimes mixed, student attitudes within groups were generally similar. Groups were often enthusiastic about their experience of functional mathematics compared to school although some felt there was little difference. This polarisation into positive and negative groups allowed contrasting cases to be compared in order to explore the causes. For the purposes of this paper two contrasting case studies will be outlined. Both groups were considered by their teachers to be ‘challenging’ but the differences in outcomes make these cases particularly interesting to compare.

Case study 1 – Lindsay’s Public Services group

Lindsay was teaching Public Services students who were aiming for careers in the armed services or emergency response teams (police, fire or ambulance). She did not view herself as an expert mathematician but as a functional skills specialist. Her own experience, prior to teaching, was in retail management where she frequently used mathematics in a practical way. She explained that functional mathematics, in her opinion, was about applying mathematics and solving problems in everyday life.

In lessons Lindsay used contexts for questions that she believed would relate to students’ lives and interests. These included tasks about the calories in burgers, alcohol consumption in England and the cost of smoking. Occasionally the context was linked to the students’ vocational course. Although Lindsay was careful to use current prices and authentic sources of information, the mathematical calculations involved in these tasks often seemed unrelated to anything the students would realistically want to work out in that scenario. This was not a concern for Lindsay because her strategy was to engage the students in the lesson through discussion about the context or scenario, even when this deviated from mathematics.

Lindsay’s students were enthusiastic about functional mathematics. In the focus group, they explained how their attitudes had changed because they saw the relevance of mathematics to their lives. The contexts were convincing for them and they readily described situations in their own lives that required mathematics. The students seemed unconcerned that the actual mathematics was sometimes only loosely linked to some of the scenarios used. The connections made to their vocational area, although limited, were valued and the students provided several examples of vocationally-related applications of mathematics that would be useful to them in the future.

The students’ perceptions of functional mathematics were of a subject with similar content to GCSE mathematics but with an emphasis on application to life that made the lessons very different. Students felt the curriculum had allowed them to concentrate on mathematics that was useful and understandable. They had enjoyed the discussions about issues pertinent to their lives and felt these had helped them engage in the lessons. As a result they believed they had learned useful skills and were more confident about using mathematics in their lives.

Case study 2 – David’s Public Services group

David also taught Public Services students. He identified himself as an engineer with high-level skills in mathematics. His definition of functional skills was about using mathematics and solving problems, but his interpretation was more about using logical processes than finding practical solutions.

David used contexts that he believed would interest students such as buying sufficient carpet for a flat or going clubbing but details such as outdated prices suggested these were contrived rather than authentic. He rarely made any links to the vocational area. The emphasis in David’s lessons was for students to ‘get on with the work', by which he meant the mathematics. There was little discussion about the scenarios or contexts used. Discussion seemed to be viewed as a distraction from the main purpose of the lesson, which was learning mathematics. He was concerned when students failed to engage with the lessons because he believed mathematics was relevant to them and they would need the skills in the future.

David's students did not enjoy their functional mathematics lessons. They saw very few differences between their school and college experiences of mathematics. These students viewed the curriculum as simply basic mathematics that they had already covered in school and were reluctant to repeat. They did not associate aspects of functionality, such as applying mathematics and practical problem-solving, with their experience in college. Some of the students welcomed the absence of certain topics from the curriculum that they had found challenging in school, such as algebra, because they thought these to be particularly irrelevant parts of mathematics.

The contexts used in lessons were perceived to be outdated and unconnected to their lives. In the focus groups, students explained how they found it difficult to see any links to their vocational area and struggled to provide any specific examples of applications of mathematics that might be useful in their future lives. Their attitudes to mathematics that developed in school remained unchanged in college and they failed to see the relevance of functional mathematics. For them, their functional mathematics course was a 'waste of time' because they had not improved or learned any new skills.

Main findings and discussion

For the students in these case studies, the functional mathematics qualification did have an 'exchange value' and therefore could be considered relevant (Sealey and Noyes, 2010) because it was required for progression to the next level of vocational training. For the students in David's group, this reason seemed to provide insufficient motivation for them to engage with mathematics. Relevance was limited to a cognitive recognition that the qualification had some value but this seemed remote to the students and attitudes remained unchanged. In contrast, the students in Lindsay's group grasped the relevance of functional mathematics and there were changes in both attitudes and behaviour. The case studies suggest that the following aspects of the students' experiences were influential in their perceptions of relevance.

Firstly, the curriculum content in college was more limited than GCSE mathematics and excluded certain topics that students considered were unconnected to life. This meant that there was a greater focus on the mathematics that students recognised as having a practical use. In isolation, this represents a rather utilitarian view of the value of mathematics (Ernest, 2004) but identifying a practical usefulness can provide a reason for students to see the relevance (Sealey and Noyes, 2011) and begin to make connections between the classroom and their lives.

Secondly, the emphasis on using and applying mathematics in 'real life' situations provided opportunities to communicate the concept of *mathematics as a 'tool for life'* rather than an academic knowledge-based subject. This emphasis on the use of mathematics relates strongly to the identified needs of the workplace (Hodgen and Marks, 2013) in which application skills are important. Students who grasped this conceptual view could see more easily the purpose of mathematics for their current or future lives. A connection to their personal goals or interests was associated with a positive emotional response and behavioural changes (Goldin et al., 2011) so they became enthusiastic about functional mathematics, engaging with lessons and gaining confidence. This appeared to be more than a transient emotional change and there was evidence of a shift in longer-term attitude towards mathematics (Hannula, 2012).

Thirdly, the *use of context* for these students was a factor that had the potential to increase the relevance of mathematics. Both teachers in these case studies attempted to use context to make mathematics relevant but with different outcomes. The examples used by Lindsay may not always have been 'authentic' in the sense used by Wiliam (1997) but appeared to be 'realistic' to students because the details of the scenario accurately matched their experience in real life. The images these descriptions created in students' minds (Van Den Heuvel-Panhuizen, 2003) seemed to be effective for learning purposes and connections to life were established.

The contexts selected by Lindsay were ones that stimulated student interest and she also reinforced these links to everyday life through discussion. This sometimes deflected the focus of the lesson from mathematics, but had a positive effect on their perceptions of relevance. Maximising the *emotional connection*, through careful selection of contexts and discussion, drew students into a closer relationship with mathematics.

In both of these case studies the teachers were working with the same functional mathematics specifications but their interpretations varied. Lindsay had a clear grasp of mathematics as a 'tool for life' that was rooted in her own experience of using mathematics in employment. She communicated this effectively and her students became convinced that mathematics was a useful tool with relevance for their lives. In contrast, David's interpretation was focussed on the need to acquire knowledge of basic rules and procedures rather than solving problems that arose from the contexts. This influenced his teaching approaches and his students saw little change from the mathematics they had learned in school. The subject remained remote and unconnected to their lives.

For Lindsay's group, her interpretation of functional mathematics as a useful 'tool for life' led to teaching approaches that had a positive effect on students' views and responses to mathematics. The emotional connections, established through contexts and discussion that linked mathematics to students' lives, added a further dimension to their perceptions of relevance than a simple cognitive recognition that the subject was important.

Conclusions

Teaching *mathematics as a 'tool for life'* increased the opportunity for the *connections to students' lives* because of the emphasis on using and applying mathematics. For these vocational students, this was an appropriate focus. They were unlikely to require knowledge of higher level mathematics but the ability to use mathematics, even in a limited number of applications, was required for their vocational pathway and their intended future careers (Hodgen and Marks, 2013).

The teaching approaches associated with the communication of *mathematics as a 'tool for life'* served to enhance the relevance for students when teachers *used contexts* that related to students' personal lives. Effective connections to students' life experiences stimulated emotional responses that helped them identify with mathematics in a different way. These students began to see mathematics not just as a useful set of skills or a qualification required to reach a personal goal, but as an intrinsic part of their current and future lives. Their view of relevance became more than just a cognitive recognition of the value of mathematics. The affective connection to their personal experience added a new strength to their perceptions of relevance, changed previously negative attitudes and led to increased engagement with mathematics.

Although it is widely acknowledged that affect and cognition are both important, the way *emotional connections* can be used to increase relevance and change attitudes to mathematics seems to have received little attention. This research suggests that a functional curriculum, taught with an emphasis on mathematics as a 'tool for life', can create opportunities to increase the relevance of mathematics through these emotional connections. The resulting changes in student attitudes and engagement mean that this aspect of relevance may be a useful area for further study.

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Problem solving tasks in mathematics classrooms: An investigation into teachers' use of guidance materials

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This paper reports on a design research study undertaken as part of the Mathematics Assessment Project (MAP). The project aims to support teachers in implementing a new curriculum in US schools, through the use of formative assessment lessons (FALs) designed by the MAP team. Here we report results of our research into teachers' use of the accompanying guidance materials as they implement problem-solving FALs, drawing on detailed case studies of lessons from a sample of UK teachers. Although we observe much variation in the ways in which teachers use the guidance, both when in and out of the classroom, we identify the provision of a 'Common issues' table, outlining likely responses from students together with advice of potential ways to respond, as one of the most valued and used aspects of the guidance materials provided.

Keywords: Problem-solving; formative assessment; guidance materials; teacher guidance; curriculum.

Introduction

Problem-solving lessons can often be challenging for teachers (Ball, 2001; Chazen and Ball, 2001; Evans and Swan, 2013; Lampert 2001; Leinhardt and Steele, 2005; Sherin, 2002). In solving a non-routine, unstructured problem, students may select one of a range of mathematical methods to apply and teachers cannot always be sure which method they will choose. Not only do teachers have to try to understand how students are making sense of the problem, they also have to try and begin to align students' disparate ideas and approaches with canonical understandings about the nature of the mathematics (Stein and Kaufman, 2010). Teachers frequently need to adjust the way they typically teach and well-designed curricular guidance has been shown to be effective in supporting teachers to make a pedagogical change (Remillard and Bryans, 2004).

As Remillard (2005) pointed out, the factors influencing how teachers use materials is critical to their development and iterative designs of curriculum materials take into account the ways in which teachers respond to and use the resource. It is known, for example, that when guidance materials have been provided, teachers often read the guidance, without necessarily engaging fully with the content. However, while some teachers do "little more than check for the materials that they need to carry out the lesson … and the activities that students are supposed to do" (Stein & Kaufman, 2010: 671), others "actively look for the mathematical point of the lesson and for information regarding how students might respond to the various tasks" (Stein and Kaufman, 2010: 671). Unsurprisingly perhaps, the latter group of teachers tend to enact the lesson more closely to the designers' intentions. In terms of enacting the lesson, Remillard's studies (1996; 2000) showed that minimal teacher learning results from merely reading the guide and that the most significant professional learning occurs when the teacher enacts the lesson in the classroom.

The challenge for designers of curriculum guidance is to promote teacher engagement with the resource, whilst resolving some of the tensions described by Davis and Krajcik (2005), such as designing for different types of teachers and determining an appropriate amount of guidance. This study addresses this issue by investigating teacher use of curriculum guidance materials when enacting problem-solving lessons in the classroom. In particular we explore which features of the guide teachers use and ignore, in an endeavour to understand more fully what teachers value most in guidance materials.

Mathematics Assessment Project

During a four-year design research project beginning in January 2010, a team of mathematics educators at the Centre for Research in Mathematics Education at the University of Nottingham, designed mathematics Formative Assessment Lessons (FALs) to support US teachers in implementing the Common Core State Standards in Mathematics (CCSSM). The lessons were developed, from research-informed draft designs to 'final' products, through an iterative process of piloting and refinement. A third of the lessons designed were 'problem-solving' lessons, aiming to engage students in the solution of non-routine tasks.

Each problem-solving FAL follows a similar structure including the following phases:

- Prior to the FAL, students attempt the problem individually. Rather than grading these scripts, which tends to promote competition between students (Black and Wiliam, 1998) and distracts attention away from the mathematics, the teacher formulates questions for students to consider, in order to improve their work on their next attempt.
- Students begin the FAL by individually reviewing their initial solutions to the problem, in light of the questions raised by the teacher.
- In small groups, students share and evaluate their initial attempts, with the aim of producing a joint solution that is better than their individual efforts, usually in the form of a poster.
- After discussing as a class the range of strategies used by different groups, students are given some sample responses to analyse. These are based on student work, but carefully written by a designer to introduce a range of problem-solving strategies. Students evaluate the sample responses, commenting on their strengths and weaknesses.
- The lesson ends with a class discussion of the various strategies seen and used and students are encouraged to compare different approaches.
- In a follow-up lesson, students engage in a final reflection on what they have learned.

Guidance materials

As expected in a design research project, the content of the guidance materials included in the FAL units, alongside the task, evolved during the course of the project, based on feedback from teachers' enactment of the lessons in US trials. In the first drafts, the suggested lesson outline section of the teacher guide was about three to four pages of text. As the project developed, the number of guidance pages increased significantly to between 10 and 12 pages, as the extent of support for the shift required in many US teachers' pedagogic practices through the introduction of the CCSSM became more apparent. For example, in February 2012 a US observer wrote:

> In working with high school teachers it seems clear to me that their vision and beliefs about teaching and learning in math classes is far from the Standards, which is why I support giving more detail and more script in the summary, as it serves as a means of presenting a different vision and challenging their beliefs.

Such requests asked for more examples of possible questions to use within the lesson, as well as recognition of the need for additional explanations for US teachers to aid their understanding of the intended use of the resources.

At the time of the research reported here each FAL teacher guide comprised:

- an overview page (outlining mathematical goals of the lesson, CCSSM addressed, a summary of the lesson structure with proposed timings and materials needed);
- advice on introducing the task *before* the lesson and assessing students' initial responses, including a 'Common issues' table (see below);
- a lesson outline detailing specific suggestions for what the teacher might do at each phase of the lesson, including questions they could ask and
- a summary solution and outline of possible approaches to the problem.

In addition to the teacher guide, PowerPoint slides containing an outline of the problem, directions for students as they work in the different phases of the lesson and the sample responses to support class discussions are provided for use during the lesson enactment.

The design team were conscious that producing the much lengthier guidance document that was emerging could result in teachers choosing not to read it all. Feedback from the US trials highlighted this: for example, a member of the US observation team reported, "the teacher stated that the lesson materials packet was 'long' and he didn't read it all the way through" (Observation Report, Dec 2012).

The 'Common issues' table

When the problem-solving FALs were first designed, the students were given the problem at the start of the lesson. However, US trials showed that it was difficult for teachers to understand and follow students' reasoning in the moment of the lesson and to be able to follow it up immediately. As a result, the teacher guide was revised with the direction to give the problem to students to tackle on their own, in class or for homework, *before* the lesson. Feedback from the teachers and observers involved in the US trials suggested that teachers would welcome more guidance with the formative assessment of students' initial attempts and to assist teachers in doing this a 'Common issues' table evolved.

At first draft stage, the contents of the table were informed by research findings of common misconceptions and the designers' own classroom experience. The table was revised and refined during the design process, based on feedback and analysis of students' attempts at the task and common difficulties students had with the task, during the trials.

The aim of the table is to help teachers identify common problem solving strategies that students might be expected to employ and the difficulties that may arise. The table exemplifies how these might be evidenced in students' work with each issue linked to suggestions of follow-up questions and prompts that teachers may use as feedback to their students. The aim is for the teacher to select questions from those provided, or devise their own, to direct students' attention to the strategies for problem-solving and help them to make further progress on the task. Teachers can use these questions and prompts to provide feedback on students' work, both at the start of the FAL and as the lesson progresses.

Central to our research study was the opportunity to examine in what ways the 'Common issues' table could be a useful mediating device, providing teachers with insight into the big ideas of the problem-solving tak and practical advice on how to proceed in the classroom.

UK Research Study

In the 2012 - 2013 academic year, eight teachers with a range of teaching experience, from three different UK secondary schools, used some of the problem-solving FALs regularly with a class of their choice. During the course of the year, a team of researchers (who had been involved in the design of the FALs) followed the teachers' journeys as they enacted the lessons in their classrooms. Each of the 66 FALs taught during the course of the academic year was videoed and teachers were interviewed prior to and after each lesson.

Whilst the FALs had been used in a trial phase in the US to provide iterative feedback for the design of materials, including teacher guidance, their use in British classrooms was in a research phase to better understand their use and in particular, teachers' use of the guidance materials. We expected that the UK teachers would view the teacher guide, not as a prescriptive set of instructions to be followed closely, but as suggestions for classroom enactment. This study provided an insight into their engagement with the different features of the guide and an opportunity to observe the extent to which the pedagogical principles suggested within the guidance were adopted.

The analysis in this study of the UK teachers' use of the guidance materials accompanying the problem-solving FALs draws on Remillard's framework for examining teachers' curriculum development (1999) where she models teachers' construction of mathematics curriculum in the classroom. The model includes three arenas in which teachers engage in curriculum development: design, construction and curriculum mapping. The design arena focuses on the decisions teachers make before the lesson, whilst the construction arena is concerned with the actual enactment of the lesson in the classroom. In the context of our study it is these first two arenas that are of relevance, as we explore teachers' use of the guidance materials both prior to and during the lesson, using them to frame our analysis. Evidence is taken from pre- and post-lesson interviews, an interview conducted near the end of the study and comments made during a final meeting, which involved teachers coming together to reflect on their experiences. The lesson videos allowed for triangulation of data.

Findings

The ways in which the UK teachers chose to engage with the teacher guidance provided with the problem-solving FALs varied from one-off decisions to patterns of behaviour that became a regular feature of the teachers' lessons.

The pre-lesson interviews provided an insight into the teachers' preparation for the lesson (the design arena) including how much time they had spent and what their focus had been. The post-lesson questions offered a way of following up from the observed FAL and an opportunity to explore the decisions the teachers had made when enacting the lesson in the classroom (the construction arena) and how these had been informed by the teacher guidance.

Use of the guide in the design arena

We found that the level of engagement with guidance materials in preparing for the lesson varied considerably from teacher to teacher. At one extreme, one of the teachers, although reporting he "found all parts of the teacher guide useful" (final interview, 1/7/13) and read the details of the 'Materials required' and 'Time needed' sections, only briefly scanned the lesson outline. On the other hand, at the other extreme, another teacher reported on one occasion, spending half an hour reading the suggested lesson outline and described how the guide "provided [him] with a starting point, enabling [him] to understand where to begin with the lesson" (pre-lesson interview, 19/6/13). He later described how he "used [the teacher guides] a lot in [his] planning … [and] always used them to give [him] an idea of what's expected" (final interview, 1/7/13).

We also found that use of the materials varied over time for individual teachers. For example, another of the teachers reported that over the course of the project, she began to understand the reasons behind the structure of the lesson and pedagogical principles expressed in the guide and as a result endeavoured to follow the guidance more closely in her lessons (final interview, 11/7/13). Other teachers also reported an increasing interest in the use of guidance materials, as their suggestions in dealing with issues likely to arise in the lessons proved to be helpful in preparing for the lesson.

The 'Common issues' table appears to have provided an important factor in engaging teachers with the guidance materials and signalling key intentions of the lesson in relation to problem solving. As one teacher commented, he found it to be "the most helpful part of the guide" (final interview, 1/7/13). Another described how he would "choose some of the questions when marking students' initial attempts at the task" (final interview, 1/7/13) and another described how he would "print off [the 'Common issues' table] to use when assessing the pre-lesson work" (pre-lesson interview, 25/6/2013). All eight teachers reported making attempts to annotate students' initial work on the task prior to the observed lesson, making reference to the 'Common issues' table.

Use of the guide in the construction arena

In the UK research study teachers' use of the guidance materials during the enactment of the lesson varied significantly. For example, one teacher always had a printed copy of the guide with him in the classroom and commented that he would use it to "[think] about the timings of the lessons and just [remind] myself [of] what's coming up next … it's nice to just have those prompts for when your mind goes blank" (final interview, 1/7/13). Another teacher would print out the teacher guide but scale it down to four pages per printed page, (likely to make it difficult to engage with, at a glance during a lesson) and another teacher would have the teacher guide accessible on his iPad "so that [he] could refer back to it" (final interview, 1/7/13). Others would print certain sections of the guide for classroom use or not print it out at all. One of the teachers regularly printed out the 'Common issues' table and had this with her in the lesson. Whilst another teacher did not print out the guide, he would often access the guide on his computer in order to be able to display the 'Solutions' section to students at the end of the lesson. Two other teachers also displayed the 'Solutions' section of the teacher guide to students during the lesson, a surprising use of the summary solution, which had been designed as guidance for the teacher, rather than a resource for students.

Some further issues also arose, that demonstrate how the teachers' enactment of the lesson in general deviated from intentions signalled in the guidance materials:

1. The guidance suggests that teachers make sure students understand the context of the task, prior to attempting the problem and gives suggestions of how to do this. In the research study, the teachers would often distribute the pre-assessment quickly at the end of an earlier lesson, refraining from giving *any* information, in an attempt to allow students to work things out on their own.
2. The guidance asks teachers to 'note different student approaches to the task'. Teachers are advised to listen and watch students carefully, to note different approaches to the task and the assumptions students make and to note any common mistakes. This information can then be used as a focus for the whole-class discussion towards the end of the lesson.

In relation to this second point, when questioned about the use of the teacher guide, one of the teachers commented:

> One thing I haven't really done, because the guide talks about things like making notes about what the pupils are saying to each other, to then sort of come back and influence the whole class discussion. I've kind of really just been listening to what they've been saying, and giving them prompts at the time. And then when it's come to sort of a class discussion, I've let them dictate ... There are a couple of times when somebody's said something that's vastly different to the rest of the class and I've wanted to make sure that they've shared that, but I haven't really used the things that they were saying to each other to form a basis of the class discussion as was suggested.

When questioned as to the reason for his deviation from the suggested guidance he explained:

> I think maybe if I'd thought about having something physical where I could jot these things down. Part of it is remembering it, especially if it's like the first group you see, and you know you're going to see another five or six groups. Remembering who's said what, so it probably would have helped to have something to write those down on … Also, I wanted the discussions to come from them, not led by me. (final interview, 1/7/13)

Whilst the teacher made this deviation from the guide explicit, some teachers, on occasion, when questioned about decisions they had made during the enactment of the lesson, would assume that they had been acting upon the suggested guidance, when at times this had not been the case. An example of this was seen when one of the teachers decided to ask the students to work in groups of four during one of the lessons. To facilitate this, the teacher spent considerable time rearranging the tables in his classroom and when questioned as to his motivation for doing this described how he wanted all groups to present their work to the rest of the class, commenting that he had "followed the suggestion from the teaching notes regarding this lesson, to allow pupils to discuss in pairs then come together in a four" (post-lesson questions, 13/12/13). On inspection of the teacher guide for this particular lesson, it was recognised that this was not actually the case and yet the teacher was confident that he had been following the lesson guidance.

Conclusions and discussion

It was evident during the UK study that teachers valued the guidance for the insight it provided into the lesson as a whole, but did not tend to use it to guide their minute-by-minute interactions in the classroom. Whilst all teachers referred to the teacher guide both prior to and during the lesson, their engagement with it varied.

It would seem that the 'Common issues' table proved to be the most useful aspect of the guidance, providing a valuable resource to the teachers within both the design and construction arenas. The inclusion of concrete examples of anticipated responses, with underlying misconceptions identified and suggestions on how to respond to the range of possible student approaches to the task, encouraged teachers to think about their practice *prior* to the lesson as well as informing teacher actions *during* the lesson. The inclusion of material on students' thinking in teacher guidance requires substantial enquiry by designers and then by teachers, into student responses to particular topics and tasks. As Ball and Cohen (1996) suggest, by designing guidance materials in this way, with the construction arena in view, it is possible to see opportunities to use teacher guidance to assist teachers' learning and practice, offering them more opportunities to learn in and from their work.

We note that the enquiry to inform the development of the 'Common issues' table in the guidance materials reflects the important 'kyozaikenkyu' phase of the Japanese lesson study cycle, in which teachers engage in classroom research into the critical role of stimulus material in generating student mathematical activity. Doig and Groves (2012) point to how this is often overlooked outside of Japan and perhaps the 'Common issues' table provides teachers with an alternative means of gaining the insight that their own research, through the use of a lesson study approach to examining their practice, might provide. It appears that the role of the 'Common issues' table in anticipating likely student responses provides an effective means of mediating important decisions within the design arena, in a practical way, that has utility and purpose to classroom teachers, both as they prepare for the lesson and in their moment-to-moment decision making during its enactment in the classroom. As remarked earlier, we found repeated use of the guidance materials and enactment in the classroom led to deeper engagement with the guide, with teachers then reporting a better understanding of the design principles for lessons.

The finding that teachers were prepared for students to engage in a problem-solving task without support in understanding the context in which the problem arises, has some significance. Although the 'Common issues' table does give advice about how to proceed in response to students' initial and subsequent attempts at problem solving, it appears in this initial phase of the FAL, teachers provide minimal support. Research has shown that lack of specific guidance about what teachers could do to support students has led some teachers to conclude that they should never directly tell pupils anything (Leinhardt, 2001; Smith, 1996) and it may be that the teachers involved in the study were so focused on not telling students *how to do the task* that they omitted to check that students *understood the context of the task*.

Producing guidance materials that encourage teacher engagement is challenging and as has been outlined in this paper, often requires a lengthy process of revision and development over time. As the materials are developed, the question as to how such teacher guidance can be made accessible to teachers in a way that is useful, without being lengthy and onerous still remains. The role of online versions, that enable teachers to be selective about what they read, may provide a solution. Whilst the use of technology in presenting guidance materials was not a feature of the design of the FAL teacher guides, its potential in this area is recognised. The content of the materials is still of paramount importance, regardless of the presentation format. Whilst it is the enacted lesson that counts, the design of guidance that prompts teachers to think about the classroom enactment and possible responses from students *in advance of* the lesson, has an important role to play in encouraging teachers to present, rich, multi-approach problems for students to solve in their mathematics classrooms.

Acknowledgement

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Using context and models at Higher Level GCSE: adapting Realistic Mathematics Education (RME) for the UK curriculum

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Since 2003, staff at Manchester Metropolitan University (MMU) have been involved in a number of projects related to Realistic Mathematics Education (RME). This originally involved trialing materials with 11-14 year olds and then, in collaboration with the Freudenthal Institute, writing materials for Foundation level GCSE. In 2012, these materials were published by Hodder Education as a series of books called *Making Sense of Maths*. Classroom trials of the original materials showed an increased willingness of students to discuss and engage with their mathematics, and to attempt to 'make sense' of what they were doing rather than simply to try to remember taught procedures. The results from the trials led, in 2009, to a further project designing materials for Higher level GCSE. The focus here is on the development of these materials, and how we have interpreted the original design principles of RME for UK schools. In particular, we focus on the use of context, the notion of progressive formalisation and the use of models. We provide excerpts from the materials that exemplify these principles and discuss the issues for teachers attempting to integrate this approach into an examination driven curriculum.

Keywords: secondary, context, models, formalisation, teachers.

Background

The Freudenthal Institute, University of Utrecht, was set up in 1971 in response to a perceived need to improve the quality of mathematics teaching in Dutch schools. This led to the development of a research strategy and to a theory of mathematics pedagogy called 'Realistic' Mathematics Education (RME) which is now used in over 80% of Dutch schools. The principles underlying this pedagogy were strongly influenced by the notion of mathematics as a 'human activity' (Freudenthal, 1983). The Netherlands is considered one of the higher achieving countries in the world in mathematics (TIMSS, 1999, 2007; PISA, 2012) and consequently a number of other countries have attempted to design curricula based on the principles of RME.

RME in the USA – Maths in Context (MiC)

In 1991, The University of Wisconsin (UW), funded by National Science Foundation, USA and in collaboration with the Freudenthal Institute (FI), started to develop the MiC approach based on RME. The initial materials, drafted by staff from FI, were first published in 1996 after extensive trialling. The scheme has undergone several revisions since then, and is currently being modified in the light of the new US Core standards.

RME in the UK – Making Sense of Maths

In 2003, the Centre for Mathematics Education at MMU purchased a set of MiC materials and trialled them with Year 7 classes in a local school. This led to a three year trial of the materials in 12 Manchester schools. (See Dickinson and Eade (2005) for an account of the trials and findings, and Hanley and Darby (2007) for an account of research into the changes in teachers involved in the project.) The reaction to the materials was extremely positive, and there was a real sense that this approach was worthy of continued exploration

As a consequence of the MiC trial, a new project, *Making Sense of Maths* (MSM), was launched in 2007 aimed at KS4 pupils. This began with Foundation tier pupils and was then extended to include both tiers of the new two-tier GCSE. This project was in collaboration with the Freudenthal Institute in The Netherlands, working with Mathematics in Education and Industry (MEI) in the UK.

The focus of this paper is on the development of the MSM Higher level materials and how we have interpreted the original design principles of RME for these materials. The issues of context, models and progressive formalisation have already been explored in relation to Foundation level work (for example see Dickinson et al., 2009), but these created some very different challenges as we began to work at a higher mathematical level. We will provide one example where the design principles are relatively easy to see, and one where we initially struggled to adopt an RME approach. For a more general discussion of RME, and other design principles associated with it (for example, interactivity in the classroom), see Treffers (1991).

Theoretical Framework

Use of context

The use of context in mathematics teaching is not a new idea. Contexts are often used as a means of providing interesting introductions to topics, and then for testing whether or not pupils can use their knowledge to answer ‘applications’ questions. In RME, however, context is used not only as a means of applying previously learned mathematics but as a means of constructing new mathematics (Fosnot and Dolk, 2002). In this respect, context is seen as both the starting point and as the source for learning mathematics (Van den Heuvel-Panhuizen, 2003), and contexts are carefully chosen to encourage students to develop strategies and models that are helpful in the mathematising process. These contexts need to be experientially real to the students, so that they can engage in purposeful mathematical activity. At Foundation level, we regularly found that we could use real-life scenarios as our context, but at Higher level these were not always immediately available. In this situation, we often found ourselves looking at the historical development of the topic, and attempting to see where the need for the mathematics first arose. This not only provided us with a range of rich contexts but is also consistent with Fredenthal’s notion of ‘guided reinvention’, which is central to RME (Freudenthal, 1983).

Use of Models

In RME, models bridge the gap between informal understanding connected to ‘reality’ on the one hand, and the understanding of more formal systems on the other (Van den Heuvel-Panhuizen, 2003). Crucially, a model that may initially simply be a model of a student’s mathematical activity has the potential to develop into a model that will support and facilitate increasingly abstract mathematical reasoning. Many such models which emerged at Foundation level are immediately recognisable (eg. the double number line) although others would often be attributed a different label in the UK (e.g. repeated subtraction, 'fitting in'). These models could sometimes be adapted for use at Higher level but, where this was not the case, it was often initially difficult to see what models would be appropriate in terms of the qualities discussed above.

Progressive formalisation

The term 'progressive formalisation' describes a learning sequence that begins with informal strategies and knowledge, and then develops into pre-formal methods that remain linked to concrete experiences, models and strategies. These models and strategies then develop progressively into more formal and abstract mathematical procedures (Webb and Meyer, 2007). This is strongly connected to the notion of 'vertical mathematisation' (Treffers,1991). Crucial to this process is a shift in how students view and use their models – from a 'model of' a situation (where the model is closely connected to the context) to a 'model for' (where the model is more abstract and can be used as the basis for mathematical reasoning) (Streefland, 1991).
An example of this, where the sharing of sandwiches evolves into a fraction bar and then a double number line, is shown below.

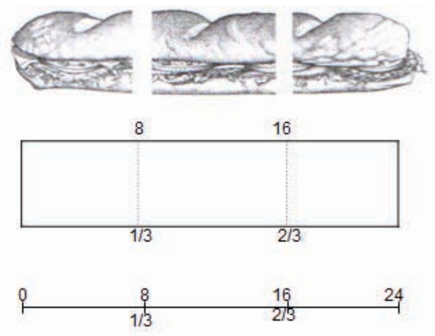


Figure 1. Progressive formalisation of models (Dickinson and Eade, 2005).

The issue of progressive formalisation is crucial, both to RME and to its adaption to the UK classroom. Clearly, a teacher wants students to be able to understand formal methods and procedures, nowhere is this more evident than at Higher level. RME does not shirk this responsibility, but offers a very different story of how students and teachers work towards this aim. For example, in relation to the teaching of fractions, the formal notion of equivalence has been categorised as being 'on the horizon' (Fosnot and Dolk, 2002) or the 'tip of the iceberg' (Webb, Boswinkel and Dekker, 2008).

Theory into practice

In some areas of the curriculum, Higher level topics are extensions of topics already met at Foundation level. Often, in these cases, the contexts and models used at Foundation level could also be extended.

So, for example, in a unit of work on Data Handling, the emergent model was a dot plot of some discrete data. The contexts used for this included estimating ten seconds, recording 100m Olympic finishing times and comparing long jump lengths. One example of the latter is shown below. This also provides a nice example of the 'model of … model for' interplay; in the diagram below, some students see the model as a representation of the 'sand pit' and have even been known to put stick drawings instead of dots (model of), while others now see this as a mathematical representation of the data (model for).

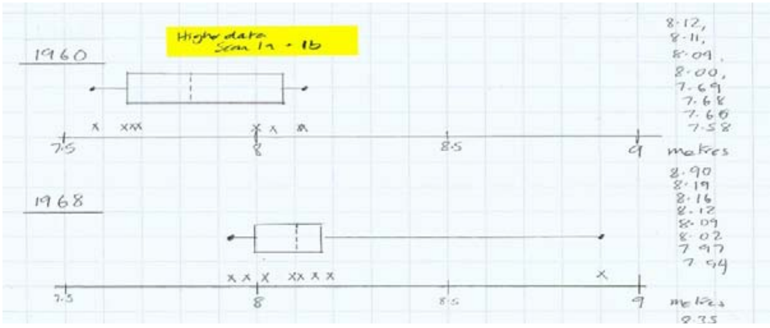


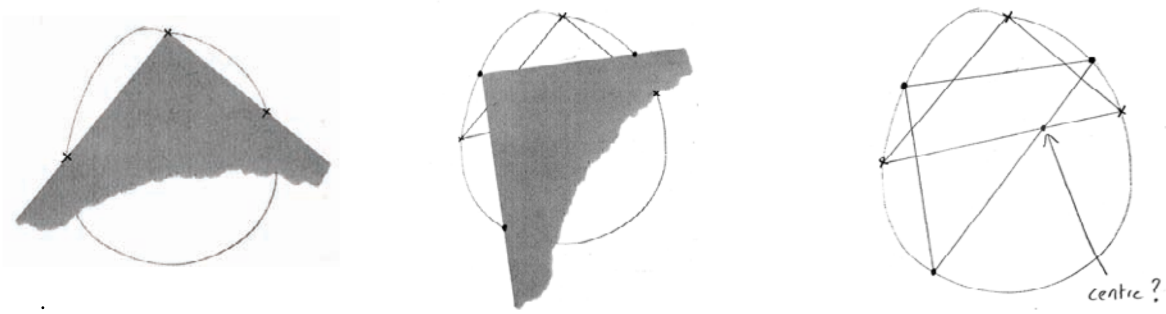
Figure 2. Using the dot plot as a 'model of' and as a 'model for'.

This model could now be extended to become a 'box plot', with consideration of measures at a higher level. Interestingly, within the context of comparing different people's estimation of 10 seconds, it became apparent that 'consistency' was very important, and from this, students began to develop measures akin to mean deviation. So in this respect, the context and model not only help to develop the mathematics on the Higher level curriculum but can go further than that.

In other areas of the Higher level curriculum, however, it was initially more difficult to see a learning trajectory based on RME. For example, part of the Shape strand concerns 'circle theorems'. No 'context' or 'model' was immediately obvious here, until we began to look at the historical development. From Thales' theorem regarding the angle in a semi-circle, it became apparent that this was also used to 'test' for a perfect circle; from this developed the context of a circle drawing competition as shown below.

> Mrs Ayad the mathematics teacher is setting an end of term challenge. She has a prize for the person who can draw the best circle using only a pencil. As you can imagine, there was great excitement in the class! Mrs Ayad had an unusual way of judging the circles. She placed a piece of paper on a circle so that the 90° corner was on the circumference of the circle.

Figure 3. Testing for a perfect circle, adapted from Thale's theorem.



She then did this again within the same circle

She said that where the two lines across the circle meet must be the centre of the circle

Through this the idea of the angle in a semi-circle emerges, but the formal maths comes from the mathematical activity of the students, and not the teacher!

Interestingly, using a fixed angle on a piece of paper can also be used to develop other circle theorems including ‘Angles in the same segment are equal’, and ‘Tangent and radius meet at 90°’. The fixed angle then becomes a model, in that it is bridging the gap between students’ informal ideas and the formal mathematics. In the classroom, some students wish to continue to use the piece of card whenever possible, some simply refer to it, while others begin to draw angles in the circle. These represent different stages of the journey from using a ‘model of’, to using a ‘model for’ and hence different stages of formalisation.

Research methodology

The most appropriate methodology for the purposes of developing RME based Higher level GCSE materials was to use ‘design study’ (Cohen, Manion and Morrison, 2011). In this sense we were designing a product that would be tested in real conditions, under observation and re-developed to take account of the findings. Fourteen teachers took part in trialling some of the six available modules, over a three-year period from 2010 to 2013. Five teachers had been project teachers in the earlier Mathematics in Context project; they had a strong sense of how students progress through the use of contexts and models with Foundation level topics. Five teachers had recently trained at MMU, four teachers were completely new to RME.

Teachers attended half-termly meetings after school hours where they were introduced to the materials and asked for feedback on the modules they had tried. Other forms of data collection included lesson observation and teacher interviews. Where possible lessons were video recorded and all interviews were audio recorded. This provided a large quantity of data which was later analysed in relation to the use of context, the use of models and a third emerging issue; that of teacher development.

Issues relating to the use of context

Experienced RME project teachers were able to recognise the importance of students’ being able to access the chosen contexts. For example, when a Ferris wheel is chosen as the starting point for a module on trigonometry, the students are first asked to imagine themselves on the Ferris wheel and to decide where on the ride their height would increase the most and the least. It emerged that students were finding it difficult to distinguish between vertical height gain and the circular direction of movement. Students were unable to estimate the vertical height gain as a distance because they did not know what distance this referred to in their diagrams. Rather than resorting to marking on the distance, the teacher went back to the context, video footage was shown and the students were asked to imagine dropping a stone out of their capsule; how would it travel, where would it land? This gave students an image of vertical height that they could now relate to.

On another occasion, the context of filling 3D Perspex objects with orange juice was used as part of a learning trajectory designed to enable students to access the relationships between formulae for the volume of a cube, a pyramid, a cone and a sphere. Through ‘fitting in’ the liquid, students could check their estimates for how many square-based pyramids it would require to fill a cube with the same sized base. The teacher with less experience in RME chose not to demonstrate this. His year 10 class were soon to do an end of year test and he felt under pressure to move quickly

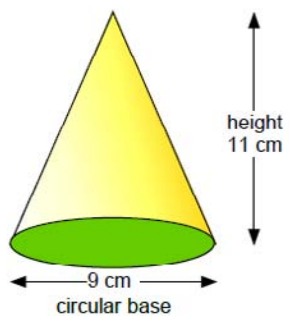


Figure 4. Formal representation of a cone

to the end of the module whereby students would engage with the traditional formal content of finding volume by substituting into the correct formulae.

We observed a student puzzling over finding the volume of this cone. When asked to describe what he saw in this picture, the student said he saw an oval shape and a triangle. He was unable to relate this representation to a real life object and would appear to be viewing this problem as one of area. Part of the sense making process in RME is to move slowly from the context (in this case the 3D Perspex cone) through informal representations of the context (student's own hand drawn 2D representations) to the more formal representation shown above (Figure 4). It would seem at Higher level, teachers inexperienced in the use of RME, driven by the need to cover formal content, at times chose to dilute this aspect.

Issues relating to the use of models

One of the issues at Higher level is that the sense making features of a particular context or model may not apply to the degree of abstraction required. For example, although the bar model works well for numerical fractions, it becomes extremely challenging to represent the addition of algebraic fractions on a bar model. The area model for multiplication, where numbers are contextualised as lengths, is much harder to conceive for algebraic expressions of the form $(x - 2)(2x - 6)$, where the use of negative numbers would imply cutting back into line lengths already drawn. Our teachers dealt with these issues in a number of ways. In schools where classes had previous experience of working with RME models at KS3, teachers were able to build on the previously established context to model experiences and extend these to algebraic thinking. For example, students who were able to conceive the addition of $\frac{1}{3}$ and $\frac{1}{5}$ as drawing a bar which would partition into both 3 and 5 parts, when asked how many pieces would be required to add $\frac{1}{a}$ and $\frac{1}{b}$ referred to needing 'a lots of b parts'.

In another case, a project teacher was so delighted with the way the context of buying hats and T-shirts had prompted sense-making models for solving simultaneous equations, that she persisted with questions relating to the context even when the equations were contextually difficult to interpret. Students faced with solving $2x - 3y = 12$ and $2x + 7y = -8$ were asked, 'Where are you getting more?' 'How much more?' and 'How much more does it cost you?' These questions acted as prompts to use the models students had previously developed.

Where classes had little experience of RME, some teachers understandably felt unable to invest time exploring situations where the models emerged from the context. Instead, they introduced their students to models as methods in their own right. For example, students were taught to expand 2 brackets by drawing a grid. The grid gave no attention to the relative 'size' of the various terms and when the material required students to describe where in the picture the 'x^2' could be seen, it was clear that they were thinking not in terms of area, but in terms of the procedure used to generate the 'x^2' term.

Teacher development

Teacher interviews conducted at the end of one or two years' engagement with the material revealed a number of common threads. Teachers were extremely positive about the use of context, not only as a motivational tool, but also as a memorable point of reference long after the module was completed. In revision sessions, teachers described themselves and their students as using the language of context to evoke imagery and associated methods, in addition to their more usual practice of revisiting routines.

The role of discussion featured heavily in the teachers' appraisal of what had changed. One teacher new to the RME approach described how discussion had enabled him to 'stop driving some elements of the lesson … you can allow people to have time to find things out for themselves, to discuss what they're doing.' He goes on to acknowledge his changing teacher identity: 'It's a lot less about me driving them through something, explaining something for them to regurgitate …. I just kind of perch and find interesting things that are going on, then periodically stop people and say, "right, let's have a look at what you're doing". This teacher would appear to be beginning to develop the complex teacher role of framing student contributions.

By contrast, an experienced RME teacher also commented on the rich opportunities for discussion but acknowledged the skill required by the teacher to 'steer them in certain directions.' She would appear to be aware of the importance of the teacher who is operating in an RME frame pro-actively steering student ideas and representations towards more formal thinking. (Gravemeijer et al, 1998)

A conflict of interest in the current educational climate

Although teachers viewed progressive formalisation and the increased status of discussion as positive features, they also expressed concern about how these would be viewed by the current monitoring bodies. Schools' interpretation of Ofsted guidelines and the National Numeracy Strategy (1999) has led to teachers being encouraged to demonstrate progress for individual students within the course of one lesson, to itemise objectives at the start of a lesson, and to ensure a lesson has 'pace'. Teachers felt there was a disparity between these requirements and RME based lessons where students were given time to express opinions, to share a range of solution strategies and to slowly make progress towards formal thinking. In addition, the then current practice of entering students for modular GCSE, plus repeated early entry, even for Higher level students, meant a lot of the KS4 curriculum time was spent preparing for tests and examinations. This on occasion resulted in teachers withdrawing from particular modules, and is an issue that we continue to work on. The increased focus on rigour in the new 2014 National Curriculum is likely to increase the pressure on teachers to deliver formal methods quickly, to emphasise acquiring the procedure and so create a further source of potential conflict with an RME based approach.

Conclusion

In conclusion, we believe that it is possible to develop materials for Higher attaining students that fit within the design principles of RME. To do this, however, requires an extension to how we might traditionally interpret the notions of context, models, and progressive formalisation. Some contexts and models extend from Foundation to Higher level, but this is not always possible. We found that looking at the historical development of topics often suggested a context and model, as in the work on circle theorems. In other areas, previous learned mathematics may become the context for new work. In terms of formalisation, it is as important as ever that students are allowed to work informally and 'make sense' of situations before developing more formal mathematics. On the other hand, teachers at Higher level report feeling under pressure to move to formal mathematics quickly, and as curriculum designers, we clearly need to reflect this in the materials we produce. Our work at both Foundation and Higher level suggests that, for many teachers, working with RME can be a challenging (though ultimately very rewarding) experience. Although the materials are designed to support the teacher in the classroom, Professional Development is essential if the materials are to be used effectively. This is particularly true where teachers feel under pressure from examinations and other external factors. We believe that an RME based curriculum is suitable in the current UK climate, but that teachers need training and direct classroom support for this to be truly effective. The nature of this CPD, and how it can be successfully managed, is an area for more study.

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How a primary mathematics teacher in Shanghai improved her lessons on 'angle measurement'

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We report on one component of a study of school-based teacher professional development (TPD) in Shanghai, China. Here we focus on an experienced primary teacher who is teaching the topic of angle measurement to 10 year-olds. Using data from the teacher's original lesson plans, her modified lesson plans, together with an expert teacher's advice and the teacher's reflections on her lesson design, we illustrate how the support of an expert teacher enabled the teacher to improve her instructional practice. This was by supporting her in thinking explicitly about the traditional classroom practice with which she was familiar and in building her 'wisdom of practice' within the context of instructional reform taking place in China.

Keywords: professional development, mathematics teaching in China, curriculum and pedagogy reforms, angle measurement

Introduction

In the 2012 PISA (Programme for International Student Assessment) study Shanghai-China (SH) has again the highest scores in mathematics, with a mean score of 613 points – the equivalent of nearly three years of schooling above the OECD average (OECD, 2013). As such, the ways in which mathematics teachers develop professionally in jurisdictions like Shanghai is of much interest. In China, the latest version of the National Mathematics Curriculum Standards (briefly called the Standards in this paper) encourages teachers in the country to develop teaching strategies to tackle the complex relationship between the teacher's leading role in the classroom and students' independent learning through classroom activities (Ministry of Education (MoE), 2011). Moreover, the Standards (MoE, 2011: 43) suggest that teachers "emphasise the implementation of the curriculum goals (e.g. students' basic knowledge and basic skills, mathematical thinking, problem solving, and interests and attitudes) as a whole through classroom mathematics activities." One of the most challenging issues for teachers in China is to adjust to this revised role and to implement the curriculum goals as a whole. There is an equivalent challenge for teacher educators and professional developers in supporting teachers as they update their knowledge and skills.

In this paper we report on one component of a school-based teacher professional development (TPD) study taking place in SH. Here we focus on a primary teacher as she works on how to improve the way she teaches the topic of angle measurement to her class of 10 year-olds. In this work the teacher is part of the school-based TPD community that includes researchers and an 'expert teacher', the latter being one of the leading primary classroom teaching research specialists from the city-centre school district in SH. Our research question focused on how the teacher utilised advice from the 'expert teacher' to improve her pedagogic thinking and instructional practice in her classroom.

The school-based coaching approach in teacher professional development

A range of recent research attempts to develop new insights into the effectiveness and impact of 'coaching' in school-based TPD. In the United States, for instance, Guskey and Yoon (2009) point out that the professional development efforts that brought improvements in student learning focused principally on ideas gained through the involvement of outside experts who presented ideas directly to teachers and then helped facilitate implementation. Furthermore, Obara (2010) highlights two aspects of coaching that are said to have positive impact on teachers' changes in their teaching practice: (1) being on-site and (2) encouraging collaboration and reflection. Nevertheless, while a teacher's beliefs about their classroom practice might undergo change as a result of the coaching process, Neuberger (2012) notes that such changes may not necessarily be stable. In Norway, Jaworski (2003) proposed another approach to teacher coaching, that of establishing a co-learning community. Jaworski is careful to distinguish between a community of inquiry and a community of practice, stating that in a community of inquiry "participants at all levels are learners" (p. 256). In the latter, what is learned can differ from one person to another, and can depend on the role a particular person might have, whereas in co-learning the learning of one participant is dependent on the participation and learning of other members of the co-learning community.

In the case of China, the experts' input is highly valued by teachers in a variety of forms of tschool-based TPD, such as apprenticeship practice (Huang, Peng, Wang and Li, 2010; Huang, Su and Xu, 2013); teacher research group (TRG) (Yang, 2009); and public lessons (Han and Paine, 2010). In view of the significant curriculum reforms taking place in China over recent years, Gu and Wang (2003) particularly highlight the critical role of experts in leading teachers to update theoretical ideas through what they term the 'Action Education' (AE) model ('Xingdong Jiaoyu' in Chinese). Here, 'experts' refers to university researchers, specialists in research on teaching (usually from the school district level), and expert teachers from inside and outside the school. In reporting on the effectiveness of such teacher/expert collaboration, Huang and Bao (2006: 292) quote a participating teacher's teaching diary as follows: "The advantages of collaborative lesson planning design are (1) to help me form an innovative teaching idea and (2) to find effective ways to handle difficulties by learning from other experienced teachers and experts".

Noticeably, in a review by East China Normal University of various Chinese TPD training programmes Yu (2009) reports that the AE model matches well the professional development needs of secondary 'backbone teachers' (who are excellent in teaching, for more see Han and Paine, 2010). However, little is known about the kinds of uncertainties ordinary school teachers may have during such a learning process, and of the kinds of teacher-expert interactions which may be effective in helping teachers to tackle such uncertainties. It is the expert teacher's direct interactions with the teacher in her learning process through the AE model that is the focus for this paper.

The study

Our school-based TPD study is being conducted in a laboratory school located in Qingpu district, a western suburb of SH (see also Ding, Jones and Pepin, 2013). The main project is a design-based experiment to study a particular model of professional development, akin to the AE approach by Gu and Wang (2003) that aims at developing the teacher's professional knowledge by absorbing and building on the accumulated "wisdom of practice" (Shulman, 1986) through iterative cycles of teachers' lesson planning, implementation, post-lesson reflection and lesson re-implementation.

The participant groups of the study were: four researchers (the four authors), an expert teacher (Mr Zhang), two teachers (one in Grade 3 (G3) and the other in G4) each with more than ten years teaching experiences in primary mathematics at the time of the study and twelve mathematics teachers from the school's mathematics teacher group (from G1 to G6, ranging from newly-appointed teachers to teachers with about ten years teaching experience). In this paper we focus on the G4 teacher named Yanzi, a pseudonym.

We analysed Yanzi's teaching notes and lesson plans reflecting on her interactions with the expert teacher as a case study (Yin, 2013) of what an experienced teacher can learn from interacting with an expert teacher. The data collection strategies and sources included the following:

(1) Field notes:
- The major data source for this study was Yanzi's lesson plans and her teaching notes before and after Mr Zhang's interventions.
- Field notes of Mr Zhang's interventions in helping the teacher to overcome her uncertainties in planning lessons (also audio recordings of all the conversations).
- Teacher reflection notes about the 'lesson plan improvements'.

(2) Video- and audio-taped conversations and study meetings with Mr Zhang and the teacher/s and the researchers, before and after lessons; and the teacher's lessons.

In terms of data analysis, we focused on Yanzi's interactions with Mr Zhang, and her progress over the different stages of the first cycle of the study. As shown in Figure 1, the case teacher worked through the several stages of the first study cycle (June 2012 – July 2013).

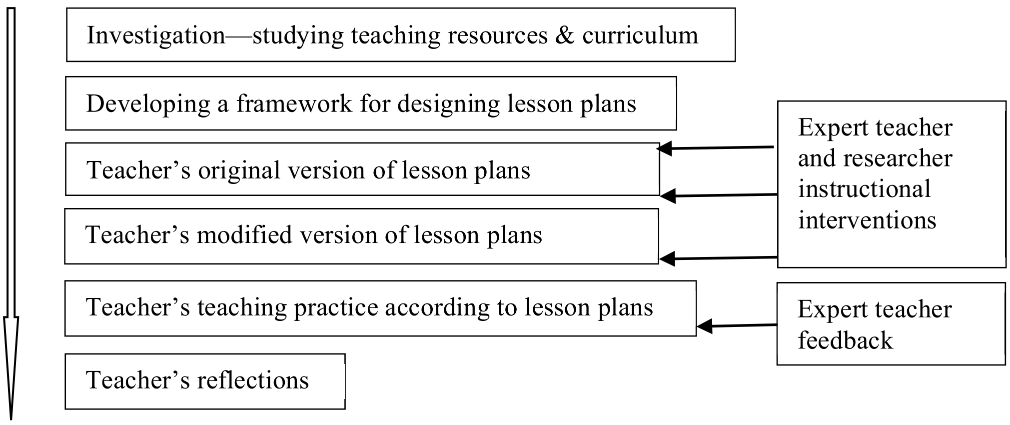

Figure 1. The first cycle of the study

During the first phase of our analysis, the first author examined all the available data across the stages of the study (see Figure 1), including relevant curriculum and textbook materials, teachers' initial lesson plans and improved lesson plans according to Mr Zhang's suggestions, transcribed lessons and meetings, field notes, and the teacher's written reflections. The first author then translated the key data from Chinese into English.

In analysing the data, following the AE model (Gu and Wang, 2003), as the research team we chiefly focused on Yanzi's self-reflection on her original lesson plan and Mr Zhang's interventions that helped her to revise her lesson plan and implement it in classroom. We identified two preliminary coding categories to illustrate the main uncertainties Yanzi had, and the suggestions Mr Zhang provided to help Yanzi overcome her uncertainties and develop her pedagogic thinking and practice in her classroom: (1) 'teacher's uncertainty of planning an activity lesson'; (2) 'using worksheet in mathematics activity'. Subsequently, the preliminary categories were used to trace Yanzi's modifications (or not) in her pedagogic thinking in her reflection notes, lesson plans and lesson instruction practice according to Mr Zhang's interventions. In what follows we illustrate how Mr Zhang helped Yanzi to improve her pedagogic thinking and instruction practice through 'reflective lesson design'.

Findings

Yanzi initially designed a sequence of four lessons (each 35 minutes long) on the topic of angle measurement. The four lessons respectively focused on the 'concept of angle', 'angle classification', 'understanding the design of a protractor', 'using a protractor to measure angles'; these being the key content in the SH textbook. To trace the teacher's progress of pedagogic thinking and practice, together with Mr Zhang's interventions, here we focus on Yanzi's initial and revised lesson plans of the first lesson.

Overcoming the uncertainty of planning an activity lesson

In planning the first lesson, Yanzi took Mr Zhang's suggestion of using the activity of playing string puzzle (PSP), a popular game that young children like to play, to introduce the learning topic of angle to the students. The lesson plan was structured as follows:

1) *Introduction of problem context (IPC)*. Here, the activity of PSP was introduced to the class. Several key questions were highlighted in the lesson plan. For instance, can you play the string puzzle game? What figures do you find in the game? What is an angle according to your playing experience?
2) *Investigation of new knowledge (INK).* Three activities were planned here. Activity one was to lead the students to review the angle names such as obtuse, acute and right angles that they previously learned. Activity two was to guide students to count angles in different conditions. For instance, count angles when two lines are crossed, when two lines are crossed by the third line, when angles have a common point, and when angles have a common side. Activity three was to explore angle addition and subtraction.
3) *Consolidation by exercises (CE).* An immediate assessment of students' learning was intended by a set of routine exercises from the SH textbook.
4) *Conclusion (C).* Here, the question "What did you learn today?" was highlighted.

Mr Zhang's interaction with Yanzi to develop her pedagogic thinking about designing classroom activity is summarised in the following three key points:

1. To distinguish two types of learning experience in a mathematical activity: thinking and reasoning experience that is quite abstract and behavioural and operational experience that is concrete.
2. To use the '*Shen Tou*' method to construct purposefully various types of activities in a lesson, or across a sequence of lessons. This teaching method entails establishing a particular relationship between the teacher's purposeful

instruction and students' gradual learning progress, from being unfamiliar at the beginning to eventually acquiring particular skills or rough understandings of a method in certain area of mathematics (for details of Chinese expert teachers' use of the *Shen Tou* method see Ding, Jones and Zhang, 2014). In particular, the teacher should have a clear instructional consideration of what specific experiences students need in a specific activity for learning related concepts or methods later on.

3. To make a conclusion of the activities in order to enhance the representations of the concepts/experiences to be learned/gained.

Based on Mr Zhang's suggestions, Yanzi revised her first lesson plan as follows:

1) *Preparation activity*. The PSP activity was introduced to the class in a worksheet. In the worksheet, several tasks were listed. First, two pictures were given to students to observe and to compare (Figure 2). Students were asked to make the same pattern in the PSP activity and then to draw the angles they found in the patterns on the worksheet.
2) *Angle discovery activity*. Two activities were planned here. Activity one was to look for different types of angles (e.g., obtuse, acute and right angles), and activity two was to observe the relationships between angles (e.g. common point of angles, and common side of angles).
3) *Moving from experiences to exercises*. Students were expected to work on the paper-and-pencil exercises selected from textbooks according to the activities they had just played.
4) *Conclusion activity.* Teacher used drawings, symbols, letters, and words to make a summary of the representations of angles students conveyed across the previous two activities.

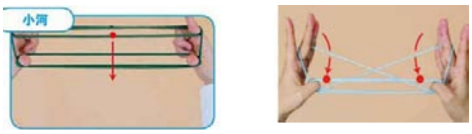

Figure 2. Two pictures of the playing string puzzle (PSP)

Using worksheet in mathematics activity

In her reflection notes, Yanzi had uncertainties about planning mathematics activity lessons:

> I was not able to develop concrete thinking of how to actualise the theoretical ideas of the activities in the classroom when I made the initial lesson plans. So the initial lesson plans could only be called plans of learning procedure. Teaching plan concerns about what teaching methods to be used and what kind of exercise and problems to enrich and to develop the learning process of students. Thus, teachers must consider the details of teaching such as teacher's questionings, hypothesising students' learning responses, what kind of knowledge foundation the teacher's question or the learning activity is based on, etc. So teaching plan is a guidance to enable teachers to have concrete ideas to conduct instruction. For instance, in the initial lesson plan of lesson one, the lesson structure (*IPC-INK-CE-C*) is a learning procedure I designed. … At that moment, I had not yet developed ideas about how to guide students to complete the learning procedure. I had little idea how to design the following aspects of the learning procedure. For example, 1) how to design the concrete activity? 2) how to design questions to guide students in activity? 3) how to design the exercises so to immediately gain students' learning feedback? (Yanzi's reflection note, 8 June 2013)

According to Mr Zhang's interactions with Yanzi, he guided Yanzi to develop her pedagogic thinking of organising mathematics activities in lessons by addressing in particular the following three strategies: (1) enable students to develop a broad understanding and 'free thinking' in the activities (not merely focus on understanding subject knowledge); (2) cultivate individual students' learning experiences as the core of teaching; and (3) use the '*Shen Tou*' teaching method.

Moreover, Yanzi conveyed the benefits she gained from Mr Zhang's constructive suggestions about the use of a worksheet in the activity instruction.

> In each activity, students were given a task worksheet. It's the first time for me to use it in my classroom instruction. In the traditional instruction, there are also learning tasks. However, the tasks are mostly stated for teachers to consider in the teaching plan, or they are explained in the teacher's oral guidance in the class. That is, the tasks are implicit in each step of learning. It is now the first time to make the tasks explicit in a worksheet. It would engage students in independent learning. In the traditional instruction, teachers often use tasks to gradually guide students to make gradual progress in learning. (Yanzi's reflection note, 8 June 2013)

Discussion and conclusion

In our study we are working towards building up a design framework to guide teachers to update their knowledge and develop flexible pedagogic thinking about teaching mathematics activities. In so doing, we aim to help teachers to enhance their teaching skills in order to be able to effectively implement important curriculum goals as a whole (see the list of the basic goals in the introduction section of the paper). In the process of building up our design framework, this paper points to the crucial role of the local expert teacher in helping teachers to overcome the gap between the teaching norms advocated in the reformed curriculum and teachers' daily-life practice in their classroom.

As shown in the foregoing session, Yanzi was uncertain about how to deal with two teaching 'norms' in her lesson plans: classroom activity and mathematical activity. That is, Yanzi's uncertainty was not about recognising students' active learning role in general classroom activities, but about how to deal with the complex relationship among three core elements in a classroom activity: students' independent learning role, teacher's teaching role and mathematics. Thus, Yanzi initially planned the teaching phases of the first lesson according to the traditional teaching norms like *IPC-INK-CE-C*. As demonstrated by Shao et al. (2013), in writing their lesson plans, teachers in China traditionally refer to what are called the Kairov five teaching phases (named after the Soviet educator since 1949). The teaching norms that characterise the five phases are: *organising teaching*; *reviewing learned knowledge*; *introducing new content*; *consolidation and summary*; *homework assignment* (p. 17). In our study, we introduced teaching norms according to our study of overseas textbooks (see Pepin and Haggarty 2001) for helping teachers to enable students to become independent in accumulating learning experiences in classroom activities: for instance, the norms of *Preparation activity*, *Independent discovery activity* and *Summary from activity experience* were deepened by studying foreign textbooks. We noted that while guiding the teacher to understand the new teaching norms of classroom activity, the expert teacher simultaneously made explicit to our case study teacher the careful use of the traditional teaching method (e.g. *Shen Tou*) to develop students' basic knowledge and skills in mathematics activity. Such findings lead us to argue that it is crucial to make explicit the mathematics pedagogy that teachers traditionally appreciate and practice within the context of instructional reform, in order to deepen teachers' understandings of ways to develop classroom pedagogy and improve student learning.

Moreover, Yanzi confessed that she was unclear about how to organise mathematics activities in her initial plan: for instance, she had questions about which teaching methods to use in designing an activity lesson; which questions to be asked; which exercises to be chosen and prepared/arranged; and what the students' existing knowledge and thinking related to the mathematics topic to be learned might be. To assist Yanzi in revising her lesson plans, Mr Zhang explained to Yanzi how to organise three mathematics activities in the lesson with concrete examples of teacher's questions and exercises. To help Yanzi to develop her teaching strategy of using one game (the PSP in the first lesson plan) across the three mathematics activities, Mr Zhang constructively guided Yanzi to use the worksheets with a list of learning tasks to enable students to develop independent learning through these activities. Mr Zhang also drew the teachers' attention to the possible use of various types of representations to enrich students' learning of mathematics concepts/experiences embedded in the activities. Noticeably, Mr Zhang applied considerable theoretical ideas both from general teaching-learning theory and the specific theory of mathematics pedagogy. Yet, Mr Zhang stated that he "can guide teachers to conduct mathematics activities teaching in their classes, but (he has) very vague ideas of the theory that underlie (his) instruction/advice to teachers". In developing a well-designed TPD study, the challenge we now face is to develop a deeper understanding of the local theories held by expert teachers, both in terms of their thinking and theoretical constructs, as well as of their practices.

For Yanzi, this paper shows how the support of an expert teacher enabled her to improve her instructional practice by supporting her in thinking explicitly about the traditional classroom practice with which she was familiar and in building her 'wisdom of practice' within the context of instructional reform taking place in China.

Acknowledgement

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Using Facebook as a tool in Initial Teacher Education

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This report leads from findings of an earlier research project in which pre-service mathematics and science teachers identified areas where they felt they needed greater support during their training year. In September 2011 a closed group was set up on the social network site Facebook as a support mechanism for a cohort of pre-service secondary mathematics teachers. The rationale was that the Facebook group, which was established as a 'secret staffroom' would provide a common, yet secure area where the pre-service teachers could share ideas and resources as well as engage in discussion about their progress on the course. Analysis of the data collected from the interactions that took place identifies the frequency and ways in which the social network site has been used during their training year and first year of teaching. These findings are contrasted with the results of a series of attitudinal questionnaires completed by the participants at various stages of these two years. The results indicate that Facebook can be used to support pre-service teachers; however there remains a challenge in using social networking to support teachers in their first year of teaching and beyond.

Keywords: STEM, social network, teacher training, mathematics

Introduction

In 2011, Edwards & Hyde carried out a research project to examine the reasons why secondary science, technology, engineering and mathematics (STEM), specifically mathematics and science, trainees may be unsuccessful on teacher training courses (Edwards & Hyde, 2011). One of the four common themes identified through individual interviews and discussion groups as having a major impact on the progress made by trainees was isolation on teaching practice. A number of factors contributed to a trainee feeling 'alone': being the only trainee in the school, perception of insufficient support from their school mentor and not having someone within their family or friendship groups who understood the pressures associated with undertaking a teacher training course and with whom they could discuss their concerns. The structure of the Post Graduate Certificate in Education (PGCE) course does not help to address the problem of isolation as trainees only spend a short period of time in university sessions before starting their initial teaching placement. There is insufficient time to form any common support group and whilst social network groups had previously been used by some trainees they were not all inclusive. In addition, the University's Virtual Learning Platform, Blackboard, which features a discussion board was seen by trainees to be restrictive.

Social network sites have been present on the internet since the late 1990s, with Facebook being founded in 2004. Since that time there have been a number of studies conducted exploring their potential in education from elementary (Lee et al., 2013), to doctoral level (Ryan, Magro and Sharp, 2011). Ellison, Steinfield and Lampe (2007) argue that using social network sites could have positive gains in social capital for individuals allowing them to draw on resources from other users of the network. In addition they found that Facebook usage interacted with measures of psychological well-being, suggesting that it might provide greater benefits for users experiencing low self-esteem. Tynes (2007) in her study of adolescent use of social network sites suggested that the act of writing in a discussion forum created time for the participant to reflect on the sequence of entries and posts made by others in order to carefully construct their own post for others to read. Such an act would be helpful in supporting the development of the reflective practitioners that we would wish all teachers to be. Furthermore, studies such as those of Pilgrim and Bledsoe (2011) and Koskeroglu Buyukimdat et al. (2011) have identified that social network sites could be used as a tool to extend and enhance pre-service teachers' professional development. The aim of this study is to evaluate pre-service and newly qualified mathematics teachers' perceptions of the usefulness of being part of a Facebook group and the ways in which they are using it.

Methodology

This report includes quantitative and qualitative findings collected from the pre-service mathematics teachers involved in the study. Thirty-nine of the forty mathematics trainees invited joined the 'virtual staffroom' on Facebook. Ethics approval was obtained and all the trainees who had joined the group received an information sheet that outlined the purpose and length of the study and information about confidentiality.

At the time of writing the trainees have been surveyed as to their perceptions of the usefulness of using the 'virtual staffroom' on four occasions; at the start of the project, halfway through their first school placement, at the end of their PGCE year and at the end of their Newly Qualified Teacher (NQT) year. The first three surveys were short paper based questions which were distributed and completed at the end of a University teaching session whilst the fourth was hosted by University of Southampton isurvey (https://www.isurvey.soton.ac.uk) and sent to the NQTs via a link in an invitation email.

At the end of each year the activity data were 'captured' from the Facebook 'virtual staffroom' group for analysis. A simple thematic coding was used to categorise participants' posts. The subjects of the posts fell mainly, into seven categories: university work, course administration, subject knowledge, sharing of resources, sharing news reports, pedagogy and course tutor input. University work included questions about the format for submissions and how to reference; Course administration included setting up a group discount for a NSPCC course, course representatives asking for comments to feed back to the student liaison committee, asking questions about the course such as what percentage timetable should be taught or how many official observations were required at particular stages of the course; Resources included sharing resources with others as well as requesting resources. An eighth category for 'other' was also created which included looking for jobs and arranging social events.

It was perceived that ownership of the group could be an issue. Harriet Swain writing in the Independent cites findings highlighted in a survey carried out by Ipsos Mori for the universities' joint information systems committee (JISC). "It showed that 65 per cent of sixth-formers hoping to go to university regularly used social-networking sites. But most failed to see how they could be used for teaching, and said that they resented the idea that they might be invaded by academics." (Swain, 2007).

Previous cohorts had formed their own social network groups keeping in contact with each other either through a social network site or by text or e-mail. These groups had however tended to be selective in membership rather than inclusive of all. Ryan (2011) found that discussions on Facebook helped to build communities amongst doctoral students. Her research shows that these discussions aided in various types of knowledge exchange, helped to minimise the anxieties of starting a new course and were useful in promoting socialisation and community amongst the students.

Tutors wanted every trainee on the course to have the opportunity to join the group and to be aware of any discussions that were taking place on the site. In this way trainees would be able to self-select their position between 'active participant' and 'silent observer'. As tutors were also part of the group they would be able to intervene and offer support when required. Swain (2007) describes how Jo Fox, a professor at the University of Durham was invited by her students to join their Facebook group through which she picked up on their discussions, subsequently following some of these up in her seminars. It was hoped that the trainees would welcome rather than reject tutor participation within the group.

As not all the trainees wanted to be included as a member of the Facebook group tutors on the course had to be careful not to use the Facebook page in a way that could disadvantage any trainee who had chosen not to take part. As a result it was decided that tutor input to the page would be minimal.

Results and Findings

Analysis of the 'questionnaires'

At the start of the course trainees were asked only one question, "How useful do you think the Facebook staffroom is likely to be to you when you are in your teaching placement?" Their responses (97% return) to this were generally positive indicating that they thought it would be helpful for keeping in touch with others on the course, sharing ideas and resources, resolving problems and asking questions that were not serious enough to trouble their mentor or tutor. A few, however, noted that whilst they saw it as quite useful they felt that as the Facebook 'virtual staffroom' page would be looked at by tutors on the course, use was likely to be more restrictive.

When surveyed halfway through their first teaching placement the trainees were asked two questions. Their response rate is summarised in Table 1.

Halfway through first teaching placement	November 2011 N=39 (100% return)
Overall, how helpful have you found the Facebook 'virtual staffroom'? *Very helpful (5) – Unhelpful (1)*	3.9
Overall, how helpful have you found sharing resources on the Facebook 'virtual staffroom'? *Very helpful (5) – Unhelpful (1)*	3.9

Table 1. A summary of trainees' perceptions of the usefulness of the Facebook 'virtual staffroom' halfway through their first teaching placement

Trainees felt that the Facebook 'virtual staffroom' worked well as a method of communication, it had been useful to have the site where they could share ideas, thoughts and resources and it was, as one trainee commented, "good to know problems are not unique to me".

When surveyed at the end of their PGCE year the trainees rated positively the helpfulness of the 'virtual staffroom' although many admitted to being 'silent observers' as opposed to 'active participants' on the page (see Table 2).

	End of PGCE year N=33 (90% return)	End of NQT year N=26 (54% return)
Overall, how helpful have you found the Facebook 'virtual staffroom'? *Very helpful (5) – Unhelpful (1)*	4.1	3.0
How would you describe your participation within the Facebook 'virtual staffroom' *Active participant (5) – Silent observer (1)*	2.4	2.0
How frequently do you visit the Facebook 'virtual staffroom' each week? *Most days (5) – never (1)*	3.1	1.7
In which of the following ways have you used the Facebook 'virtual staffroom'?		
• Asking questions about the structure of the course	56%	15%
• Asking questions about the documentation for the course	67%	23%
• Organising events	19%	36%
• Sharing articles with other trainees	19%	8%
• Sharing resources with other trainees	48%	64%
Do you feel the Facebook 'virtual staffroom' has helped your confidence whilst on school placement/in your NQT year? *Very helpful (5) – Unhelpful (1)*	3.4	2.6
Do you anticipate using the Facebook 'virtual staffroom' during your first/next year of teaching? *Very likely (5) – Unlikely (1)*	3.7	2.5

Table 2. A summary of trainees' perceptions of the usefulness of the Facebook 'virtual staffroom' at the end of their PGCE and NQT years

Their comments spoke of being reassured by its presence, making them feel part of a group and it being a good tool for communication. More than one of the trainees admitted to having read all of the posts yet rarely contributing to the discussion unless they were very sure of the answer and some trainees only visited the page if they had a question to ask themselves. The ways in which the trainees admitted to using the Facebook 'virtual staffroom' mirrored the results found through analysing the data from the site (see Table 3). However it also became apparent that the trainees had set up a second private PGCE Facebook page through which they had been organising social events. It is not known whether or not all the trainees were included within this group. Some trainees made specific comments about how they felt having the page had helped to build their confidence whilst on school placement,

> Just knowing that you have a private way of talking to people builds confidence, especially when you are feeling down.
>
> It has been useful, it definitely was a good idea and I like having it there for support etc. but didn't really think I used it effectively.

Others however viewed it as having been a useful place where they could gather information rather than support. Trainees had mixed feelings about whether or not they would continue to use the page during their NQT year for anything more than catching up and gossiping!

Percentage posts/comments	University work	Admin	Subject Knowledge	Resources	News	Pedagogy	Tutor	Other
PGCE year	27.6	25.1	0.1	32.1	0.3	0.1	1.2	13.5
NQT year	0.0	0.0	11.7	38.9	0.0	0.6	2.5	46.3

Table 3. Percentage of posts in the 'virtual staffroom' for each category

The comments made by the teachers who responded to the online survey at the end of their NQT year were similar to those made at the end of their PGCE year, indicating that they felt that the Facebook page was "very good to act as a support network and a place to share resources tips and experiences". However one teacher observed that, "as not many people posted on it, I didn't use it much". They noted that interest in using the page tailed off as the year progressed. Despite this, the teachers' responses to the survey clearly indicated that they valued the presence of the page,

> It has given me access to a support network and a place to see and ask the questions that come up throughout the year
>
> Good to know others are around
>
> It has been helpful although not as significant as face-to-face contact with colleagues and former trainees from the PGCE course
>
> It helps to know you're not alone and others are going through *it too!"*

One teacher observed that the lack of use may have been because students still used their shared Dropbox and had less need to share resources whilst another commented that they perhaps should have used it more during their NQT year to ask questions and get support. None of the teachers' indicated that they planned to use it in their subsequent year of teaching beyond contact for social means.

Analysis of the Facebook 'virtual staffroom' data

The data gathered from their 'virtual staffroom' page were analysed to find out how and for what the trainee/NQTs had used the Facebook group during each of the two years. Each month was divided (roughly) into four, week-long periods and the number of posts and comments on each post recorded (see Figure 1).

During their training year, 55% of the 39 trainees made original posts and 84% made one or more comment on a post. Of the 26 trainees who started the NQT year, 35% made posts and 58% commented on posts. As might be expected the teachers instigating discussions were those who had used the site most frequently during the PGCE year. During their NQT year there were 30 posts and 136 comments on those posts as compared with the 133 posts and 657 comments on those posts made during the PGCE year. As can be seen from Figure 1 the frequency of usage of the site declined over the two year period.

The data were also analysed to find out what medium the trainee/NQTs were using to access Facebook. Just under 10% of the posts during their training year and 7% during their NQT year were made from a mobile phone, the remainder having been made from a computer

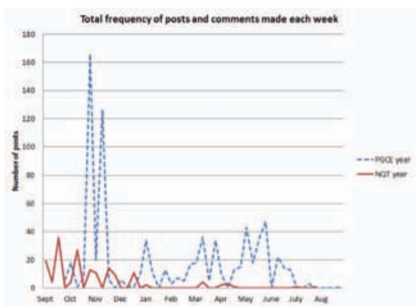

Figure 1. Total frequency of posts and comments made each week during the PGCE and NQT years

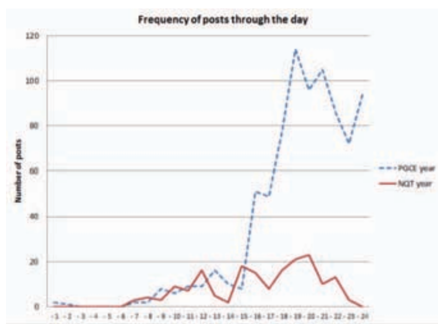

Figure 2. Frequency of posts through the day

The frequency of posts and comments through the day is summarised in Figure 2 above. The number of posts and comments increased dramatically from the end of the school day reaching a peak between 1900 and 2200 with some trainees still engaging in discussions into the early hours of the morning. In contrast during their NQT year conversations had ended by midnight and were more likely to occur during the working day.

The total number of posts and comments on posts each month were reasonably consistent throughout the PGCE year with two significant outliers. During the last week of October the trainees had university tasks to complete which led to five distinct posts amassing over one hundred comments. These contributed to the first peak visible in the graph in Figure 1, with the second peak related to setting up the Dropbox account.

The posts relating to university tasks occurred shortly before the deadline for an assignment. The content of these posts tended to be generic questions about referencing and paperwork required for submission. Similarly the posts relating to course administration fitted with the requirements of the course at any particular time.

In total there were four posts relating to subject knowledge. The first two received little or no comments were an observation about the definition of a directed number and a request for a glossary of terms of 'teaching jargon'. During their NQT year the two posts made instigated some interesting discussion:

> Nearly bottom set year 8s, BIDMAS for the first time, they're getting the hang of it, they follow the rules of BIDMAS and I feel successful, improvise big question on the board, after a few operations I'm left with this: 25 - 2 + 2. Addition comes before Subtraction in the (flawed) BIDMAS system, how would you explain the answer isn't 21? Fortunately someone said the answer was 25 and I said yes and moved on briskly, but it was scary for a moment there

> Can a kite be a rhombus? Is a kite with four equal sides still called a kite? Any ideas?

Trainees shared resources frequently through the course of the year. Early in their first school placement they set up a Dropbox account so they could share resources,. Most were generic websites for mathematics resources such as the Guardian, Nrich and the Khan Academy although some were sites with enrichment activities for the history of mathematics, mathematics and science, mathematics jokes and You Tube clips. Trainees also shared editing and presentation software as well as scanned copies of other resources they had found interesting and useful. On several occasions a request was posted asking for a resource relating to teaching a particular topic or for a resource that they had seen used in a university session. Other requests for resources were to support the teaching of data handling, linear graphs, circle theorems, trigonometry, indices, loci and construction. Most of the resources posts during the NQT year were repeated requests for resources that the teachers had seen during their PGCE year, the most common of which was for a specific piece of geometry software. Only one new resource was shared without being requested.

Tutor input over the two year period was minimal as on most occasions the trainees resolved any problem or issue that had been raised themselves. Course tutors posted on the page on six occasions through the first year. Early in October when the page was first set up to encourage the trainees to use the 'virtual staffroom', in late October in response to a large volume of discussion relating to university tasks, in early December to remind trainees about an ICT conference and finally in the first week of March. This last posting was related to a point in the course when tutors were aware that some trainees may be feeling overwhelmed. Tutors posted a comment pertaining to this to which several trainees responded, gratefully saying that they were comforted to know that the way they were feeling was not uncommon. One trainee wrote “your advice and contribution this week was a great relief and comfort. Thank you”. Tutors posted three times during the second year: to wish the new teachers good luck for the forthcoming year, to remind them to complete the NQT survey and to remind them to keep in touch

Finally, the ‘other’ category included sharing advertisements for jobs, the organisation of an end of year social event, the arrival of PGCE certificates, NQT induction events and the management of the Dropbox.

Concluding remarks

From the data gathered during their PGCE year it is clear that the main reasons the trainees used the ‘virtual staffroom’ were asking for support and guidance on university based tasks, asking administrative questions and sharing resources. These findings support those of Wong et al. (2012) who identified that first year degree students were using Facebook for group discussions on assignments and projects, concluding that the social network site was being used effectively to promote an online community and enhance online learning among students.

The most notable reasons for posts, questions relating to university work and course administration, ended with the PGCE course. The group maintained their shared Dropbox account into the NQT year and, as developing teachers it is assumed that the participants were starting to establish their own personal set of classroom resources and rely less on others for support in this area. Assumptions can also be made that as employed teachers the participants were developing their own personal support network within their department or school. Their comments however clearly show that they continued to value the ‘secret staffroom’ as a place they could go should they need support, thus contributing in some small way to combating isolation and increasing confidence. There is no evidence from the teachers’ posts that using the site helped their development as reflective practitioners. This may be related to the level of participation of the members of the group and the tutors on the course. Sample size is a limitation for any generalisations made from the findings of this study, as are the dynamics of the group. Tutors on the course kept their contributions to a minimum and some of the cohort set up their own independent Facebook group. It is impossible to know how this alternative site was used, by whom and whether or not its existence impacted on the way in which the ‘virtual staffroom’ was used. As a ‘security blanket’ for novice teachers the ‘virtual staffroom’ was successful. However there is clearly scope for its development both within and beyond the training year.

Berg (2007) proposed her university could use Facebook in a number of areas including tutoring, study groups and counselling. Schwartz (2012) examining how teachers use social networking notes that “general social networking sites, as well as education-focused sites, have emerged as powerful tools for teachers in the last several years. They combat the isolation of the classroom and can provide forward-thinking teachers with a community that shares their views”. Bewell, a tutor on the PGCE course at the University of York advocates the use of social media to help “teachers keep up to date and enhance their CPD” (2013). Research by Pilgrim and Bledsoe (2011) and Koskeroglu Buyukimdat et al. (2011) into the use of social networks as a tool to expose trainee teachers to trends and issues in education as well as online access to professional organisations support this claim. However in each of these studies, as with the work of Ryan (2011) the tutors were not passive. In each instance they 'led' the trainees.

Each academic year a new cohort of mathematics teachers join this 'virtual staffroom' bringing with them fresh ideas. All members of the group need to consider how and when it can be used to maximise its potential: who will post and their reasons for doing so. As an established shared community the opportunity for continued professional development is boundless.

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Acquisition of mathematical skills in trigonometrical concepts through project based learning in junior secondary schools in Calabar municipality of Cross River State, Nigeria.

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This paper focuses on the interactive development of junior secondary three (JS3) students' knowledge and performance in Pythagoras theorem using project-based learning skills (collaboration and critical thinking). The purpose therefore, is to acquaint the students with the necessary mathematical skills that enrich their knowledge of formation and solution of Pythagorean triple triangles. A non-equivalent pre-test post-test quasi-experimental design was used on a sample of 280 JS3 students from two private and two public schools to ascertain the knowledge level and cognitive achievement before and after exposing them to project-based learning strategy. Students were also interviewed. Special lesson plans were developed and the teachers were trained on the use of this strategy. Mean scores, standard deviations and dependent t-tests were used to analyse the data. It was discovered that students' performance was enhanced and they were able to construct and solve Pythagorean triple triangles easily and quickly. Recommendations were made based on these findings.

Keywords: acquisition, mathematical skill, project-based learning, trigonometrical concepts.

Introduction

There has been a drastic reduction in the standard of performance by students at all levels of education in Nigeria in the past decades. This fall in standard of education is attributed to many factors, one of which is wrong method of teaching (Emaikwu and Nworgu, 2005; Onah, 2012). Emaikwu (2012) attributes the fall in standard of performance at secondary school level to pedagogical approaches adopted by teachers. The methods commonly used may not be effective as learners are passive listeners in the teaching and learning process. This leads to low achievement in both internal and external examinations. Researchers have made several efforts towards designing techniques for more effective teaching in mathematics.

At the moment, there are a number of innovative instructional techniques advocated which include constructivism (Ekon, 2013), project-based learning approach (Ekwueme, 2013), co-operative learning and problem solving approach (Ekwueme, 2006). Ekwueme, (2013) stated that these innovative strategies, especially project-based learning, not only help students learn and retain information but have a positive effect on the students' attitudes towards studying mathematics. Ekwueme (2006) stated that mathematics is more about seeing and doing than hearing since we can easily forget what we hear. She further stated that mathematics is made more real when students are fully involved in the activity of arriving at a conclusion.

She advocated project-based learning approach as an innovative approach that exposes students to collaboration and critical thinking.

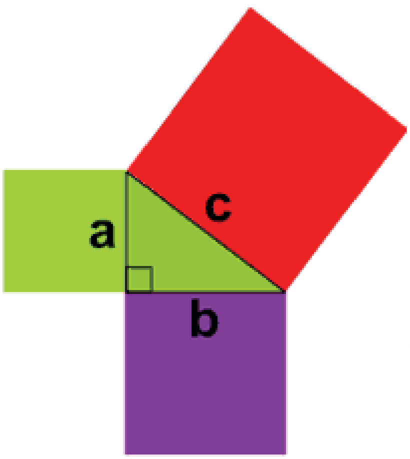
Project-based learning is an individual or group activity that goes on over a period of time, resulting in a product, presentation or performance. It is an effective method of teaching mathematics for inter-disciplinary transfer. Here, the teacher engages students actively in projects involving mathematical skills and knowledge. Learners are engaged in varied, meaningful practice, the type that can be fostered through real-life applications in order to gain skills required for competent and flexible performance on independent tasks. John (2000) described project-based learning as a model that organises learning around projects. He stated that there are instances where project work follows traditional instruction in such a way that the project serves to provide illustrations, examples, additional practice or practical applications for materials taught initially by other means. Meaningful applications of concepts rather than rote recall need to be practised until they become second nature.

Project-based learning varies from classroom to classroom, with the teacher in the role of facilitator rather than leader. A high level of intrinsic motivation and active engagement are essential to the success of a problem-based learning lesson. Students are actively engaged in doing things rather than in learning about something. Project-based learning is one of the best teaching strategies for engaging students in realistic learning activities. This is because it engages the minds of the students and they think critically. It is an individual or group activity. In project-based learning, students go through an extended process of inquiry in response to a complex question, problem or challenge. Rigorous projects are carefully planned; managed and assessed to help students learn key academic content and practise 21st century skills such as collaboration, communications and critical thinking.

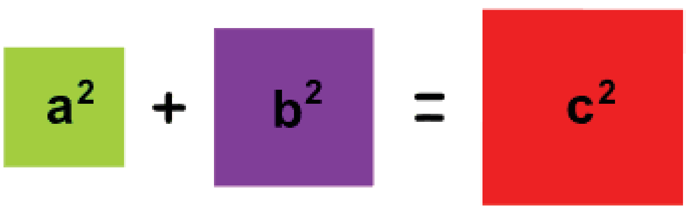
In mathematics, Pythagoras' theorem is mostly taught as the formula *a2* + *b2* = *c2,* where *a*, *b*, and *c* are the lengths of the sides in a right angled triangle. Finding a missing length is taught as a procedure to be learnt. In a problem-based learning approach that incorporates the formation of Pythagorean triples, there are possibilities for developing mathematical understanding.

A **Pythagorean triple** consists of three positive integers *a*, *b*, and *c*, such that *a2* + *b2* = *c2*. Such a triple can be written as (*a*, *b*, *c*), and a well-known example is (3, 4, 5).

Figure 1: Pythagoras' theorem (http://www.mathisfun.com/index.htm)



When you make a triangle with sides **a**, **b** and **c** it will be a right angled triangle



Note: **c** is the **longest side** of the triangle, called the "hypotenuse"; **a** and **b** are the other two sides

Purpose of study

The purpose of the study is to acquaint the students with the necessary mathematical skills that will help them master, solve and form Pythagorean triple triangles with ease. Specifically the study seeks to:

(i) Acquaint the junior secondary school three students with the mathematical skills of constructing Pythagorean triple triangle with ease.

(ii) Solve related problems in Pythagoras' theorem using any of the stated patterns.

(iii)Create an opportunity for collaboration and critical thinking in students.

Methodology

A quasi-experimental design was used for the study. The study covered four secondary schools (two private and two public) with a sample of two hundred and eighty students. A simple random sampling technique was used to select the four schools from the forty-five secondary schools in Calabar Municipality of Cross River State, Nigeria. A class that has been taught Pythagoras' theorem conventionally was selected for the intervention in each of the selected schools.

The two instruments used for data collection were a Mathematics Skills Acquisition Test (MSAT) and a Preference Teaching Strategy Interview (PTSI). These were used to measure students' mathematical skill acquisition (before and after the project based learning intervention) and their preference for teaching approach, respectively. The MSAT was in two sections: Section A consists of questions on skill acquisition based on constructing Pythagorean triple triangles, while section B consists of problem solving using Pythagoras' theorem and patterns in Pythagorean triples.

Students worked in small groups on different methods of constructing Pythagorean triple triangles. Each group used a different method. A representative of each group made a presentation of the group's work. Students were interviewed individually about their preference for either a conventional or project-based learning approach. Their responses to each of the methods were recorded. Four mathematics teachers from the selected schools were trained to use the project-based learning approach to teaching Pythagorean triples.

Experimental Procedure

Training programme for teachers

The four teachers were trained on how to use the project-based learning approach to construct and solve Pythagorean triple triangles. The training exercise lasted for two days. The researchers highlighted the need for better instructional methods to improve students' academic performance in mathematics, and how project-based learning could help. They introduced the Pythagorean triple triangles lesson, and the different methods that could be used in their formation. The training included micro-teaching by the teachers.

Research questions

1. What is the students' general assessment of the project-based learning approach?
2. What is the mean difference in performance of students in pre-test and post-test scores?

Research hypothesis

There is no significant effect of the project-based learning approach on academic performance of JS3 students in solving Pythagorean triple triangles.

Data analysis

Data were analysed using percentages, mean difference and dependent t-test analysis as presented in the tables below.

Research question 1

What is the students' general assessment of the project-based learning approach?
The students' responses to the interview questions are shown in table 1.

Table 1:Response of Students on Preference and Assessment for Teaching Method

Teaching method	Frequency of students' response	
	Private (n=100)	Public (n=180)
Conventional approach	30 (30%)	32 (17.8)
Project-based approach	70 (70%)	148 (82.2%)

From table 1, more of the students in both Private and Public schools indicated a preference for the project-based learning approach with 70% and 82.2% respectively. They preferred this approach because it exposed them to the real meaning of Pythagorean triple triangles and easy solution of right angle triangle problems without memorisation of the theorem.

Research question 2

What is the mean difference in performance of students in pre-test and post-test scores? Their mean scores and standard deviations were calculated as shown in table 2.

Table 2: Mean scores and standard deviation of the pre-test and post-test

	N	X	SD
Pre- test	280	40.03	9.54
Post-test	280	71.61	17.99

From table 2, the mean and standard deviation scores in pre- and post-tests were 40.03 and 9.54; 71.61 and 17.99 respectively. This implies that students performed better in their post-test after experiencing project-based learning than in their pre-test as can be observed from their mean scores of 71.61 and 40.03. However the variation in the results was greater. To find out if the observed difference in mean was statistically significant, the corresponding hypothesis was therefore tested.

Hypothesis

There is no significant effect of the Project –Based Learning Approach on academic performance of JS3 students in solving Pythagorean triple triangles.

The scores of the pre-test and post-test were used to calculate the mean, standard deviation and dependent t-test as shown in table 3.

Table 3: Dependent t-test analysis for pre- and post-test scores on MSAT

	N	X	SD	Df	t-Cal	t- table	Dec
Pre-test hypothesis rejected	280	40.03	17.99	558	100.9	1.645	Null
Post-test	280	71.61	9.54				

.05 Significance level

Since the calculated t-value of 100.9 is greater than the critical t-value of 1.645 at an alpha level of 0.05 with 558 degree of freedom (Df), the null hypothesis of no significant difference was rejected and the alternate hypothesis upheld. This showed that there was a statistically significant effect of the project-based learning approach on academic performance of JS3 students in solving Pythagorean triple triangles.

Discussion

The findings revealed that there was a statistically significant difference in the mean scores of the students' pre-test and post-test. The high mean score of 71.61 after exposing students to the problem based learning approach is an indication that the experience helped the students to acquire mathematical skills required for real understanding of Pythagorean triple triangles.

The active involvement of the students caused them to interact with one another, work in groups and construct Pythagorean Triples individually. This finding is in line with Ekwueme (2013) who observed that the problem-based learning approach enhances students' performance in mathematics. In the individual interviews most students expressed a preference for the problem-based learning approach over a conventional approach when constructing Pythagorean triple triangles. They felt more confident about using a variety of strategies for constructing Pythagorean triple triangles.

Recommendations

Based on the findings of this study, the following recommendations have been made.

1. The research indicated that majority of the students preferred the project-based learning approach to a conventional approach. Mathematics teachers should try to use this approach.
2. Regular seminars and workshops should be organised for mathematics teachers to acquaint them with innovative teaching techniques to improve their effectiveness in the classroom.

3. Mathematics teachers should be exposed to innovative methods of teaching through in-service training programmes organised and delivered by appropriately qualified personnel.
4. Teachers should make lessons that will challenge students to be actively involved by choosing tasks that will provoke their curiosity and interest in mathematics.
5. Teachers should give students opportunities to express themselves in a variety of ways.
6. Teachers could adopt the problem-based learning approach used in this study to secure better understanding of solving Pythagorean triple triangles and greater achievement in mathematics.

Suggestion for further studies

Based on the findings of this study, further studies should be carried out in other places and subjects to investigate the efficacy of the problem-based learning approach.

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The Role of Sample Pupil Responses in Problem-Solving lessons: Perspectives from a Design Researcher and Two Teachers

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The benefits of learning mathematics by comparing, reflecting on and discussing multiple approaches to a problem are well-known (Silver, 2005). However, teachers using non-routine problem-solving tasks designed to encourage multiple approaches face challenges: understanding how pupils make sense of the problem, especially when pupils use unique or unanticipated approaches and helping pupils make connections between disparate approaches and aligning these with lesson goals. In an attempt to address such challenges an extensive set of problem solving lessons were developed. Each lesson includes a range of sample solution-methods that expose pupils to multiple perspectives. A detailed teacher guide supports each lesson. This paper focuses on the use of these sample solution-methods. It explores their development from initial design to final versions. We analyse the varied interpretations and use made of sample solution-methods, in both US classrooms and by two UK teachers, and reflect on how these interpretations align with the designers' intention.

Keywords: Problem-solving, design research, multiple solutions.

Introduction

This paper arises from a study on the development and use of formative assessment lessons. Between 2010 and 2014, a team at the University of Nottingham designed over one hundred Formative Assessment Lessons (FALs) to support all grades in US High and Middle Schools implementing the new Common Core State Standards for Mathematics (http://www.corestandards.org/Math). About a third of these lessons involve non-routine, problem-solving tasks. The aim of the lessons is to assess and develop pupils' capacity to apply mathematics flexibly to unstructured problems, both from pure mathematics and from the real world. The tasks can be solved using a range of methods. The lessons are also intended to support the assessment and development of pupils' capacity to understand, evaluate and compare up to three Sample Pupil Responses (SPRs) to the same problem. During the third year of the project, the latest version of the FALs were taught by eight secondary school teachers in UK classrooms, two of whom are co-authors. Design researchers (known from this point as designers) observed the lessons, one is a co-author. This paper focuses on SPRs. It explores the development of the design of the SPRs and analyses the variation and evolution in interpretation of the use of SPRs both in trials in US classrooms and by the two UK teachers. It reflects on how these interpretations align with the designers' intentions.

Background literature

The literature is clear on the potential benefits to learning of each pupil working with multiple approaches to a problem (Silver et al., 2005). Different approaches can facilitate connections to different elements of knowledge; creating or strengthening networks of related ideas and enabling pupils to achieve 'a coherent, comprehensive, flexible and more abstract knowledge structure' (Seufert et al., 2007: 1056). However, this may be challenging for some teachers; to tease out and connect the important structures, representations and ideas (including misconceptions and errors) embedded in pupils' various solution-methods (Stein et al., 1996). Teachers need to monitor pupil work with a desire to understand possibly unanticipated or nonstandard approaches; discerning the mathematical value of these approaches in order to scaffold learning; purposefully selecting solution-methods for a plenary whole class discussion; orchestrating this discussion to build on collective sense-making of pupils by intentionally ordering the work to be shared; helping pupils make connections between and among different approaches in order to advance the instructional goals of the lesson (Chazan and Ball, 1999; Lampert, 2001).

Teachers need support to develop skills to work in this way. Research has shown that lack of explicit guidance on how teachers can encourage and support mathematical learning means some teachers believe they should not press pupils for justifications or directly tell them anything, instead they should allow pupils to work things out on their own (Swan, 2006). So that whole-class presentations of pupil solution-methods may be little more than a "show-and-tell" (Ball, 2001).

This not only takes up a lot of teaching time but also precludes a major benefit of working with multiple solution-methods namely; comparing them. There are usually advantages and disadvantages of most approaches, depending on the context. For example, numerical representations are often familiar to pupils and form a convenient bridge to other representations. They may help pupils understand a problem but may miss important aspects of a problem and not lead to generalisation. Graphs may provide a visual approach and are often intuitive (at least on the surface), but may lack the accuracy necessary to solve the problem. Symbolic algebra is powerful, being both concise and general, however it may obstruct meaning for some.

However, it is not enough to simply suggest that solution-methods should be compared (Chazan and Ball, 1999); teachers need more guidance. There is little research in mathematics education about this, although there are some case studies of expert mathematics teachers who emphasise the value of encouraging and comparing pupils' different solution-methods (Silver et al., 2005).

Content and Structure of the Problem-Solving Lessons

The remit for developing the FALs was that the problem-solving lessons should draw on a range of important mathematical content, be engaging and cognitively demanding. There should be multiple entry points and solution strategies, allowing different pupils to approach the task in different ways based on their own prior knowledge, leading to a sense of pupil ownership. The tasks should provide opportunities for pupils to conjecture, reflect and make connection between concepts they have already learnt and show how mathematics can help make sense of the world. Finally, the tasks should maximise opportunities for pupils to make visible to themselves and the teacher their current understanding and reasoning. In so doing teachers are able to be particularly adaptive and responsive to pupil learning needs over the course of a lesson as well as facilitate their planning of future lessons (Black and Wiliam, 1998; Swan, 2006). Each task includes both a problem and SPRs.

The structure of each problem-solving FAL is broadly consistent: pupils are given a problem to tackle on their own *before* the main lesson. The teacher reviews this work and formulates further questions for pupils to answer in order to improve their approach. The lesson starts with pupils using these questions to review their own work. Next, in small groups, pupils evaluate fellow pupils' approaches, with the aim of producing a joint approach that is better than their individual efforts. Pupils also have the opportunity to discuss strategies used by their peers and used in the SPRs. In a plenary, these responses are discussed and finally, through a questionnaire, pupils reflect on their work.

Detailed lesson guides designed to support the teacher accompany each task. The purpose of the guide is not simply to offer instructional ideas, but to serve as a catalyst for local customisation (Remillard, et al., 2004). The guide states clearly the designers' intentions; it suggests formative assessment and problem-solving strategies, and provides examples of issues pupils may face. The designers expect teachers to take account of the talents, interests and limitations of their pupils.

Lesson design: the development of Sample Pupil Work

The design of each lesson is grounded in the research literature and supported by evidence from lesson trials. The materials have been developed, from initial draft designs to latest versions, through an iterative, process of lesson design, enactment in three to five US classrooms, analysis, and redesign (Cobb, Confrey, diSessa, Lehrer, and Schauble, 2003). Redesigns are based on detailed feedback from observers of the materials in use in classrooms using an instrument developed by the designers. There are at least two revision cycles per FAL. In the third year of the project, the latest version of FALs were taught in UK classrooms. UK teachers taught between six and ten of the FALs over the course of a year. Researchers observed and videoed most lessons and interviewed the teachers both before and after each lesson.

From the US trials of the FALs the designers soon discovered that within any class, although each task had the potential to be solved using a range of distinct methods, this generally did not happen. Classes tended to use one or at the most two methods per task. It is unclear as to why this was the case.

It is possible to speculate that because many classes in the US focus on one subject area for several months, e.g. algebra, pupils are highly likely to draw on the same subject area for a solution to a problem. Alternatively, pupils may choose a solution-method they assume the teacher values. This is supported by a conjecture made by a US observer:

> Due to the "traditional" approaches generally used here in the States, many teachers believe that "geometric" or sketches are NOT showing rigor or intelligence and that number is the best way and students have internalized this… I have seen teachers, especially secondary teachers scorn the use of manipulatives or actual constructions to make sense of a problem situation. (Observer 1, 2013)

The school year of pupils could also be a factor. As the problems are rich, non-routine and unstructured, the mathematics used to solve them needs thorough understanding, especially as potential methods often require the building of mathematical connections. Consequently, the lessons are aligned to school years that allow pupils to draw on mathematics from previous years. Without this alignment the breadth of potential solution-methods would not be readily available to pupils. This same issue arose in the UK classrooms. Most UK pupils preferred a numerical method, very few attempted algebraic or graphical methods.

> This presents both teachers and designers with a dilemma – how does one reveal the power of, for example, an algebraic approach to a particular problem without

> 'forcing' pupils to use it, in which case the lesson is no longer a true problem-solving lesson, but a mere exercise in algebra? (Malcolm Swan, lead designer)

The designers' response was to introduce two or three pre-written SPRs for each task. These SPRs draw on a range of mathematical ideas and were to be given to pupils once they had already worked in small groups on the problem.

The solution-methods were not for the teacher to present to the class, as although this approach does guarantee pupil exposure to a wide range of methods, the authority for the mathematics would remain with the teacher and the materials. Presenting the strategies as if they were written by pupils introduces a third party to the classroom, a third authority for the mathematics, an authority unknown to the pupils. This anonymity can be advantageous as pupils do not know the mathematical prowess of the author. If it is known that a pupil with an established reputation for being 'mathematically able' has authored the solution then it is possible to conjecture that most will assume the solution is a good one. In which case, its evaluation depends more on who is thought to have written it rather than whether or not it is a worthy one. Anonymous 'pupils' authoring the sample responses also reduces the emotional aspects of peer review. Feedback from the US trials indicated that sometimes pupils were reluctant to voice comments that might be perceived as negative.

Additionally, encouraging pupils to compare methods may help teachers recognise that not all mathematical ideas are equally valid, as already mentioned there are advantages and disadvantages to most approaches depending on context. This appreciation may in turn help teachers when orchestrating a whole class discussion.

Further to this, solution-methods written by 'pupils' rather than presented by the teacher provides designers with the opportunity to include common errors or misconceptions, write incomplete SPRs or SPRs that lack clarity. Thus, SPRs can be used for a number of interconnected and overlapping purposes, including:

- to help pupils struggling to get started or making little progress with a problem by introducing a new approach
- to help pupils understand and critique unfamiliar approaches. This may develop pupils' capacity to flexibly solve problems, make connections between different concepts and reflect, critique and improve their own approach
- to facilitate the confrontation of a common pupil misconception by embedding it into a SPR
- to provide an opportunity for pupils to compare and critique approaches.

Discussing SPRs can make visible what pupils value and their own learning goals. For instance, do pupils regard the criteria for success the correct answer as opposed to a clear, efficient, elegant or/and generalisable method? This can provide useful feedback as to whether there is a mismatch between the teacher's and pupils' learning goals (Leahy et al., 2005).

Pupils working with the Sample Pupil Responses

When lessons were first trialled in the US the guide contained suggestions on how the teacher could introduce the SPRs to the class: by writing the following instructions on the board:

> Imagine you are the teacher and have to assess this work. Correct the work and write comments on the accuracy and organization of each response. Make some specific suggestions as to how the work may be improved.

Feedback from the US trials indicated to the designers that these instructions were inadequate. Teachers and pupils were not clear on what mathematics should be discussed, for example, US teachers were asking the designers questions such as:

> What is the math we want to have a conversation about? Do we want pupils to explain the method? Do we want each piece to stand-alone or should pupils compare and contrast strategies?

US observers suggested that pupils were not digging deeply into the mathematics of each SPR and unless asked a direct question by the teacher, pupils often worked in silence simply looking for errors without evaluating the overall solution strategy. Some pupils mimicked the feedback they received from the teacher, providing comments such as 'Awesome', 'Good answer' or 'Show a little more work'. A clear message came from the observers; the prompts in the guide needed to be more explicit and more focused on the mathematics of the problem; scaffolding was required. The decision was made to include more specific questions, such as:

> What is the point of figuring out the slope and intercept?

The designers hoped that such questions would also emphasise the important mathematics in the lesson. The feedback from the US observers was encouraging:

> I think the questions or prompts about each piece of student work, really focus the students on the thinking, bring out the key mathematics and are a great improvement to the original lesson…Last year students just made judgment statements, but this year the comments were focused on the mathematics. (Observer 2, 2010)

However, this contrasted with the approach taken by the two Year 9 teachers in the UK. Understanding and critiquing SPRs was new to them and their pupils. Working collaboratively was also new to some pupils. Envisaging that questions specific to each SPR could exclude some pupils from participating in discussions the teachers preferred initially to simply ask pupils to explain the approach; describe what the pupil had done well and possible improvements. The teachers noted that most pupils readily engaged with each other in answering these questions. But often, as in the US, pupils focused on finding errors or superficial features such as spelling or neatness of handwriting. The latter could be regarded as unfortunate distractors easily remedied by the designers, however these features could also serve as a useful platform for discussion: what is of value in a particular approach? The errors presented a more complex issue. The designers include them in some SPRs as a means of access into the solution, to enable most pupils to engage with the task. However, it became clear that they could also distract pupils from attempting to understand the overall strategy. As one UK teacher commented, error-free SPRs often resulted in a more holistic evaluation of a method.

Lack of understanding of SPRs was a feature of many UK lessons throughout the project. Pupils tended to focus on the step-by-step processes, not the underlying reasoning behind a strategy. For example, when pupils were asked to complete one particular SPR, most were able to do so, they understood the processes, and were able to work out the correct numerical answers, but then encountered difficulties interpreting the resulting figures in the context of the real-world situation. Furthermore, asking questions, such as 'Explain Harry's method' seemed to encourage pupils to concentrate their energies on the detail of each step of the SPR rather than also attempting to understand the overall strategy and its purpose. This was aptly demonstrated in another lesson when UK pupils explained the multiplication of 20 x 30 = 600 gave a correct answer, but did not think about the meaning of the answer within the context of the problem. Only when prompted by the teacher did they appreciate that the 'pupil' was calculating an area.

With encouragement from the UK teachers, many pupils noted lack of explanation in some of the SPRs, but usually with cursory suggestions for improvement, for example simply stating 'she needs to explain it better' or 'the diagrams should not be all over the place'. This could be remedied by prompting pupils to improve the SPR rather than just make suggestions on the improvements.

Number of Sample Pupil Responses (SPRs) used

The design brief was to write a problem-solving lesson that lasted one hour. Initial feedback from US observers indicated the lessons were taking a lot longer; teachers were giving out all pieces of SPRs, but there often was not sufficient time to successfully evaluate and compare the different approaches. In response to this issue, designers decided to include the generic text in all lesson guides:

> There may not be time, and it is not essential, for all groups to look at all sample responses. If this is the case, be selective about what you hand out. For example, groups that have successfully completed the task using one method will benefit from looking at different approaches. Other groups that have struggled with a particular approach may benefit from seeing another pupil's work that uses the same strategy.

These instructions provide the potential for pupils to critique and reflect on unfamiliar approaches, to explicate a process and to compare their own work with a similar approach; this, in turn could serve as a catalyst to review and revise their own work.

However, in US trials this approach was not always successful; it turned out that comparing two SPRs or evaluating an unfamiliar method was often limited to a few pupils. This raised the question of whether pupil differentiation should be by task rather than support? Should all pupils have the opportunity to engage with all the mathematical ideas incorporated in the SPRs? If teachers limited some groups of pupils to working with, for example, just one SPR while others work with two or three, then when the teacher holds a whole class discussion about the different SPRs inevitably some pupils will be disadvantaged; it can be challenging to understand 'in the moment' of explaining, a SPR. However, giving all pupils all SPRs may not be a practical, as it could result in the task extending over several lessons.

Also, feedback from US trials highlighted the tension many teachers encountered when faced with struggling pupils. To allow pupils to struggle often conflicted with ingrained beliefs of what it meant to be a teacher. Yet the guide consistently encouraged teachers to permit pupils to persevere collaboratively with the task in hand; to struggle productively (Swan, 2006). The above instructions may inadvertently encourage teachers to do the opposite, as seen in this US observer comment:

> We have some teachers who give all the sample student work and let students choose the order and the amount they do. This might be less common. Others are very controlling and hand out certain pieces to each group….Others like a certain method to solve problems and like to use that one to model. I think this is a function of the teacher's comfort level with control and students' expectations. (observer 3, 2013)

The designers continue to struggle with this issue. It was also a concern for the UK teachers. At the start of the project they were reluctant to give pupils all SPRs at the same time, they were concerned that pupils would be overwhelmed. They believed that pupils would find it difficult to 'get into the head' of another pupil. No pupils in the study had previously had the opportunity to edit and dissect another's thinking. Also, the teachers regarded the purpose of SPRs as a catalyst for improvements of pupils' own solution and a key formative assessment tool; primarily to help pupils reflect on and critique their own work.

In a UK teacher's first lesson she gave one piece of work to each group. Those that had struggled with the problem were given what she regarded as the easier SPR. Her instructions were, as already mentioned non-SPR specific. For example: 'What has the pupil done well? What if they did it again could they do better?'. She also asked pupils to do the same for their own work. In the following whole class discussion, although limited to just a few participants, her priority was not to compare SPRs, but to bring to the attention of the class not just the answers to questions mentioned above, but how the pupils had gone about arriving at the answers. Some pupils had reworked the answer to the problem for themselves using the same method as in the SPR, others had checked the SPR line by line. Her goal here was to help struggling pupils develop strategies in order to understand and critique a piece of work whether their own or someone else's. This goal became less of a priority as the project progressed, but for both teachers using the SPRs as a tool to help solve the problem remained a fixed goal throughout the project. However, surprisingly, there was little evidence of pupils changing their work apart from when they noticed a numerical error. Some pupils acknowledged their work needed improving but did not take the next step and improve it. Observers reported the same in the US, only pupils that were stuck were likely to adapt or use a strategy from a SPR.

Conclusion

Using SPRs was a new and challenging experience for both UK teachers and pupils. However, at the end of the project both teachers noted that the SPRs acted as professional development; serving to improve their own flexible knowledge of mathematics and of pupil's thinking when solving problems. This in turn helped them resist insisting or explaining the "one" way for pupils to "get" the answer.

In the study it became clear that not all of the designers' purposes for introducing SPRs were enacted in the classroom. Teachers had their own reasons for using SPRs and these reasons evolved as the project progressed. One of the designers' purposes was to help pupils struggling with the problem, yet there was little pupil written evidence of this happening. Pupils were able to acknowledge mistakes in their own work, but generally did not take the next step and write it down. It is possible to speculate this was due to timing, most pupils are given the SPRs at the same time, after they have worked on the problem for some time, so had 'moved on'.

Another purpose for SPRs was to help pupils understand and critique unfamiliar methods. The UK teachers did consider their pupils' capacity to do so improved a little and they anticipated with more experience of SPRs pupils would move beyond explaining the detail of the step-by-step processes within a SPR to also explaining the overall reasoning. Revisions to the materials may support this move.

A third purpose for the introduction of SPRs was to provide an opportunity for pupils to compare methods. The UK teachers rarely overtly encouraged pupils to compare SPRs and it may not have been appropriate to do so, especially as, as already mentioned, pupils found understanding another, unfamiliar, pre-written solution very challenging. UK pupils were consistently given the opportunity to compare their own work with one piece of SPR but not to compare SPRs. This was also the case in the US trials, although pupils often received all SPRs there was little evidence that they compared them. However, the UK teachers now plan, once pupils are accustomed to analysing another person's solution-method, to use the powerful tool of comparison, especially when pupils are critiquing work.

More exploration is also required into how the use of SPRs affects pupils' own capacity to solve problems. For example, one might have expected that the range of distinct methods used to solve a problem would increase as the project progressed; there was scant evidence of this. One teacher did note that some of her pupils now write fuller explanations when solving the problems for themselves.

There are no major studies that focus on how teachers work with a range of pre-written solution-methods for a range of rich problems. This one-year study raises many issues and in so doing acts as a launch pad for further more detailed studies

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An exploration of primary student teachers' understanding of fractions

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The purpose of this study was to discover the individual and distinct ways in which each student teacher understands fractions and their strategies for working with them. A phenomenographical approach was adopted in order to provide insight into each student teacher's subject knowledge of fractions. This study involved detailed scrutiny of six self-selected small groups, which enabled a range of rich and honestly reflective data to be collected. Groups undertook two collaborative tasks involving the sequencing of fractions by magnitude, followed by reflective interviews. Each group also undertook a diagnostic interview, considering a range of questions, which they had ordered in terms of their perceived difficulty. A constructivist perspective was adopted giving students the opportunity to reconstruct their own understanding of fractions through the explanation and discussion of their existing ideas. A range of successful strategies was demonstrated, especially the use of mathematical anchors and the use of residual or gap thinking as a means of comparison. Improper fractions and reunitising were the main difficulties cited by many in the group. A common assumption was that there was a particular 'correct' method to be adopted. The study helps to identify misconceptions that can be addressed within teacher training.

Key Words: fractions, primary mathematics, student teacher subject knowledge, phenomenography

Introduction

This study developed from a professional and personal interest in the learning and teaching of mathematics, in particular fractions, and a desire to make it more effective. It is acknowledged that there has already been considerable research focusing on the difficulties encountered by pupils in primary and secondary schools in many countries, especially the UK, USA and Australia. There have been fewer studies that focused specifically on student teachers' understanding of fractions (for example: Ball, 1990; Miller, 2004; Domoney, 2002; Toluk-Ucar, 2009). This study focuses on thirteen student teachers' understanding of fractions and the related areas of mathematics. It considers the aspects in which they feel confident as well as those that they perceive as problematic. The purpose of this study was to discover the 'qualitatively distinct ways' (Steffe, 1996: 321) in which students understood fractions.

Research design and methodology

One of the underlying assumptions of this study was that the mathematical understanding of the students and their ability to share that understanding with their pupils is integral to their perceptions of themselves as mathematicians and teachers.

Aubrey (1997: 3) claims that, "If teaching involves helping others to learn then understanding the subject content to be taught is a fundamental requirement of teaching".

The main methods employed were the use of observed shared tasks followed by reflective discussions and diagnostic group interviews. It was intended that the student teachers would explore, explain and possibly reconstruct their own understanding of fractions. These methods involved the explanation and discussion of their existing ideas and a consideration of any elements, which possibly caused confusion. Apart from exploring and explaining their present understanding it provided opportunities for them to develop a more effective, relational understanding (Skemp, 1989) of fractions and its related areas of mathematics.

Through the use of collaborative tasks and reflective discussion, it was intended to mirror the process described by Carpenter and Lehrer (1999: 20) through which mathematical understanding is promoted; "constructing relationships, extending and applying mathematical knowledge, reflecting on the experience, articulating what one knows and making mathematical knowledge one's own". In this way the study adopted a constructivist perspective. This is the belief that "knowledge is actively constructed by the cognising subject, not passively received from the environment" (Von Glasersfeld (1989: 162) in Ernest (1991). Ernest (1993: 63) described a social constructivist theory of learning mathematics which suggests that "both social processes and individual sense making have central and essential parts to play in the learning of mathematics".

The use of phenomenography, which is "the empirical study which seeks to understand how individuals experience, apprehend, perceive, conceptualise or understand the world", (Marton 1994: 4424) provides a valuable means for understanding learning from a student's point of view. Phenomenography has been used in a range of mathematical studies that considered the learning of children (Neuman, 1997) and of adults (Asghari and Tall, 2005). Phenomenography "takes human experience as its subject matter" (Marton and Neuman, 1996: 315). The study is based on the underlying premise that although participants are all undertaking the same task, there will be a number of qualitatively different ways of experiencing or understanding the question or problem which can be observed and identified. Each participant brings his or her prior experience and learning to the task and this affects the way in which it will be undertaken.

The intention of this study was to discover the nature of these differences. A convenience sample of thirteen student volunteers participated in the study. The group was predominantly female aged between 21 and 26. The majority were in their second year of a BA(Hons) degree in Primary Education, there were also two pairs from the Primary Postgraduate course. Two collaborative tasks were observed, where the student teachers worked in self- selected groups of two or three. The first involved the sequencing according to magnitude of a series of fractions, percentages and decimals and matching their equivalents, which was intended to be introductory and provide a supportive start to the student teachers engagement with the study. A second task followed the same style but focused purely on fractions with the option of some further fractions to be included for those who felt confident. These tasks focused on the part-whole and the measurement context of fractions (Kieren, 1976). After each observed tasks there were reflective discussions considering their strategies and any areas that were perceived to be particularly difficult.

A series of diagnostic interviews was conducted with individuals and pairs. These interviews were based on a range of questions that considered the findings of the observed tasks and the research literature. There were 20 questions, a sample of the type of questions can be seen below with the rationale and source for each one.

Ordering and Magnitude of Fractions	
Question 6 What fractions come between $\frac{2}{5}$ and $\frac{3}{5}$?	This question developed the idea of continuity and fraction density. It was intended as an extension of sequencing activities. The open nature of the question was intended to promote discussion. It was based on a KS2 National Curriculum Test question for level 4.
Unitising and Reunitising	
Question 3 X X X X X X = $\frac{3}{2}$ of the unit. How many is there in a unit? Question 11 These circles represent $\frac{3}{7}$ of a unit. How many is the whole unit? ● ● ● ● ● ● ● ● ● Question 12 These circles represent $\frac{3}{4}$ of a unit. How many is $\frac{2}{3}$ of the unit? ● ● ● ● ● ● ● ● ●	Unitising refers to the process of constructing chunks in terms of which to think about a given commodity. It is a subjective process (Lamon 2005: 78). It is a natural process and plays an important role in several processes needed to understand fractions especially in partitioning and in equivalence. The questions are based on examples from Cramer and Lesh (1988). Rational Number Knowledge of Pre-service Elementary Education Teachers. The progression of questions building on the slightly simpler version (question 3). This progression of the questions was based on the views of students undertaking the pilot.
Approximation of magnitude of fractions and benchmarking task using near equivalence to 1.	
Question 5 Which is the best estimate for $\frac{12}{13}+\frac{7}{8}$? a) 1 b) 2 c) 19 d) 21 (multiple choice)	Question used with US grade 5 children, by Mitchell (2004) as it had also been used in National assessments and earlier research studies. The use of multiple choice was intended to provide a greater level of discussion.

Figure1 A Sample of Questions from the Diagnostic Interviews

At the start of each interview the participants were asked to indicate the questions about which they felt most and least confident, these perceptions were used to structure their interview. This was intended to follow the phenomenographical approach and ensured that each interview followed an individual path depending on the student's choices and enabled them to begin with the questions with which they felt more confident. A questionnaire was also used to provide further information about each participant's qualifications and feelings about their own learning in mathematics. The videoing of these activities focusing on the table and the writing produced a huge amount of data which was reviewed and coded and cross tabulated to identify the student teacher's perceived strengths and areas of difficulty.

Findings

Areas of Strength

The student teachers demonstrated confidence in a range of areas. Some of which were as expected, for example, the use of percentages, finding fractions of quantities and the use of equivalent fractions, usually by continuing an established pattern. There was also a range of strategies that were demonstrated across both observed activities and within the answering of questions within the diagnostic interviews. One such successful strategy that was adopted by individuals in several groups, was the initial placing of the more common fractions and their equivalents to provide a structure. This reflected the use of *mathematical anchors* (Spinillo, 2004). These were often used as a guide and referred to as 'boundaries' or 'markers' by the participants. For example, in activity one when placing $\frac{2}{5}$ on the number line;

> **Iris:** Two fifths feels bigger than a quarter and it must be less than a half, because that would be two and a half fifths. So it goes in the middle but nearer the half.

This comparison, to the more accessible numbers, for example, $\frac{1}{2}$ or 1 is also referred to as *Benchmarking* by Clarke at al. (2008). The use of mathematical anchors for comparison reflected the use of a half, as an anchor; made by eight and nine year old children when adding fractions (Spinillo, 2004) where it was considered to further facilitate their understanding. This strategy was also sometimes combined with the use of residual thinking (Clarke et al., 2008) where a learner refers to the amount needed to make a fraction up to a more accessible number, usually one or, in the following case, a half. In this example it enabled Anne to deduce that $\frac{45}{80}$ was larger than $\frac{55}{100}$.

> **Anne:** They are both five away from a half, but five hundredths is equal to a twentieth, and five eightieths is the same as a sixteenth. One twentieth is smaller so it will be nearer to the half.

When further explanation was needed by her student colleague, she elaborated with, "That (pointing to $\frac{45}{80}$) is…(greater) because it is a half and a sixteenth and that is bigger than a half and a twentieth." This was an interesting and natural use of a mathematical anchor, where her prior knowledge of a half and its equivalents was used effectively to enable her to make a comparison between apparently more complex fractions. This strategy was employed by six of the thirteen students, in a variety of ways, within the sequencing activities to establish the magnitude of a fraction by comparing it with one that they felt more certain about.

There were also some very personal strategies and approaches adopted, one example of this was demonstrated in response to the following question;

> At the ferry port, one quarter of the passengers are travelling to France, one third are going to Germany, what fraction are travelling to Holland?

Gill had drawn a circular diagram but was uncertain about how to establish what the remaining piece might be.

> **Ellen:** Well, a quarter looks like this…. Like a clock and if you drew a third, then this bit is Holland…
> **Gill:** So that bit is quarter past… and….
> **Ellen:** I couldn't work it out straight away like this… (new drawing)
> Can you see the 5 minutes… round the clock? $\frac{1}{12}$ and $\frac{2}{12}$ (pointing to each 5 minute section). Does that makes sense? $\frac{1}{4}=\frac{3}{12}$ and $\frac{1}{3}=\frac{4}{12}$ so that is $\frac{5}{12}$, it looks like 25 to…
>
> **Gill:** Hmmm, yes
> **Ellen**: I only know 'cos my dad used to teach me fractions on the clock.

Such moments within the diagnostic interviews proved enlightening as the interviewee took on the role of teacher and in sharing their thinking with their student colleague gave an insight into the strategies that they used to answer the question. A range of contrasting methods were employed, for example, Lynne favoured considering all fractions as decimals and had a wide range of what were described as 'known facts' such as $\frac{1}{9}$ is $0.\dot{1}$ and $\frac{1}{8}$ is 0.125 which enabled her to convert most fractions into 'something like a real number'. Fractions were considered too difficult to deal with so this was an approach developed at secondary school, which was then applied to most questions asked; it was mostly successful.

Areas of difficulty

When asked to identify their areas of difficulty, the participants suggested working with improper fractions and re-unitising to be the most problematic, though analysis of the data showed that whole number bias was also a particularly evident difficulty.

Whole Number Bias

Whole number bias (Ni and Zhou, 2005), which is the inability to view a fraction as a single quantity was evident in most students' responses. For example, see the following question where there was a considerable range of answers.

Which is the best estimate for $\frac{12}{13}+\frac{7}{8}$?

a) 1 b) 2 c) 19 d) 21?

Three out the thirteen students gave accurate and immediate answers. Four students displayed some evidence of whole number bias, they initially responded to the question by considering the numerator and denominator separately taking each as a natural number. Two initially responded with "nineteen over twenty-one" (Iris and Gill), having added each pair of numerators and denominators. One student (Betty) was very uncertain. "It must be quite big… probably 19, but I can't remember what you are supposed to do".

This was a tendency to respond to the numerator and denominator separately, taking each at face value. This relates to children's early experience of natural numbers where each number has its own unique value, which can be counted as a discrete quantity and is represented in a systematic way. Sometimes this initial understanding of number over-rides the undertaking of the fraction as a number in some children and adults (Mack in Carpenter et al., 1993). In the reflective discussions following this question, several participants considered that when faced with each number separately they would have said it was, "close to one" but when addition was included, there was an assumption in many cases that a method was required and they were uncertain as to what that might be. "My number sense deserted me," (Jane) "it is quite obvious when you look at each number separately" This was a recurring theme with more formally laid out questions; the belief that there was an established method which should be employed but they were unable to recall it from their secondary school mathematics lessons.

Unitising and Reunitising

Unitising involves *partitioning* a whole, into *equal* parts whereas reunitising involves reconstructing the parts back to create the original whole (Lamon, 2005). Questions which required the participants to unitise and re-unitise in a flexible way, proved a particular problem for eight of the students. The focus of questions (see figure 1) was re-unitising, which involved identifying the original whole, with only one question involving unitising once the whole had been established. This focus was selected as a range of studies suggested that this remains a problem in some secondary school pupils and adults (Kieren, 1993; Lamon, 1999). This also seemed to be the case in this study, the students were generally confident with unitising but the concept of re-unitising seemed unfamiliar to the majority of the students Re-unitising was considered an important strategy whether undertaken physically or mentally and the inability to identify the base unit was considered as a factor in inhibiting development of a greater level of understanding (Mitchell, 2004). By including three questions, this developed into a learning opportunity and in most cases, a suitable strategy had been employed and the third question was approached with more confidence. This was typified by Betty, who after a less confident start tackled the question logically, using the same type of jottings she had used successfully on the earlier questions.

> **Betty:** Oh no, not another unit one! So… to make it a whole you would need seven sevenths, so nine is three sevenths. Then divide that into that, so each little bit… (circling three dots at a time) hmmm… so I need some more sevenths, so add three and… (counting up in threes). Is it twenty-one?

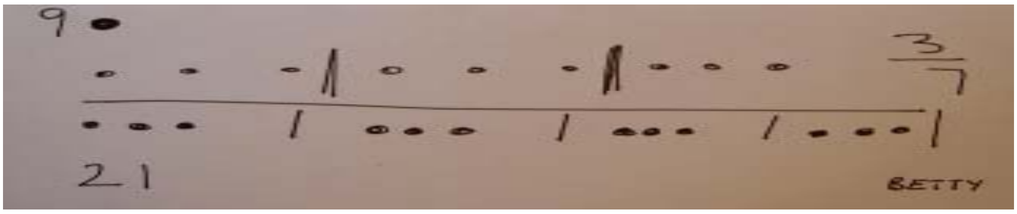

Figure 2. Betty's jottings in response to the question: These circles represent $\frac{3}{7}$ of a unit. How many is the whole unit? ● ● ● ● ● ● ● ● ●

Working with improper fractions was considered problematic by several students who felt they lacked experience in using them and were often referred to in a different way to 'proper' fractions, for example as "thirty-nine over ten" or "three over two". In general they did not appear to apply their understanding of proper fractions when working with improper fractions. In several cases they seemed to adopt a very procedural approach to aspects of the tasks and interview questions that included improper fractions. There were a range of other difficulties experienced by a smaller proportion of the participants and some very specific individual responses, which also shed light on adult understandings of fractions.

Conclusions and Professional Impact

The adoption of a phenomenographical approach proved very effective in understanding the learning of fractions from a student teacher's point of view. The students, although undergoing the same experience, brought their prior learning and attitudes to the task, which resulted in a range of qualitatively different responses. It is acknowledged that this is a very small research project based on only thirteen volunteers, however it has revealed that student teachers' knowledge and understanding of fractions is complex, individual and varied. The establishment of a supportive environment in which participants can articulate and explore their individual beliefs, understandings and misconceptions has contributed to a better understanding of the difficulties which they experience with fractions and related areas. It has highlighted several areas that have impacted on the primary mathematics initial teacher education courses in my current institution and with our work with teachers in the partnership schools. A greater level of consideration has been given to the process of reunitising and linked to the more usual focus on unitising. Another aspect, which has been identified as particularly beneficial by students, has been the inclusion of fractions when studying other areas of mathematics. The intention of this is to emphasise fractions as numbers and to reinforce their place within the number system. This has been especially effective when considering place value and measurement.

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'Can't you just tell us the rule?' Teaching procedures relationally

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It is now almost 40 years since Skemp's (1976) seminal division of understanding into 'instrumental' and 'relational' categories, yet the current political direction of mathematics education in the UK is decidedly towards the traditional teaching of 'standard algorithms' (DfE, 2013). In this research paper, I draw on a lively staffroom discussion about different approaches to the teaching of quadratic equations, in which one method used was derided as 'a trick'. From this, I discuss reasons why certain mathematical processes are often regarded as inherently and irretrievably 'procedural'. Informed by recent theoretical interpretations of procedural and conceptual learning in mathematics, which increasingly stress their intertwining and iterative relationship (Star, 2005; Baroody, Feil and Johnson, 2007; Star, 2007; Kieran, 2013), I make a case that stigmatising particular methods and censoring their use may deny students valuable opportunities to make sense of mathematics. I argue instead that encouraging students to take a critical stance regarding the details and the value of the procedures that they encounter can cultivate in them a deeper awareness of mathematical connections and a more empowered sense of ownership over their mathematics.

Keywords: Algorithms; Conceptual knowledge; Instrumental understanding; Procedural knowledge; Quadratic equations; Relational understanding; Student autonomy

Introduction

Don't waste time learning 'tricks of the trade'. Instead, learn the trade.

James Bennis

In a classic article, written almost 40 years ago, Skemp (1976) outlined what has become a highly-influential distinction between instrumental and relational understanding. By relational understanding, he meant "knowing both what to do and why" (p. 20), in contrast to instrumental understanding, which was merely "rules without reasons" (p. 20) – something we would not normally characterise as understanding at all. Since then, the terms procedural and conceptual learning have been widely adopted, and more recent theoretical interpretations of these in mathematics have increasingly highlighted their interweaving and iterative relationship (Star, 2005; Baroody, Feil and Johnson, 2007; Star, 2007; Kieran, 2013; Star & Stylianides, 2013). Indeed, "the wider debate is starting to move away from the opposition of conceptual understanding from factual and procedural knowledge", seeing the two as mutually reinforcing rather than antagonistic (DfE, 2011: 67). Nonetheless, there remains a wide consensus among mathematics educators that a classroom focused predominantly on the competent performance of algorithms does not offer students an authentic experience of mathematics and that the use of richer tasks is essential for developing the necessary relational understanding of the subject

(Mason and Johnston-Wilder, 2006; Watson, 2007; Sullivan, Clarke and Clarke, 2013). Teaching students to do mathematics by applying a set of memorised algorithms is viewed as hindering their mathematical development, because they are able to achieve correct answers without an understanding of the underlying mathematical principles.

Despite this, the current UK political climate shows a decided preference for the traditional teaching of 'standard algorithms', with its emphasis on practice for fluency (DfE, 2013). Indeed, the first stated aim of the new mathematics programme of study for key stage 3 in the national curriculum for England is:

> that all pupils become *fluent* in the fundamentals of mathematics, including through varied and frequent practice with increasingly complex problems over time, so that pupils develop conceptual understanding and the ability to recall and apply knowledge rapidly and accurately. (DfE, 2013: 2, original emphasis)

Here, procedural fluency is promoted as a route to conceptual understanding, yet the flavour of the prescribed curriculum as a whole is widely perceived as lying more towards the procedural side. Pope and Cotton (2013) express concern about "the heavy reliance on practice as a principal teaching approach", concluding that the "curriculum as presented will result in more attention spent on developing technical competence in outdated written methods for arithmetic at the expense of developing secure foundations for progression through mathematical concepts and skills" (p. 9).

I have previously argued that the ideological valuing of procedural knowledge has a tendency to fragment the curriculum into meaningless, bite-sized facts and skills, learned with little relational understanding (Foster, 2013a). In their most recent report on mathematics, Ofsted (2012: 18) comment that they observed few "lessons that were helping pupils to gain a better understanding of mathematics", as opposed to those with "a strong focus in teaching to the next examination". The powerful backwash effect of high-stakes assessments understandably leads many students to ask, 'Can't you just tell us the rule?' Indeed, much within the culture of the UK mathematics classroom (perhaps even the name 'exercise books') predisposes the teaching of procedures.

In this paper, I consider the potential value and dangers of teaching mathematical procedures. I base the discussion on a lively staffroom conversation about the teaching of quadratic equations and explore possible reasons why some mathematical procedures may be designated 'tricks'. Are (some) mathematical procedures inherently harmful? Are students better off not being taught standard algorithms? Or can procedures be taught in non-damaging (or less-damaging) ways?

A staffroom conversation

I draw on a spontaneous staffroom conversation, overheard in a UK secondary school, relating to the teaching of quadratic equations. I was a 'fly on the wall' observer for this unanticipated discussion, which I noted down afterwards. I do not present this episode as data; rather as an extract that illustrates the wider debate in a local context and is included for communicative purposes rather than as an evidential base. The names are pseudonyms.

Prior to the discussion, Jack had shown his Year 10 class (14–15-year-old students) how to solve quadratic equations by factorising, if possible, or by completing the square, if not. He was then away from school for a lesson, and a non-mathematics colleague, Jill, had taken the class in his absence. Jack had heard from his students that Jill had told them that when she was their age she always used the quadratic formula $x = (-b \pm \sqrt{b^2 - 4ac}) \div (2a)$, which they had not heard about. She had shown them the formula and some of them had remarked that they liked this method much better than Jack's methods because they found it quicker and easier, and it was one technique to remember rather than two – and they completed all of Jack's set work using Jill's formula. Jack was now back in school and Jill was telling him with some pride that she showed his class how to use the quadratic formula, because 'they didn't seem to know about it'. However, Jack was unhappy with her comment, seeming to take it as a criticism of his teaching:

Jack The quadratic formula is just a trick.

Jill What do you mean 'a trick'?

Jack They just bung numbers into a formula without thinking about what they're doing. Here's the formula; stick the numbers in. It could be any topic. It's got nothing to do with the ideas behind quadratics.

[Jill looks unsure how to respond, but other mathematics colleagues who are present begin to join in.]

Mike Completing the square is just a trick; steps you go through. Everything in maths is just a trick.

Jack No, the principles you use in completing the square are powerful and I mportant mathematically: the algebraic manipulation, solving an equation by doing the same things to both sides. And factorising is a big idea in maths that I want them to understand.

Mike Substituting into formulas is powerful – it's all they seem to do in science exams these days.

Helen [*to Jack*] Don't you teach your classes the quadratic formula then?

Jack I do, but later. If you teach it first, then they've got no motivation to learn any other method. And they have to learn completing the square first anyway, otherwise how do you derive the formula?

Helen I find that my classes are not really that interested in proofs, and the quadratic formula one is really fiddly – much too hard for them.

Mike I agree. If you asked me to sit down now and prove the quadratic formula for you, I'm not sure I could. But why do I need to? I know it and I can use it, and that's what matters.

It seems clear from this discussion that there is a significant difference between Jack's pedagogical intentions and those of his colleagues. Jack's pejorative use of the term 'trick' implies that he sees something illegitimate about the quadratic formula – that although it may be an efficient means of obtaining the correct answer it fails to expose students to 'the ideas behind quadratics'. Mike responds that the alternative methods might also be regarded as tricks, and that substituting into formulae is an important skill, but Jack maintains his position that factorising and completing the square are powerful methods which he is passionate about his students experiencing. What is it that leads to a method being derided as 'a trick'? Is Mike correct that every mathematical process is a trick?

Solving equations

Vaiyavutjamai and Clements (2006) comment on the lack of research into students' difficulties with quadratic equations, and since then a number of studies have explored this area (Kotsopoulos, 2007; Lima and Tall, 2010; Didiş, Baş and Erbaş, 2011; Olteanu and Holmqvist, 2012; Tall, Lima and Healy, 2013). Lima and Tall

(2010) report that teachers taught the methods of factorisation and completing the square but 'moved on quickly to the use of the formula in the belief that this would enable [their students] to solve any quadratic equation that would be given in a test' (p. 1). This is interesting in the light of Pólya and Szegö's (1972: viii) famous statement that "An idea that can be used only once is a trick. If one can use it more than once it becomes a method". By this definition, the quadratic formula is certainly not a trick. Indeed, Bossé and Nandakumar (2005) point out that a randomly-chosen quadratic expression with integer coefficients is extremely unlikely to be factorisable, and thus advocate completing the square and the formula as more reliable methods.

Jack's use of the word 'just' suggests that he may see a trick as something utilised in a thoughtless, reductive way, such as is implied with the jingle reported by Wu (2011: 375) for dividing fractions: "Ours is not to reason why, just invert and multiply". If students 'just' use the quadratic formula to obtain the answer, without any deeper sense of what is going on, their understanding would rightly be described as 'instrumental'.

The factorising method might be promoted on the grounds of developing students' understanding of the zero-product property, but Didiş, Baş and Erbaş (2011) found that students took only an instrumental approach to factorising. Solving quadratic equations by factorising can be reduced to finding two numbers which multiply to give a certain amount and add to give another, and then putting them inside brackets after writing 'x +'. This would seem to be just as instrumental as substituting into the formula. The third method, completing the square, is a demanding process, involving careful algebraic manipulation. It forms the basis for the derivation of the quadratic formula, yet Wu (2011) warns that:

> When students see the technique of completing the square merely as a trick to get the quadratic formula rather than as the central idea underlying the study of quadratic functions, their understanding of the technique is superficial. (p. 380)

Anecdotal evidence suggests that although students may prefer 'the formula', and think that it is easier and more reliable, they frequently make errors in using it (such as miscalculating b^2 as $-b^2$ when b is negative) and obtain incorrect answers. Even when a quadratic equation is already in factorised form, students will sometimes expand the brackets, simplify and use the formula, leading to multiple opportunities for error and demonstrating a lack of appreciation of mathematical structure.

However, there is more for the teacher to consider than the efficiency of obtaining a solution to a given equation. Giving students a formula, especially if they are hazy about where it comes from, may position them as recipients rather than authors of mathematics. For instance, it would be perfectly possible to construct a formula for the solution of *linear* equations:

> The solution to the equation $ax + b = cx + d$ is given by $x = (d - b) \div (a - c)$, $a \neq c$.

But it is very unusual to see linear equations taught in this way, presumably because the pedagogical purpose in this topic in not so much to find out as efficiently as possible what x is, as to learn about algebraic equality and solving equations at a more conceptual level. Instead of this formula, students are more likely to be told to 'do the same operations to both sides'. This itself might be regarded as a procedure, yet one arguably giving much greater scope for students to experiment and explore, and thus not, by most definitions, an 'algorithm' (something requiring no judgment [MacCormick, 2012]). However, students may be taught to apply balancing algorithmically, 'dividing by the multiplier', etc. I have seen a student who always carried out this step, even if the multiplier was 1, so that she would convert '$1x = 5$'

into '$x = 5$' by dividing both sides of the equation by 1. When questioned about this, she said that she knew that the value would not change but believed that she had been taught that this was the 'formal' way to do it (see Feynman, 1999: 5-6, for a similar account).

The value of procedures

It cannot be denied that mathematical procedures have considerable instrumental value. No one would want to have to differentiate from first principles every time or derive every formula on each occasion that it was used. A mathematician who wishes to divide fractions will almost certainly 'invert and multiply', but without the 'ours is not to reason why' prohibition quoted above. Yet 'reasoning why' every time would doubtless get in the way of fluent performance of the operation and distract from the wider purpose for which it is being done. So, while it is necessary for the mathematician to retain awareness of the conditions under which procedures are valid, facility with an appropriate algorithm tends to preclude conscious awareness of the details.

However, the best procedures are more than a pragmatic means to a calculational end. Indeed, even algorithms – the most rigid and prescribed of procedures – can be said to have mathematical beauty (MacCormick, 2012). As Crary and Stephen Wilson (2013) put it, "At the heart of the discipline of mathematics is a set of the most efficient – and most elegant and powerful – algorithms for specific operations". There is something neat about the careful construction of an effective algorithm, and many would regard Euclid's algorithm or Dijkstra's algorithm, for instance, as possessing considerable mathematical beauty. Algorithms, like proofs, consist of a series of prescribed steps with a clearly-designated outcome, so if proofs can be beautiful, why not algorithms too (MacCormick, 2012)?

It would seem then that procedures, even strict algorithms, are not inherently harmful in and of themselves. Their rigidity does not have to be experienced as oppressive and destructive to original thought; indeed the affordance of automation may simultaneously open up greater opportunities for originality within a wider context. For Brousseau (1997), "It is the didactical function and didactical presentation which retain or remove the value of a procedure. More exactly, it is the nature of the contract which takes shape on their behalf" (p. 40). If the teacher implies that there is a standard known method for solving a particular problem, this can block the 'devolution of the problem' to the student. Where preferred methods are privileged by their presentation as 'best', without the student coming to see their value for themselves, this is indeed likely to be disempowering. As Gutiérrez (2013) comments:

> when schools demarcate which algorithms are valid when learners are asked to show their work, the practice can lead to immigrant students discounting the knowledge of their parents who have learned mathematics in other countries, even if those 'foreign' algorithms are correct. (p. 44)

It seems essential that students are introduced to useful procedures and acquire facility in their use while at the same time feeling ownership and control over them, but how can this be achieved?

Conclusion

Farmelo (2009: 300) describes how Paul Dirac regarded a mathematician as someone who "plays a game in which he invents the rules". Inventing rules and taking ownership over them are critical elements of doing mathematics, but in order to experience this students' attention must be freed up from the minutiae of incidental procedures. Boaler and Greeno (2000: 185) describe how students (particularly girls) became alienated from mathematics when understanding was side-lined:

> [The students] were capable of practicing the procedures they were given and gaining success in the classroom and on tests, but they desired a more connected understanding that included consideration of 'why' the procedures they used were effective. (p. 185)

But that does not mean that procedures must never be used without consciously thinking about the details. Fully internalising an important procedure so as to develop an intuitive 'expert-induced amnesia' about it shifts the process into implicit memory. This enables the student to operate faster than would be possible with conscious thought and frees up working memory for other things (Syed, 2011).

So I suggest that an algorithm has two possible legitimate roles to play in the teaching of school mathematics:

1. *An object of focus in its own right*: students develop an algorithm to achieve a particular end or take a critical approach to given algorithms, comparing, modifying, inventing and evaluating;
2. *An incidental tool for pursuing a wider mathematical problem*: here the students' attention is deliberately on a larger problem and the algorithm is merely a means to an end.

On the one hand, the algorithm can be probed analytically, and on the other it can be utilised for a grander purpose, where the goal of procedural fluency may be embedded in a richer more worthwhile problem – what I have described elsewhere as a 'mathematical étude' (Foster, 2013b). Burying procedural practice within a more interesting problem may have the advantage of taking attention away from the algorithm, perhaps aiding the development of fluent mastery. What must be avoided is the all-too-common situation where the focus is on the algorithm, but not in order to probe and understand its workings, or to fulfil some greater purpose, but simply in order to perfect its performance.

Viewed in this way, none of the methods mentioned in the staffroom discussion is ruled out *per se*. The quadratic formula is taught, but students attend to its construction and interrogate its components by considering questions such as:

- What happens if $a = 0$? or if $b^2 - 4ac = 0$? or if $b^2 - 4ac < 0$? Why?
- What happens if a, b and c are all multiplied by the same factor k? Why?
- How does the formula compare with an alternative such as $x = 2c \div (-b \pm \sqrt{b^2 - 4ac})$?
- What values of a, b and c will make *both* values of the formula positive/negative/zero? Or lead to one positive and one negative value?

Exploring such questions takes students well beyond simply being on the receiving end of a proof. Similarly, with other methods, such as factorising, critical thought can be encouraged with questions such as:

- How can you solve an equation like $(x - 2)(x + 3) = 8$ for integer x?
- What is the value of $(x - a)(x - b)(x - c)\ldots(x - z)$?

Even more important than posing these questions is encouraging students to ask their own questions about the mathematics (Foster, 2011). If students are to regard methods such as the quadratic formula, 'invert and multiply' for fractions, 'cross multiplying' and so on as more than a trick, they need opportunities to probe and question those methods in order to gain insight into how and why they work.

Injunctions to adhere rigidly to somebody else's rules may be perceived by students as disempowering. The sense of not being trusted to work things out for themselves can lead students into learning to accept rules that make no sense to them. As Noyes (2007: 11) puts it, "Many children are trained to do mathematical calculations rather than being educated to think mathematically". I argue in this paper that the answer is not to eschew procedures wholesale but to ensure that students encounter them in a critical, questioning spirit and then, when convinced of their value, internalise them to the point that they regard them as useful tools with which to pursue more interesting mathematical problems.

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Mathematics at home and at school for looked-after children: the example of Ronan, aged eight

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In several countries, there is concern about the low levels of educational attainment achieved by many children in public care. This paper outlines some of the reasons why looked after children's average attainment in mathematics is poor. Using the case of Ronan, aged 8, I examine the experience that this child's schools offered him in mathematics, as he moved from a school designated 'satisfactory' to one acclaimed as 'outstanding'. Whilst his experience in most areas of school life improved, his mathematics lessons became less effective. I explore the mathematics he did at home with his foster carers, and note that there was little co-ordination between school and family in mathematics.

Keywords: children in care; low attainment; classroom organisation; home-school links.

Introduction

Children in public care (also referred to as 'looked after children') are a tiny minority of the school population: about 0.6% in the UK (Cairns and Stanway, 2013). Most will have experienced family breakdown, neglect, abuse or trauma, and significant loss. Once in care, a high proportion suffers from instability, for example because of frequent changes of care placement. These experiences adversely affect many children's achievement in school (Jackson and Sachdev, 2001). This is illustrated by children's results in public examinations and in national assessments: for example, in Key Stage 2 National Curriculum assessments at age 11 for children in England in 2010, 79% of *all* children reached the 'target' of level 4 or above; 64% of children eligible for free school meals (a recognised indicator of poverty) reached that level; but only 44% of looked after children attained level 4 or above (DfE, 2010; ONS, 2011).

A report from the Social Exclusion Unit (2003) summarised issues that affect looked after children, in addition to the socio-economic factors that contribute to low attainment for many children from similar backgrounds to those who come into care. As well as identifying instability and time out of school as particular problems, the report noted a lack of sufficient remedial help for children having difficulties; that foster carers were not expected or equipped to provide support or encouragement at home; and that looked after children needed more help with emotional, mental and physical health and well-being.

The more hierarchical aspects of mathematics (such as number) are likely to be particularly susceptible to gaps in schooling, caused by moving school, exclusion or absence (all of which are more common for looked after children). The effects of trauma and loss (including bereavement) may include loss of concentration, poor memory, depression and anxiety (Worden, 1996), with consequent effects on the child's ability to learn. Additionally, attachment disorders (Howe, 2006) are common amongst looked after children, resulting in children having difficulty in forming positive relationships with other children, teachers and their foster carers.

Once in care, many authors (for example, Davies and Ward, 2012) have pointed out that looked after children need support to compensate for past disadvantages. At present, there are considerable differences in outcomes for children in different local authorities, and between children in the same local authority: Brodie (2010) in her review of research and practice aiming to improve educational outcomes for looked after children, noted the need for further detailed study in this area.

Methodology

This paper uses data on one child from a larger study, where I undertook case studies of five looked after children aged 7 to 11, each of whom had been identified as being the lowest-attaining pupil in their class. I aimed to examine the ways in which different elements of the children's experience of learning mathematics fitted together, and to explore ways of working with them that might improve their situation (reported elsewhere). My fieldwork covered a period of about twelve months, including the second half of one school year and the first half of the following year, as this would commonly provide a picture of the child working with two different teachers.

I used assessments of the children's work in number; clinical interviews with the children (Ginsburg, 1997); interviews with class teachers, teaching assistants, head teachers, and social work staff; an examination of the children's written and drawn work in the classroom; and interviews with foster carers. Common themes were identified and explored within the analysis across the five case studies.

My study concentrated on number (and specifically on counting, addition, subtraction and place value) as being an aspect of mathematics seen as important by teachers, children and parents or carers. It is also an area where the issue of hierarchy is important, as children need to understand earlier concepts before they can move on successfully (Denvir and Brown, 1986).

Findings and discussion

Ronan was aged 8 and in Year 3 at the beginning of the study. He had been placed with experienced foster carers a year before, along with his siblings. This was initially a temporary placement, but the foster carers had applied to adopt the children, and were waiting for a court hearing. Ronan was attending Brookhouse Primary when I first met him, close to his birth mother's former home, which meant he had a daily taxi journey of about 25 minutes each way to school from his foster carers' home. His foster carers were keen for him to move to their local school, Cranfield Primary, as soon as possible, but this was not arranged until Ronan had been with them for a year.

The two schools that Ronan attended in Year 3 and Year 4 were very different. Here is the summary information from the Ofsted (Office for Standards in Education, Children's Services and Skills) website:

Brookhouse Primary: for ages 3 to 11, 440 pupils on roll; 38% on Free School Meals (FSM). Last three Ofsted reports to 2011: all 'Satisfactory' (a designation now referred to as 'requires improvement').

Cranfield Primary: for ages 4 to 10, 430 pupils on roll; 13% on FSM (i.e. less than the national average). Last three Ofsted reports to 2011: 'Good', 'Outstanding' and 'Outstanding'.

Ronan in Year 3 at Brookhouse Primary with his teacher, Claire Berry

I interviewed Claire in the Summer Term, when she had had Ronan in her class for almost the whole school year. She had been teaching for about three years, and said she enjoyed being at Brookhouse. She taught her class of 30 children as a mixed ability class with five groups based on attainment. Ronan was in the 'bottom group'. There was one boy in the class who was autistic, and he had full-time one-to-one support from a teaching assistant (TA); Ronan and one other boy sat with them. This was also the arrangement for literacy lessons each day, and for reading support. Consequently, Claire tried to provide Ronan with other company for the rest of the day. However, she said that although they were happy to sit with him, other children did complain that he copied their work. Claire said, "I do give him opportunities to say, 'I can't do this, I need help', that he normally takes up.", but Ronan still copied frequently and without trying to hide the fact, "He's not sly about it at all!"

Claire talked knowledgeably about Ronan's work in mathematics, giving details about particular things he could do. His work did not follow the same plans as the rest of the class, because Claire felt that was too difficult for him. His targets were to be able to add two single-digit numbers, and to use a number line; she was also concentrating on helping him to avoid writing numerals in reverse. Claire set the work for Ronan to do each lesson and marked his exercise book. She had tried to encourage him to use counters and cubes for counting, and to draw, but he did not seem to want to do this.

Claire had set homework for Ronan in the previous school term, for example to practise writing his numbers correctly, but it had not been completed, so she no longer set any. She said she was surprised, as she felt the foster carer was very conscientious, but she had not contacted the foster mother, as she thought the school's SENCO (Special Educational Needs Co-ordinator) was in touch with her, and she did not want to complicate matters.

Claire spoke throughout about Ronan with obvious affection and interest, and she had made special arrangements to help him feel settled in her class (for example, by letting him keep his new bag under her chair, until he felt confident about hanging it on a peg like everyone else). Her assessment of his attainment matched the view I had gained through my clinical interviews, and the work she had provided for him seemed to be at an appropriate level. Claire was concerned that Ronan spent too little time working, too much energy on avoiding engagement with his work, and his pace was slow, but she was not sure how to tackle this within the whole class, because he was so far behind everyone else.

Ronan and mathematics at home: the view of his foster mother, Debbie

I interviewed Debbie a few weeks after Ronan had started at his new school, Cranfield. Her view of what the previous school, Brookhouse, was able to offer Ronan was expressed in generous terms, as she said she thought that his new school was able to pay him more attention because they had fewer children with difficulties. Debbie felt that at Brookhouse, Ronan sat with other boys who were naughty, "so he didn't do an awful lot of work". She also said that he had not had homework at Brookhouse; she thought perhaps the teacher did not set it for the lowest attaining children. Debbie commented on the long taxi journey to Ronan's previous school, saying that actually the children were too tired by the time they got home to do any homework.

The transition to the new school, Cranfield, had been managed very carefully, organised by the foster carers with the school, and using several visits to help the children feel comfortable with the move (including during the school summer holiday, when Ronan had met the SENCO each time to borrow games and books). Once the new term started, there was still close contact between school and family; the SENCO frequently came out to talk to Debbie at the end of the day. Ronan was in a special group with the SENCO for reading, and Debbie was spending five minutes a day with Ronan, practising spellings set by the school. However, Debbie said the school did not feel he needed extra help in mathematics, outside normal lessons, "because he's on the low table over maths, and they've got a sort of helper for the table. So he's getting his boost in the class… He's managing, he's coping."

Debbie and her husband John both helped Ronan with homework when it was set, but they had different experiences as to how willing Ronan was to co-operate. With John, if Ronan said he could not do something, then John would explain but then give up. Debbie felt she put more pressure on Ronan to try, and he did do more:

> [I thought], this must have been exactly the same at Brookhouse. He could do them, he was capable of a lot more than he gave… [From John] it was all. 'OK then, if you can't do it, you can't.' But I'm a little bit more 'You *will* do it, I *know* you can do it. You're going to sit and do it! And I suppose I shouldn't have done, but [I'd say] 'If you're going to muck about here then you're not going to the park' and he done it, no problem.

Debbie described herself as "never a great achiever [at school]. I never passed my eleven-plus. But [maths] wasn't anything that I ever dreaded." Even though Debbie had not been provided with mathematics homework for Ronan, she engaged in quite extensive mathematical activity with him, largely in playful and informal contexts. She commented that Ronan had especially enjoyed using a calculator:

> He loved it. He was so proud because he worked out for himself he could check his sums. Before, I don't think he connected: like two plus two – he wouldn't have realised you could put it in and get the answer to come up. Absolutely loved it.

Debbie played games with Ronan (including Ludo and card games), and practised counting with him by asking him to fetch small numbers of household items (for example, pegs or cutlery), and she encouraged him to count with his three-year old sister, ostensibly to teach her to count. When the family went out in the car at weekends, Debbie would ask Ronan to read the numbers on road signs; at home, they sat together, practising writing numbers, as his numbers, "used to be constantly upside down, back to front, and they're not now."

Ronan's foster mother was knowledgeable about what he could do, and what he needed to learn, and she seemed inventive and thoughtful about the methods she could use. For example, she said she had previously given the children a 50p coin for their pocket money each week, but had realised that it was better to give them five 10p pieces, so that they had to count them to check they had the right amount. She was also helping Ronan to learn to recognise different coins.

Debbie felt that the change of school had changed Ronan's opinion of himself, including about his appearance, "He's checking his hair. His pride in himself has changed". She did not mention the biggest change of all – that during the summer holiday, the children had been told they would be staying with Debbie and John, and would be adopted. The next time I met Ronan, this was the first thing he mentioned; the permanency of the placement was very important in making him less anxious.

Ronan in Year 4 at Cranfield Primary with his teaching assistant, Alanna Coates

Ronan's new school had two parallel classes in each year group, and the children in those two classes were separated into two sets for mathematics, depending on their previous attainment. Ronan was in the 'bottom' set; since his class teacher took the 'top' set, he had a different teacher for mathematics. The bottom set was further grouped according to attainment, and Ronan was in the 'bottom' group, with three other children. As his foster mother had told me, there was a teaching assistant who would normally work with this group, in the same classroom as the rest of the class and the teacher. However, because Ronan had seemed to have difficulty in settling down to work in this class, from September to January he had largely been taken out of the class by the TA, Alanna Coates – sometimes with the other three children, and sometimes on his own.

I approached both Ronan's class teacher and his mathematics teacher to find out more about his work in mathematics, but they both felt that Alanna knew most about his work, so I should interview her, which I did in February. Alanna said the group was spending more time in the classroom now, but she still took Ronan out:

> He's still quite a live wire. He wants to be the centre of attention and he would talk for England if he could, so keeping him on track can be tricky at times if he's in one of those moods. He'd like to go to the toilet regularly if it gets him out [of class].

Alanna described several other 'diversionary tactics' that Ronan used; in common with other low-attaining children in my study, he had a wide repertoire of techniques to avoid engaging with his lessons (Griffiths, 2013).

The work for the 'bottom' group was set by the teacher, but Alanna would often change what Ronan did, because she felt he needed easier work, or to provide variety. She used worksheets that she had photocopied from books in the school, items downloaded from the internet, and problems and examples of her own. She provided Ronan with counters, cubes and base ten equipment, but he did not use them.

Alanna showed me Ronan's exercise book, and we talked through the pages. From my interviews with him, I knew he could not yet reliably add two single digit numbers within ten, so I was surprised at the range of topics attempted in his book: there was some work on counting and simple addition, but also work using numbers up to a thousand, on decimals, and on finding equivalent fractions. There was a new topic each day. Alanna did not know how far Ronan could count successfully, but said she did think he had an understanding of 'what is less and what is more'. However, the one worksheet he had completed on this showed a lack of understanding, as although the three questions marked as being 'completed with adult help' were correct, the next three were all incorrect. There was little evidence that Ronan had completed any piece of work during the previous six months successfully on his own.

Alanna said that the children in her bottom group were not given homework. Ronan had not engaged with his class teacher or his mathematics teacher during the year, and the TA had effectively been given sole charge of his work in mathematics. It did not match his level of attainment, and was sometimes marked as correct when it was actually wrong. In many cases, these pages had "Well done, Ronan!" written at the bottom, because Alanna was trying to be encouraging. Alanna had said that she thought he was beginning to 'catch up', but I could not see any evidence that this was the case.

Key issues from Ronan's case

Ronan's chief preoccupation in mathematics was to 'survive' each lesson, using a self-confessed mixture of copying and guessing to complete a minimal amount of work, alongside avoidance tactics such as wandering, trying to strike up conversations on other topics, or otherwise 'opting out' of the lesson (Houssart, 2004). At Brookhouse, his class teacher had recognised this, and although she had not yet found a way of overcoming Ronan's reluctance to engage in arithmetic, Claire was persistent in encouraging him to ask her for help, with tasks that were at a suitable level of difficulty for him. Since she taught Ronan all day, she was sometimes able to provide extra help with counting and number at times outside of mathematics lessons (including at break and lunchtimes).

Ronan's foster mother, Debbie, was able to work with Ronan individually at home. She had a growing emotional bond with him, and had realised that she could insist on his completing a piece of work. She also recognised and enjoyed Ronan's pleasure when he discovered something for himself (for example, when he was adding with a calculator). Debbie was keen to work alongside both schools, and it was unfortunate that she did not have the opportunity to talk to Claire at Brookhouse about the activities she was trying at home. The school's decision to channel all communication through the SENCO was understandable, with the aim of simplifying Debbie's task of keeping in touch, but it did not give an opportunity to pay closer attention to Ronan's mathematics.

At Cranfield, too, the major responsibility for day-to-day communication from the school to the foster carers was undertaken by the SENCO. She knew a great deal about Ronan's progress in reading, but not about his work in mathematics – other than the reassuring comments from the TA. In effect, Ronan's work in mathematics at Cranfield had been delegated twice. His class teacher did not teach him mathematics at all; she did not make any additional opportunities during the day to give him extra help. The teacher of the 'bottom' set had effectively delegated the entire teaching of Ronan to the TA, who had no previous experience of working with a child with Ronan's difficulties in mathematics. This had resulted in inappropriate, dull and sometimes confusing or mathematically incorrect work being provided.

Although Ronan was receiving one-to-one support, it was not effective because the TA was not sufficiently skilled. As Blatchford, Russell and Webster (2012) describe, Ronan was separated from the teacher and the curriculum of the mainstream mathematics class. The time that Ronan spent on mathematics was also less at Cranfield than at Brookhouse, with no supplementary time outside of lessons, and with time spent on searching for a place to work on some of the occasions when Ronan was with the TA.

Both his teacher, Claire, and his TA, Alanna, at some point talked about Ronan's *ability* rather than his *attainment.* The distinction is arguably particularly important for looked after children, who have had disrupted, distressing and traumatic lives; their level of attainment is likely to have been depressed by the times when their education was interrupted or affected by their experiences. It was not something I was able to explore with individual adults across my larger study with all five children, but I suspected that some adults felt that 'ability' is fixed and innate, and they were already convinced that the child they were working with was always going to work more slowly than others. This reduced their expectations of the child.

Ruthven (1987) concluded that 'ability stereotyping' was common amongst teachers of mathematics, and the view that pupils' cognitive capability was fixed was in evidence even amongst teachers who favoured 'mixed ability' teaching. Research in the last decade has further challenged this view, and indicates that cognitive capability can be enhanced (Goswami and Bryant, 2010), but for many teachers, their belief may be that their difficulty in teaching a child is due to the child's lack of ability, rather than due to their own lack of success in finding appropriate methods to promote the child's learning. As to how to change this belief, Ruthven suggested:

> The development of a pedagogy which improves the quality of information about individual pupils, which makes more effective use of this information to remediate learning difficulties and to select appropriate learning experiences, and which reduces inappropriately differential treatment, enabling pupils to learn more successfully, is likely, in itself, to discourage stereotyped perceptions and expectations of pupils. (Ruthven, 1987: 252)

The child's own belief in their potential is often affected by their teacher's view, and the status of the groups in which they are placed (Boaler, 2009). Persuading a child that they *can* be successful, when the child has a long history of failure and avoidance, is not an easy task – it does require persistence, and time with a good teacher.

Conclusion

Looked after children are a small but very vulnerable group of pupils, and many teachers and other adults working with them will not have experience of the level and types of difficulty that the children may present. Certainly, as O'Neill, Guenette and Kitchenham (2010) discuss, a better understanding of the effects of trauma and attachment disruption would be helpful.

Additionally, many teachers would benefit from time to work with a child individually, so that they can gain a better picture of the child's understanding in mathematics (as well as being able to build a better relationship with the child). As Claire commented, she would also have welcomed more expert advice on methods that might be useful with Ronan. It was not clear within her school or local authority that such advice was available.

Ronan's foster carer was not unusual in her interest in his schoolwork, and in her willingness to support him herself. However, Debbie did provide an unusually wide range of activity for Ronan, all embedded in family life. There is a great deal that schools could learn from families about children's interests and activity outside school, and having school and home work together more closely would obviously benefit the child.

Both schools were rightly concerned to improve Ronan's literacy skills, but neither seemed to provide a similar focus on his mathematics. It seemed possible that this was partly linked with the role of the SENCO, as someone in both schools who taught reading but not mathematics.

A recent inquiry by the APPG (All Party Parliamentary Group) for looked after children and care leavers into the educational achievement of children in care (2012) noted the importance of identifying children's needs early, and providing support as soon as possible and for as long as necessary. There is still a need for further support for teachers to improve their skills and understanding of how best to provide the help that is needed – and this may be as much the case in an 'outstanding' school as in one that is in more challenging circumstances.

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Improving students' understanding of algebra and multiplicative reasoning: Did the ICCAMS intervention work?

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In this paper we report on the intervention phase of an ESRC-funded project, Increasing Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS). The intervention was designed to enable teachers to use formative assessment in mathematics classrooms by evaluating what students already knew, then adapting their teaching to students' learning needs. A key feature was the use of models and representations, such as the Cartesian graph, both to help students better understand mathematical ideas and to help teachers appreciate students' difficulties. Twenty-two teachers and their Year 8 classes from 11 schools took part in the intervention during 2010/11. Pre- and post-tests in algebra, decimals and ratio were administered to the students of these classes, and compared to a control group of students matched from the ICCAMS national longitudinal survey (using propensity score matching). The students in the intervention group made greater progress than the matched control.

Keywords: Algebra, Multiplicative reasoning, Formative assessment

Introduction

Increasing Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS) was a 4½ year project funded by the Economic and Social Research Council in the UK.[1] Phase 1 consisted of a survey of 11-14 years olds' understandings of algebra and multiplicative reasoning, and their attitudes to mathematics (Hodgen et al, 2010). This survey involved both cross-sectional and longitudinal samples. Phase 2 was a collaborative research study with a group of teachers that aimed to improve students' attainment and attitudes in these two areas (Brown, Hodgen and Küchemann, 2012). In this paper, we report on Phase 3 of the study in which the intervention developed in Phase 2 was implemented with a wider group of teachers and students. This paper provides on overview of the intervention and results targeted at the BCME audience. A full consideration of the results will be the subject of a longer paper.

ICCAMS was funded as part of a wider initiative aimed at increasing participation in STEM subjects in the later years of secondary school and university. Our research team at King's College London, having considered the existing research on participation in mathematics (e.g. Matthews and Pepper, 2007; Brown, Brown and Bibby, 2008), felt that the main obstacles to participation lay in negative student attitudes; most students did not want to carry on with their mathematical studies because they believed they were not 'good at mathematics', and 'did not understand it'. They also found it 'boring' and 'unrelated to real life'.

Two mathematical areas which are a key part of the age 11-14 curriculum but which seemed to cause particular problems to students were algebra and multiplicative thinking (ratio and the multiplicative use of rational numbers). Algebra, although not perceived as useful by most students and adults, is particularly important in relation to further study in mathematics and in subjects that draw heavily on mathematical modelling. Multiplicative thinking is central not only in mathematics but in the application of mathematics in employment and everyday life, especially using percentages and proportions.

Hence, ICCAMS aimed at increasing student participation through improving their understanding of these topics, and, through this, their confidence in their ability to do mathematics. Additionally it also aimed at demonstrating the importance and power of mathematics and its real-life applications.

The ICCAMS approach

Research suggests formative assessment is an effective approach to increasing attainment and engagement (e.g. Black and Wiliam, 1998). Yet, despite widespread take-up of formative assessment nationally and internationally, there is evidence that teachers have considerable difficulties implementing these ideas (e.g. Smith and Gorard, 2005). This may be because formative assessment has been described vaguely and is thus difficult to implement (Bennett, 2011). It may also be because formative assessment has largely been described generically rather than in subject-specific terms (Watson, 2006). Teachers' ability to use formative assessment in mathematics is limited by their knowledge about key ideas, and the likely progression of student learning in them. Thus if teachers focus on teaching mathematical procedures they may find it difficult to see what is causing problems for students in mastering and applying these, and may thus have difficulty responding to the students' difficulties (Hodgen, 2007; Watson, 2006).

In order to address these issues and provide a 'better' and more didactic description of formative assessment, the ICCAMS team drew on the extensive research literature about developing thinking in multiplicative reasoning (e.g. Confrey et al., 2009; Harel and Confrey, 1994) and algebra (e.g. Mason et al., 2005; Watson, 2009). In addition, we developed a set of design principles for which there is research evidence to indicate they are effective in raising attainment (Brown, Hodgen and Küchemann, 2012). These included connectionist teaching (e.g. Askew et al., 1997; Swan, 2006), collaborative work (e.g. Slavin et al., 2009; Hattie, 2009) and the use of multiple representations (e.g. Streefland, 1993; Gravemeijer, 1999; Swan, 2008). In particular, multiple representations, such as the Cartesian graph and the double number line (see, e.g. Küchemann, Hodgen and Brown, 2011), are used both to help students better understand and connect mathematical ideas and to help teachers appreciate students' difficulties.

The ICCAMS teaching materials

The final set of teaching materials consisted of 20 whole class assessment 'starter' activities and 40 lessons.[1] Each assessment starter was designed to inform a pair of linked lessons. For example, the first algebra starter and lesson-pair address the concept of variable and the notion that letters, and expressions involving letters, can represent a range of values simultaneously. The starter, *Which is larger, 3n or n+3?*, is intended to be used some time before the lessons to allow the teacher time to consider the students' approaches prior to teaching. In each of the two lessons, two linear expressions are compared by considering different representations, first, in the context of a boat hire problem, and, second, by returning to the 'pure' context of $3n$ and $n+3$. In both lessons, students are asked to construct Cartesian graphs of the two expressions and then compare these to tabular, word and symbolic representations of the expressions. Lesson notes provide a description of the lesson together with background materials. The first two pages of the *Boat Hire* lesson are attached as Appendix 1.

Methods

The intervention

The main Phase 3 intervention took place over the academic year 2010/11. Twenty-two Year 8 classes from 11 schools in Hampshire and London took part. Although schools and teachers volunteered to take part in the intervention, the sample of schools included a range of high and low attaining schools. The participating classes were not specially chosen and were the Year 8 classes allocated to the participating teachers. Teachers were asked to use the ICCAMS materials as an alternative to their ordinary teaching of algebra and multiplicative reasoning. Most teachers taught around half of the materials. Teachers also attended six whole day professional development sessions during the academic year led by Hodgen, Kűchemann and Brown.

The tests

Tests in algebra, decimals and ratio were administered to the intervention students as a pre-test in October 2010 and again as a post-test in July 2011.[2] The tests were first used in the Concepts in Secondary Mathematics and Science (CSMS) study in the 1970s (Hart et al., 1981). The tests were designed to assess students' conceptual understanding. The algebra test, for example, is designed to assess students understanding of variables.

The intervention focused on the topics more broadly both to ensure the topics covered related to the algebra and multiplicative reasoning topics within the curriculum more generally and to avoid 'teaching to the test'. So, for example, although a key focus within the algebra strand of the intervention was on the use of the Cartesian graph to develop understanding, the Cartesian graph does not feature on the algebra test or either of the other tests. Hence, the tests can be considered to assess the impact of the intervention on students' understanding more broadly.

The sample: Identifying a matched control group

Just over 600 students took part in the Phase 3 intervention. Not all of these students took both the pre- and post-test in all of the tests, although most students took pre- and post-tests in at least one of the tests.[3] The same tests were used in the Phase 1 longitudinal survey. This survey was administered in July 2008, 2009, 2010 and 2011 to a total of 912 students from six schools (including non-intervention classes from the Phase 2 schools). This survey had a dual purpose: it acted as a comparison for the intervention, and was also used to track students' progression across Key Stage 3. As a result, there was no matched control or comparison group in the design. In order to deal with this problem, we used propensity score matching (PSM) to construct matched comparison groups, based on pre-test score and age at pre-test.

PSM is a statistical method first developed by Rosenbaum and Rubin (1983) for estimating causal effects in studies without random allocation. It enables a comparison to be made between treatment and 'control' groups that is based on subsets of both that are well matched on a number of observed characteristics. Under many conditions PSM can achieve comparison groups that have almost identical distributions on a number of variables simultaneously. This can potentially reduce the problems of trying to interpret differences in outcome measures between two groups that were initially quite different. For this analysis, PSM was set up using logistic regression to predict group membership (intervention/comparison) using pre-test score and age at pre-test as predictor variables. The sample sizes compared in the PSM analysis are shown in Table 1.

Algebra	Number	Ratio
282	292	306

Table 1: Sample sizes for propensity score matching

Results

In this paper we report, first, overall changes in mean score and age, and, second, rates of score gain per year. Other methods, including the use of regression analysis, were also used, but are not reported here. These gave similar results to those reported here, and are discussed more fully in Coe and Hodgen (2012).

Table 2 and Figure 1 show a comparison of the mean pre- and post-tests scores by mean age for both groups (intervention / comparison) in each of the tests (Algebra, Decimals, Ratio)[4].

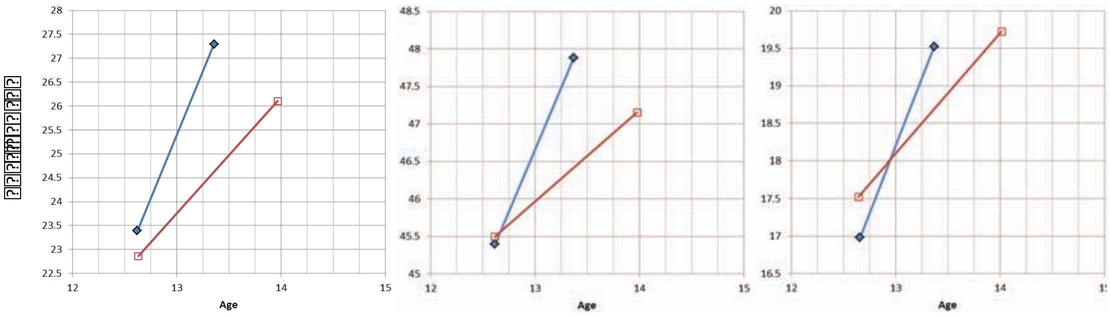

Figure 1: Mean pre- and post scores by mean age for intervention students (Blue) and comparison group (Red) based on propensity score matching for all three tests.

Test	Group	N	Pre-test score	SD	Post-test score	SD	Pre-test Age	SD	Post-test Age	SD
Algebra	Intervention	282	23.39	11.42	27.29	12.74	12.62	0.31	13.96	0.30
	Comparison	282	22.86	12.02	26.10	12.70	12.63	0.37	13.97	0.59
Number	Intervention	292	45.40	14.07	47.88	14.11	12.62	0.30	13.37	0.29
	Comparison	292	45.49	14.47	47.15	15.71	12.61	0.39	13.98	0.58
Ratio	Intervention	311	16.98	8.87	19.52	10.34	12.66	0.35	13.37	0.30
	Comparison	311	17.52	10.07	19.72	10.77	12.64	0.44	14.01	0.62

Table 2: Mean pre- and post- scores and ages for intervention students and comparison group based on propensity score matching for all three tests.[6]

For the comparisons using propensity score matching, the initial pre-test scores of intervention and comparison groups are, not surprisingly, a good deal better matched. At the time of pre-test, in all but one of the comparisons, the intervention group test mean is actually just below the mean for the comparison group. By the end of the intervention, in all cases the intervention group mean is above that of the comparison group, despite the fact that the average time between tests is over six months shorter for the former.[5]

Mean growth rates, which were obtained by dividing the mean change in test scores by the mean time interval, are shown in Table 3.

	Pooled SD of raw test scores	Rate of growth of mean scores (marks per year)		Standardised rate of growth	
		Intervention	Comparison	Intervention	Comparison
Algebra	12.37	5.29	2.43	0.43	0.20
Number	15.10	3.29	1.22	0.22	0.08
Ratio	10.43	3.57	1.60	0.34	0.15
Weighted mean		***4.03***	***1.74***	***0.33***	***0.14***

Table 3: Growth rates for both interventions and matched comparison groups with pooled (weighted) mean for the main Phase 3 intervention highlighted. The standardised growth rates are calculated by dividing the growth rate (marks per year) by the pooled standard deviation.

It can be seen from these figures that the main Phase 3 intervention groups increased their test scores by an average rate of 4.0 marks per year, compared with a rate of 1.7 marks per year for the comparison groups. These increases correspond to standardised growth rates of 0.33 and 0.14, respectively. Overall, therefore, pupils in the intervention groups have shown roughly twice the rate of increase in test scores of those in the comparison groups. In other words, students in the intervention groups have made the equivalent of about two years' normal progress in one year.

Discussion and implications

These results indicate a substantial effect for the ICCAMS intervention equivalent to a doubling of the annual rate of learning. This effect is of a similar order to that found for formative assessment (Wiliam et al., 2004). However, a major criticism of that original study was that formative assessment was described largely generically and, thus, is difficult for teachers to implement in mathematics education. The ICCAMS intervention fills this gap by providing support and guidance to enable such implementation. Hence, this study provides further weight to the evidence on formative assessment, although we note that the ICCAMS intervention drew on the mathematics education literature more broadly in its implementation of formative assessment.

It is important to express a number of caveats to these findings. First, the design did not include a matched control or comparison group and, hence, it is likely that there may be important unobserved differences between students in the intervention and the matched groups. In addition, tests for the intervention classes were administered twice in the same year (at the start and end of the year) whereas for most of the comparison group the tests were all taken at the end (June/July) of the academic year. Hence, the comparison group may suffer a disadvantage due to the summer break. We note, however, that, in a previous study of progression in primary school, the rate of learning during the summer appeared to be on a par with that for the rest of the year for Key Stage 2 students (Brown et al., 2008). Second, the teachers involved in the interventions were self-selected volunteers. Third, the intervention, and in particular the professional development for teachers, was undertaken by the design team.

Hence, whilst these results provide sufficiently strong evidence to justify further evaluation of the ICCAMS intervention, any interpretations of these differences as causal effects of the interventions must be cautious. The results are best interpreted as indicating the need for a further evaluation involving randomised allocation of students and teachers to intervention and control groups.

Endnotes

1. We are grateful to the ESRC for funding this study (Ref: RES-179-34-0001).
2. The full title of the Decimals test is Number 2 (Decimals and Place Value). The Ratio test is titled Test R to avoid indicating the items involve ratio.
3. The numbers of Phase 3 intervention students with both pre- and post-tests are 363 (Algebra), 401 (Decimals) and 399 (Ratio). Pre-tests for one Phase 3 school, involving two classes and approximately 60 students, were not carried out due to time constraints within the school.
4. The maximum scores on the tests are 51 (Algebra), 73 (Decimals) and 24 (Ratio). See Hart et al. (1981) for further information. Note that the year on year gain is relatively small (Hodgen et al., 2010).
5. In all cases, the difference in the growth rates of the two groups is statistically significant (Coe and Hodgen, 2012).
6. The comparison group students tend to be older as a result of the test administration. This is a discussed as a potential limitation later in the paper.

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Appendix 1: The first two pages of the teaching notes for Lesson 1A, The Boat Hire Problem.

Boat Hire

Olaf is spending the day at a lake.
He wants to hire a rowing boat for some of the time.

Freya's Boat Hire charges £5 per hour.
Polly's Boat Hire charges £10 plus £1 per hour.

Whose boat should Olaf choose?

Summary

In this lesson, the boat hire problem is used to explore the two algebraic relationships underlying Freya's and Polly's different hire charges.

A variety of representations are used to express the relationships:
everyday language,
algebraic expressions,
tables of values,
points on a Cartesian graph.

Outline of the lesson

1. Display the Boat Hire problem and ask students for their immediate responses.
   - Ask students to consider the problem further in small groups.
   - Collect numerical data on the total cost for various numbers of hours (and listen to students' arguments and conclusions - but don't pursue these at this stage).
   - Represent the data
     - 'randomly' on the board
     - in a (randomly ordered) table.
   - Represent the data in an ordered table
     - try to prompt the need for this, rather than simply produce such a table.

2. Ask students to represent the hire-rules as algebraic expressions (eg $5a$ and $10+a$), or algebraic relations (eg $b = 5a$, $b = 10+a$).

3. Ask students to represent the data as points on a standard (Cartesian) graph.
   - Are Freya's and Polly's charges ever equal?

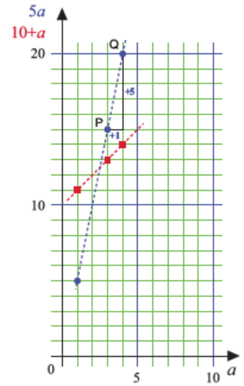
4. Discuss, use, make links between the various representations and between them and the story.

not ordered

a	$5a$
1	5
4	20
3	15

ordered

a	$5a$
1	5
3	15
+1 4	20 +5



Overview

Mathematical ideas

When we work with equations, we often think of a letter as representing a single number as yet unknown. Here, we are working with relations between two variables (the number of hours and the charge) and we think of the letters as representing a set of numbers.

A Cartesian graph is a particularly powerful way of representing variables, since it allows us to represent a range of values simultaneously.

Students' mathematical experiences

Students might discover some of the following
- for some values of a, Freya's hire charge ($5a$) is larger than Polly's ($10+a$), but for others it is smaller
- when $a = 2.5$ the expressions are equal
- if a increases by 1, then $5a$ increases by 5, but $10+a$ only increases by 1

  each set of points on the graph forms a pattern: each lies on a straight line.

Students might discuss
- different slopes and how these relate to the hourly charges
- continuity, ie whether some or all points on the line fit the relationship.

Some students may want to change the scales of the axes. Discourage them from doing so. The expressions have been deliberately chosen to be represented on an equally-scaled graph.

Key questions

When is Freya (or Polly) cheaper?

How could we record this more systematically?

What happens to the cost as the number of hours increases?

What if Olaf hired a boat for 1.5 hours?

Assessment and feedback

Be flexible over the organisation and timing of the lesson. Some teachers have taught this lesson over two periods.

Choose some students to contribute to a subsequent discussion. Some (less confident?) students' contributions may be more coherent if you "rehearse" with them beforehand: "That's a great idea. I'm going to ask you to explain that to the class. Let's have a go at preparing what you'll say."

Allow students time to generate algebraic expressions, but if they really struggle you may want to provide the expressions for them.

Some students may have difficulties constructing a Cartesian graph. Observe the students and decide whether you need to spend some time with either a group or the whole class teaching these skills.

You may need to prompt or challenge some students to consider whether the charges can ever be equal.

Adapting the lesson

You might want to adopt a different context for a subsequent lesson. For example, the price of an ice cream cone for different numbers of scoops, or the yearly cost of belonging to a swimming club (based on a membership fee and a cost for each visit). Choose the numbers carefully - for example, keep the multipliers small if you want to graph the relationhips, and think about where the values coincide - do you want this to occur for a simple whole number (4 ice cream scoops, say), or something more obscure (3.4 scoops, say)?

Trajectory into mathematics teaching via an alternate route: A survey of graduates from Mathematics Enhancement Courses

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We report survey data collated from past Mathematics Enhancement Course (MEC) students. The survey is part of a larger project involving three UK institutions offering the MEC, a UK-based government initiative to address the shortage of secondary mathematics-teachers: whereby non-mathematics graduates enter the teaching profession via a subject enhancement course. We have reported qualitative aspects of this route (Adler et al., 2013; Hossain et al., 2013). Here we discuss the survey conducted to ascertain MEC graduate's experiences of appointment, retention and progression, and our interpretation of this as to whether the MEC with its particular focus on understanding mathematics in-depth is 'fit for purpose'. We asked ex-MEC students from three institutions to respond to an on line survey and 118 participated. Findings include that a large majority (100/118) secured and retained teaching posts, with some progressing in their positions in schools. Our study of subject enhancement courses as an alternative route into teaching emerges at a critical juncture given the cultural and political scepticism concerning such routes and their longevity in the current UK education climate.

Keywords: Mathematics Enhancement Course (MEC); Subject Knowledge Enhancement (SKE)

Introduction

We report survey data collated from past students on Mathematics Enhancement Courses (MEC). The MEC is one of the Subject Knowledge Enhancement (SKE) programmes available for graduates who need to develop their subject knowledge to teach pupils in secondary school. The survey is part of a larger project involving three UK institutions offering the MEC, one since its inception in 2004, and the other two since 2006 and 2007 respectively. For anonymity they are referred to in the paper as institutions A, B and C respectively. The MEC is a UK-based government initiative to address shortage of secondary mathematics-teachers, whereby non-mathematics graduates can enter the teaching profession by taking a subject enhancement course prior to their acceptance for a Post Graduate Certificate in Education (PGCE). We have reported qualitative aspects of this route; provided a detailed literature review; theoretical discussions about the MEC as a social practice and the students' talk of connectedness, reasoning etc. elsewhere (e.g. Adler et al., 2013; Hossain et al., 2013). Although there is some debate that the shortage of teachers may lead to employment/promotion and therefore does not fully reflect the capabilities of the teachers concerned - this remains a question for wider discussion. Here in this paper we present and discuss the survey conducted to ascertain MEC graduate's personal experiences of *appointment, retention and progression* only. We view these three aspects of participation in the profession as indicative of whether the MEC, with its particular focus on understanding mathematics in-depth, has provided its students with sufficient subject knowledge for their future roles as secondary mathematics teachers, and thus whether the MEC can be considered 'fit for purpose'. Although it would have been interesting to compare MEC graduates with non-MEC graduates who enter the teaching profession - this was not the particular focus of the survey data and scope of this paper.

We begin with a discussion of the notion 'fitness for purpose', and how our survey relates to the recent report *Evaluation of Subject Knowledge Enhancement Courses* published by the Department of Education (Gibson et al., 2013). This will highlight that our survey data on entry, retention and progression extends beyond that discussed in the DfE report, providing further substance to the notion of 'fit for purpose'. We provide a brief history and description of the course before we turn to our on-line survey – its methodology and results. We will show that a large majority (100/118) of survey participants (i.e. past MEC students) secured and retained teaching posts, with some progressing in their positions in schools. We defined progression in terms of whether ex-MEC students are teaching A-level and/or holding positions of responsibility e.g. Head of Department, Head of Key Stage 3 (KS3). From this (and wider) data from MEC graduates who entered the profession between 2006 and 2012, we conclude that the MEC is, in terms of our definition, fit for purpose.

Alternative routes and their 'fitness for purpose'

The notion of 'fitness for purpose' has been extensively debated as an indicator of quality in higher education (see for example, Harvey and Green, 1993). It is not our intention to rehearse these debates here. Instead, we use 'fitness for purpose' to refer to whether the MEC meets its overall aim: to enhance the subject knowledge of prospective teachers so that they can take up subject teaching positions in secondary schools. We have found it useful to draw in the ambit of the DfE report, and its evaluation aims and related criteria, and then add to these from our survey to fill out our notion of 'fit for purpose'.

The DfE evaluation report aims, set in 2009, included:

> Assess the effectiveness of the SKE courses in *preparing trainees sufficiently in their subject areas* to meet the Qualified Teacher Status (QTS) standards.
>
> Evaluate the effectiveness of the SKE courses in *equipping trainees to become subject specialists in schools.*
>
> Investigate any differences between traditional entry and SKE candidates during all stages of becoming a teacher (*successfully completing training, commencing teaching in schools, becoming high-quality teachers and progressing within the teaching profession*). (Gibson et al. 2013: 10)

Fitness for purpose as implied in the above, while still broadly defined, is that SKE courses provide students with sufficient subject knowledge to be able to embark on a PGCE ITE course as a mathematics teacher. The DfE report concludes that:

> There is clear evidence in this evaluation of the success of SKE courses in preparing teacher trainees sufficiently with the subject knowledge they require, equipping them to specialise in teaching a subject in schools, providing an alternative route into teaching which is on a par with traditional entry teacher training and supporting the supply and quality of teachers into the profession. (Gibson et al., 2013: 16)

Close reading of the report reveals that much of it is based on reports of past SKE students, and, interestingly, this was done at three points in time: at the end of the SKE, during the PGCE and as a Newly Qualified Teacher (NQT). There is thus no data in the report that enables conclusions to be reached about retention in the profession and/or progression. Our view is that ‘fit for purpose’ must go beyond entry to include retention and progression. Our survey goes beyond the DfE sample to those in practice. We define ‘fit for purpose’ based on criteria for success according to the DfE (2013) survey study above (note the italicised elements in the aims), together with additional aspects of what is considered progression in teaching i.e. teaching A level classes, additional responsibilities related to mathematics teaching, belonging to professional organisations, holding other positions of responsibility, mentoring student and novice teachers and involvement in CPD.

MEC history and description

The Smith Report (2004) concerned with mathematics in the UK - identified the limited numbers studying mathematics at post-16 and degree level - envisaging a significant shortage of mathematics teachers unless action was taken. In response the DfE implemented at national levels the subject-knowledge enhancement course i.e. the MEC for aspirant teachers who did not hold a mathematics degree. This six-month (24 week) ‘mathematics subject knowledge for teaching’ course is completed before commencing ITE programmes. MEC courses have now been running across a number of institutions in the UK for about eight years. Graduates entering these programmes are required to have an A-level pass in mathematics, or some indication of post secondary study of mathematics. This varies across institutions. The MEC programme has provided an interesting context for the QUANTUM-UK study.

QUANTUM-UK project

The QUANTUM-UK study, funded by King’s College London, extends the on-going QUANTUM project. QUANTUM focused on qualifications for teachers under-qualified in mathematics (hence the name) in South Africa. Since 2003, QUANTUM has been investigating the kind of mathematics courses on offer across institutions - focusing on how mathematics comes to be constituted in and across courses that have both mathematical and pedagogical intentions. The rationale for the MEC in UK is to prepare non-mathematics graduates for ITE as secondary mathematics teachers. MEC programmes are focused on both deepening *and* extending mathematical knowledge in ways seen as appropriate to the profession of teaching. The significance of the MEC as an empirical site is not only that it provides an alternate route into teaching, but also that it offers an example where the specificity of Mathematics Knowledge for Teaching (MKT) is expressed through an explicit commitment in course materials to a ‘profound understanding of fundamental mathematics, emphasising deep and broad understanding of concepts, as against surface procedural knowledge’ (TTA 2003: 3).

The QUANTUM-UK study primarily contributed to the area of MKT. However, when presenting and discussing findings within the wider community we found there were strong underlying issues. Namely whether the six-month subject knowledge enhancement course could sufficiently cover the depth and breadth required for teaching mathematics. To what extent would successful MEC students be able to hold their own compared with mathematics graduates entering the teaching profession. Would they be viewed differently by employers and experience issues in securing teaching posts, and would their career trajectories be different? Given these questions and the gap in the literature - especially in terms of studies/reports providing statistical data about MEC students or comparisons with non-MEC students, a survey study to gain knowledge of and insight into the positions and practices of ex-MEC students and their career trajectories as teachers was considered as one of the key parts of the QUANTUM-UK project.

At the same time the QUANTUM-UK study was unfolding - an evaluation of Subject Knowledge Enhancement (SKE) courses was being conducted by the Department of Education (DfE). The DfE report (Gibson et al., 2013), recently published, confirms aspects of our QUANTUM-UK research, in particular that the MEC is mathematically focused – appropriately on in-depth knowledge (Adler et al., 2013) and that it prepares the students well for ITE. While our survey findings were intended to fill a gap – in this paper we focus on elements that were not covered by the DfE study in terms of: ease of entry into the profession, retention and progression, i.e. the extent to which MEC is 'fit for purpose'.

Research Methodology

The empirical work for the QUANTUM-UK study commenced in 2009, following ethical approval from King's College London (and at a later date from Brunel University) and consisted of three phases. In phase one, in 2009, four MEC tutors and 18 MEC students across three institutions were interviewed. Phase two involved re-interviewing these students during their PGCE year in 2010. Phase three involved conducting a survey between March-April 2012, of these students (who would have now completed their NQT year and be in their first year of teaching) and other MEC qualified teachers in the profession in 2012. In this paper we only describe the survey study conducted and report our findings from phase three of the study.

Survey Study

A questionnaire was designed as a survey instrument that aimed to elicit the following information:

- Names and contact details to create a database of ex-MEC students so contact could be made for any follow-up enquiry/study.
- Background information - when and where they completed their MEC course and PGCE; whether they are currently in their NQT year; details of any further study pursued since completion of PGCE.
- To distinguish between ex-MEC students who had progressed into the teaching profession and those who had not. For ex-MEC students who had not progressed, questions were asked about any teaching experience and reasons for not pursuing the profession.
- For ex-MEC students who had progressed into the teaching profession – questions were designed to elicit information in the following four areas:
    - o Ease of entry into the profession (Likert scale) and information/reflection about obtaining their first teaching post.
    - o History of employment i.e. schools and types (state, independent, 11-18), duration of employment, positions, key stages taught, including work as a supply teacher.
    - o Ease of progression within the profession (Likert scale) and whether this includes teaching A level classes; having additional responsibilities related to mathematics teaching within the school; progression in the wider community

in terms of belonging to professional organisations, holding other positions of responsibility; mentoring student and novice teachers and involvement in CPD.

- Information about other mathematics staff in their current department i.e. number of teaching staff and their background qualifications e.g. MEC plus PGCE; mathematics degree plus ITE/PGCE; non-specialists teaching mathematics; number of MEC trained teachers teaching A Levels.

The questionnaire was made available via an online survey tool. MEC tutors from three UK institutions (A, B and C) who were both participants and co-researchers/collaborators on the study located datasets containing names and email addresses of their ex-MEC students from cohorts ranging from 2004-2010 (A: 192 ex-students across 2004-2010; B: 144 ex-students across 2006-2010; C: 120 ex-students across 2007-2010). These included the 2009 cohorts from which we had interviewed 18 students during the MEC (June 2009) and followed them into the PGCE year (June 2010). Each MEC tutor sent a personal email inviting their ex-MEC students to participate in the study and directed them to the online survey.

The total number of students targeted through these email distribution lists was 456 - however it was acknowledged that given the time lapse not all email addresses would work. A total of 134 ex-MEC students initially accessed the online survey and there were 118 valid responses.

Survey results

Sample participant information

First we present some background information of the 118 participants. This is in terms of when and where they completed their MEC course and their subsequent PGCE course; whether they are currently in their NQT year and details of any further study pursued since they completed their PGCE.

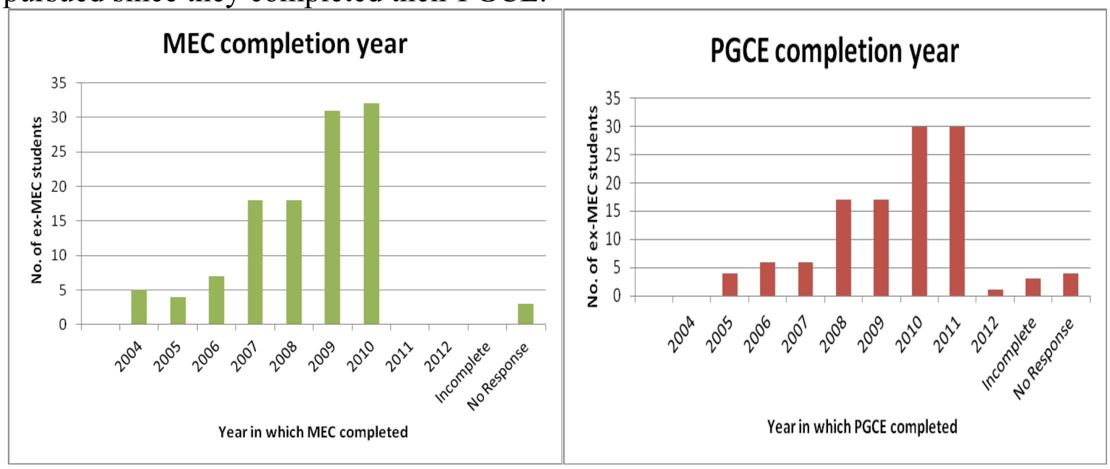

Figure 1 MEC and PGCE completion year

115 participants completed the MEC between 2004 and 2010. The higher frequencies from 2007 are because institutions B and C did not offer the MEC until 2006, whereas institution A piloted the MEC from its inception in 2004. The increased frequencies in 2009 and 2010 reflect substantial increases in allocations to MEC courses. 114 participants completed their PGCE between 2005 and 2012.

62 participants completed the MEC and PGCE at the same institution and 56 completed the PGCE at another insititution. Figure 2 shows where participants

completed the MEC. 36 completed both the MEC and PGCE at institution A compared to 10 and 16 at B and C respectively. The high number of participants (29) who completed the MEC in instution B and then completed their PGCE in another institution was because institution B is part of a consortium and many students moved to another institution within the consortium to complete the PGCE.

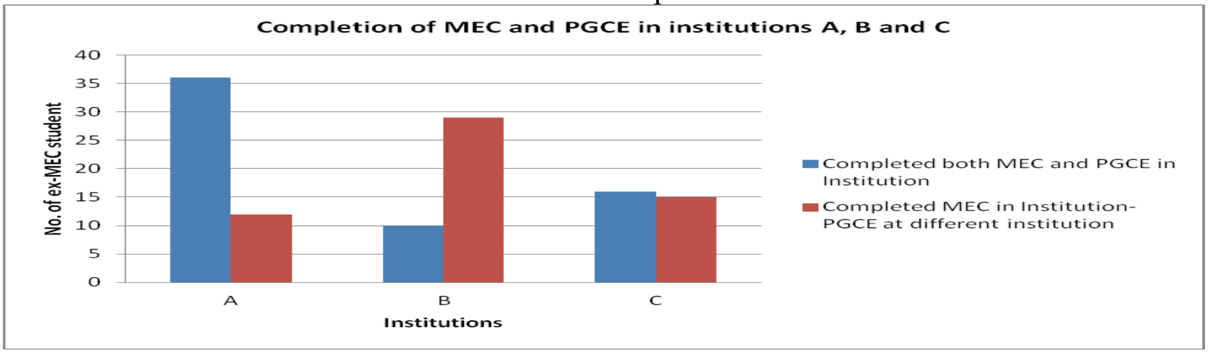

Figure 2 Completion of MEC and PGCE in same/different institutions

Given that the three institutions participating in this study were located across England i.e. North and South - with two from the opposite sides of London and the sample information shows our study participants - included MEC students from all of these three institutions over a period of time (ranging from 2004 and 2010) - we consider this to be a small but representative sample of the ex-MEC students' population.

NQTs in sample and further education

25 respondents were in their NQT year. They would have completed their PGCE in 2011 which suggests that five had not yet secured a post, or decided not to continue into teaching.

23 indicated that they had undertaken further study. The majority had pursued Masters in general teaching and learning/education but seven had studied mathematics - this included BSc in Mathematics (via Open University (OU)), module in pure mathematics (via OU) and Teaching Advanced Mathematics (TAM) course.

Participants within the mathematics teaching profession

A low number of participants 18 of 118 indicated that they had not pursued mathematics teaching. Their reasons included travelling, starting a family, behaviour issues in school and teaching being too stressful. These responses were considered to be generic in nature as they could pertain to any person deciding not to pursue a teaching career. Furthermore because these did not relate to the MEC or the participant's alternative route as a barrier to entry into and retention in the profession we do not elaborate on these students any further in the paper. We focus and present findings related to those who have progressed into the profession.

A total of 100 participants indicated that they had progressed into mathematics teaching and are currently working in schools. The number of years teaching experience varied from less than one to seven. A majority 63 of 100 participants indicated that they had at least two years' teaching experience. Participants in the profession responded to survey questions about their employment history. These included providing details about their NQT post to their most current post. Former MEC students are teaching in a wide variety of schools including secondary comprehensive (state schools); independent schools; academies; faith schools; mixed and single-sex schools.

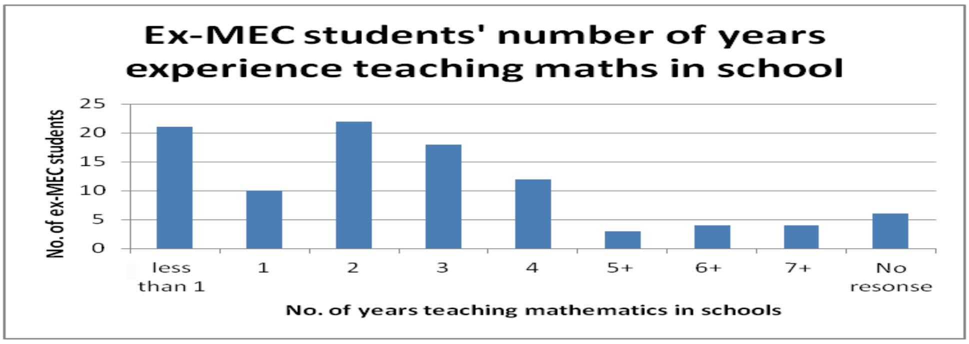

Figure 3 Number of years as a teacher of mathematics

In addition to securing posts as mathematics teachers, some hold positions of responsibility, including: head of department (2); deputy head of department (2); deputy head of house and senior tutor year group (1); assistant head of department (1); second in mathematics department (4); assistant faculty leader for mathematics (1); KS3 coordinator and foundation leader (2); subject/curriculum leaders in mathematics (3); head of year (1); examinations officer (1). All were teaching pupils aged 11 to 16. 32 indicated that they were also teaching A level mathematics.

First teaching post

Most participants reported that it had been easy to secure a teaching post whilst 12 indicated that it was difficult/very difficult. The survey also found:

- One in four secured a post or was offered a post in their PGCE placement school.
- 34 participants secured their first interview after one application. 32 participants made between two and six applications and seven participants made seven or more applications before being invited for interview.

Progression within the profession and in the wider community

83 participants responded to the question about progression within their teaching career. Of these, 45 participants indicated that progression had been easy/very easy for them; 10 indicated it had been difficult/very difficult. 25 participants indicated that they belong to a professional organisation compared to 62 who do not. Only 27 indicated that they mentor student and novice teachers. 51 reported that they are involved in various CPD activities.

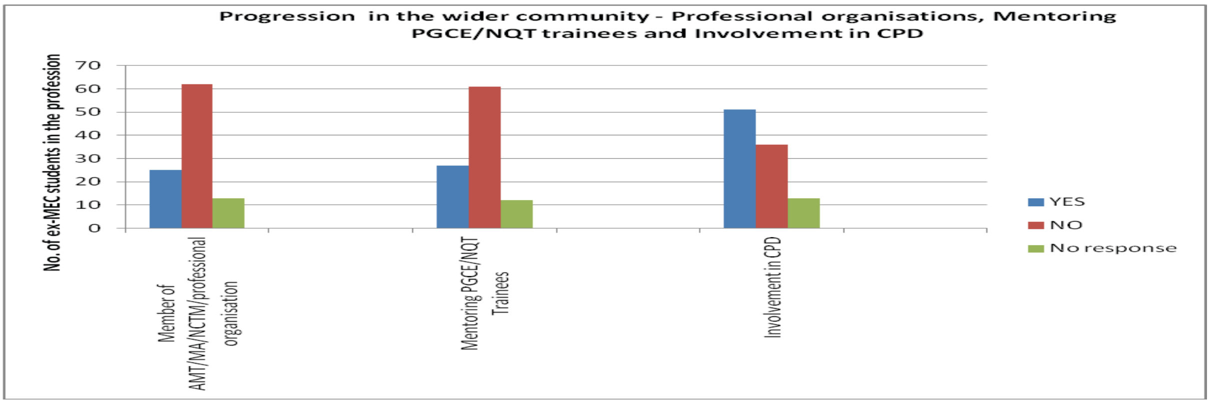

Figure 4 Progression in the wider community; professional organisations; mentoring; CPD

Discussion and conclusion

Our survey sample includes MEC students from 2004 to 2012, with teaching experience ranging from less than one year up to seven years. We go beyond the DfE (Gibson et al., 2013) study that focused on MEC, PGCE students and NQTs. We provide insight into the retention and progression of former MEC students. We found the majority of MEC students (100 of 118) progressed into the mathematics teaching profession. Most secured their first teaching post easily: 25 employed or offered a post in their PGCE placement school; approximately one in three indicating they had been invited for interview after their first application. Our results suggest MEC students did not face barriers to entering the profession despite their MEC route. This is confirmed by the DfE study (Gibson et al., 2013) in which their NQTs also indicated the ease of entering the profession. A large majority of our participants indicated that progression had been relatively easy. Many hold positions of responsibility in mathematics and pastoral care. The DfE survey explored participants' career aspirations and our survey evidence confirms these aspirations are realised in practice. The DfE report (Gibson et al., 2013) suggests that science and mathematics teachers have sufficient subject knowledge to teach 11-16. Although some NQTs teach A level mathematics, confidence about teaching at this level seems less secure. Our data shows approximately 30% of participants are teaching A level mathematics.

Overall our results indicate MEC students' entry, retention, progression in the profession seems to be 'normal' i.e. they generally do not face barriers/challenges given their alternative route. Hence confirming the MEC is 'fit for purpose' (as defined and interpreted within the context of this paper) and supports the DfE finding that SKE courses provide "an alternative route into teaching which is on a par with traditional entry teacher training and supporting the supply and quality of teachers into the profession" (Gibson et al., 2013: 16). We hope this paper provides evidence about the value of these courses in enabling non-traditional entrants to become subject specialist teachers. We also acknowledge it would be interesting to pose questions in relation to trainees with mathematics degrees i.e. whether MEC students apply for the same posts as other PGCE students? And ask wider questions as to whether MEC courses run in parts of the country with a significant shortage of mathematics teachers - as areas for future study.

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How working on mathematics impacts primary teaching: Mathematics Specialist Teachers make the connections

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We draw on analysis of assignments by primary teachers as part of the assessment for the Mathematics Specialist Teachers programme (MaST). In the assignment teachers are asked to work on some mathematics themselves, write up the mathematical part of their work then write about how this experience has impacted on their practice as a primary teacher. We focus first on case studies of teachers who included algebraic work in the first part of their assignments and look at what they say about the connections between this and their practice as primary teachers. Connections are made in a range of ways, but an overall finding is that teachers tended to focus more on the process of *doing* mathematics and the consequences this had for their practice rather than *knowing* mathematics. A further theme was *feelings* about mathematics, entailing positive consequences for practice, even where the initial feelings included negative dimensions. Examination of assignments on other aspects of mathematics confirms the presence of these three themes. Across all the assignments there was strong evidence that this experience of doing mathematics impacted positively on how teachers worked mathematically with their primary classes.

Keywords: Mathematics Specialist Teachers Programme (MaST), primary mathematics, subject knowledge

Introduction

The purpose of this paper is to explore the connections that teachers make between their experience of doing mathematics themselves and their practice as primary teachers. Our interest in this is linked to the idea of the importance of 'profound understanding of fundamental mathematics' (Ma, 1999) in relation to the teaching of mathematics in primary schools. We hoped that analysis of the connections teachers made themselves, as a result of engaging with mathematical activities, would shed some light on how this profound understanding might impact classroom teaching. This paper is based on analysis of written accounts by primary teachers, drawn from assignments they submitted as part of the assessment on the Mathematics Specialist Teachers Programme. We therefore start by looking briefly at the thinking behind this programme and the related issue of teachers' subject knowledge.

Background

The Mathematics Specialist Teachers Programme (MaST)

The MaST programme is a national initiative in England conceived following recommendations by the Williams review of primary mathematics (Williams, 2008). The review envisaged a Masters level programme which would enhance both the subject knowledge and pedagogical knowledge of participants. In particular it is specified that teachers on MaST programmes should '… explore some mathematics in its own right' (Williams, 2008: 25). Following the review, MaST programmes were developed by eight consortia across the country. A key requirement was that each programme should contain three strands, matching the three key objectives of the programme to develop subject knowledge, pedagogy and skills in working with colleagues (Walker et al., 2013). For the purposes of this paper, we focus mainly on two of these three strands, subject knowledge and pedagogy. The development of teachers' subject knowledge in mathematics and the relationship between this and their pedagogy has been the subject of extensive research and debate, which we consider briefly below.

Mathematics subject knowledge

The connection between teachers' mathematical knowledge and their teaching has been the subject of a considerable body of research across the years, much of it drawing on the work of Shulman (1986, 1987). He established seven categories of teacher knowledge, four of which are general and three that are subject-specific. The latter consist of subject matter knowledge, pedagogical content knowledge and curriculum knowledge. These categories have been adapted and developed over the years, for example by Ball and her colleagues in the USA (Ball and Bass, 2000; Ball, Hill and Bass, 2005) who distinguish between common content knowledge and specialised content knowledge, with the latter being required for teaching mathematics. Rowland and colleagues in the UK (Rowland, Turner, Thwaites and Huckstep, 2009) also develop categories of knowledge and put forward the Knowledge Quartet consisting of foundation, transformation, connection and contingency. Ways of conceptualising teacher knowledge are numerous as considered by Petrou and Goulding (2011) and by Ruthven (2011). Despite the volume of work in this area, Watson and Barton (2011: 67) point out that "The question of connection between knowledge and teaching is still open." Our intention is not to attempt a direct answer to this question, but rather to examine in detail the connections that a group of teachers make.

Teachers doing mathematics

In contrast to the extensive literature on teacher knowledge, there has been less investigation of teachers engaging in mathematical activity themselves and how this might impact their practice. One example of such work is a professional development project with secondary teachers in New Zealand carried out by Barton and Patterson (2009). The teachers engaged in this project gave a range of examples of how learning mathematics themselves directly affected their teaching practice. The authors suggest that one impact is that the teachers increase the variety of learning opportunities offered to their students in mathematics lessons. They also point to the effect of students seeing their teachers as learners and the importance of teachers thinking deeply about their classroom practice. Another key piece of work in this area is by Davis and Renert (2014) who looked at ways teachers worked together to develop their mathematical understanding through sustained collaborative investigation which they call 'concept study'. The authors argue that this impacts what they call the 'pedagogical problem solving' of the participants. They conclude by developing the idea of 'profound understanding of emergent mathematics'.

Methodolgy

We started with an initial sample of 57 assignments written by teachers following their experience of working on some mathematics themselves. In the first part of the assignment they wrote up the mathematics, accompanied by diagrams, calculations, tables and formulae as appropriate; in the second part they wrote about the connection between working on mathematics themselves and their practice as primary teachers. Our first step was to categorise essays according to the aspects of mathematics carried out, based on the first part of each assignment. A second step was to look more closely at the second part of the assignments. As a starting point we did this with those assignments identified in our first sorting as containing algebraic work. This was because our informal reading of the assignments suggested many teachers had made use of algebraic notation and formulae and hence we expected some interesting examples in this category. The original intention was to look at aspects of algebraic subject knowledge, for example finding unknowns or looking for relationships and see how teachers made use of their increased knowledge of these aspects in their practice.

The sub group identified for initial, more detailed, analysis consisted of assignments which either included the word algebra in the title, or announced a clear focus on algebra in the first part of the work, for example in stating the search for a formula. Looking at the second part of each assignment and focusing specifically on connections made by teachers, we were able to identify the category knowledge and understanding of mathematics that the literature led us to expect. However, this accounted for only a small part of the teachers' discussion of connections and hence further broad themes were identified. The second of these concerned ways teachers changed their practice in response to their preferred approach to doing mathematics themselves. This category contained approaches such as the need for time, the place of talk and approaches to getting stuck. The third broad category concerned connections made based on how the teachers felt about doing mathematics themselves and their increased awareness of affective factors in the classroom.

Findings

Summary of Findings

The three categories are described in detail below, drawing on extracts from the work of four teachers who explored algebraic ideas. Pseudonyms are used throughout.

Knowledge and understanding

Trevor is chosen as an example here because the connections in his essay included explicit mentions of knowledge and understanding of algebraic ideas. Trevor had elected to work on sections of a book on algebra aimed at teachers that offered readers tasks to try (Mason, Graham and Johnston-Wilder, 2005). In the first part of his assignment, headed 'Improving my own algebra' he presents his work on these tasks, using algebraic notation and identifying themes in his own approach such as specialising and generalising, reflecting the approach to algebra implicit in the book he was using. When Trevor moves on to write about connections between his experience of doing this and his practice as a primary teacher, he initially focuses on algebraic themes identified in part one of his assignment.

> I am trying to introduce generalisations explicitly into my lessons ... During feedback from a recent observation, my head expressed surprise that I had asked pupils for generalisations relating to multiplication facts, but delighted that they knew exactly what I meant and were frequently able to respond appropriately. So far, I have only tentatively introduced the idea of symbols to represent values. One area in which pupils have indicated that they understand the concepts is in two-part problem solving. The pupils are encouraged to use a cloud symbol ... to represent the link between the parts of the problem, the answer to the first number sentence becoming part of the second part. One advantage, in this situation, of the cloud, is that the pupils are able to fill the cloud with the appropriate number when they have found it. This may not strictly be algebra, in that answers are not general, but it is an introduction to the idea that the unknown can be worked on.

In the extract above, Trevor shows a deep understanding of key algebraic ideas, such as generalisation and the ability to transfer this to his practice at a level appropriate for the pupils he teaches. He also shows awareness of the fact that symbols can be used for both unknowns and for variables. However, having acknowledged these aspects, Trevor moves on to suggest that changes in his approaches to teaching are much broader than the incorporation of basic algebraic ideas.

> I feel that the algebra I have done so far has only slightly improved the level of algebra at which I am working. However, it has made a significant difference to the way that I look at and teach mathematics, I now expect the pupils to look at maths in a much more open way, to focus on why particular problems produce the answers that they do and to be moving towards making far more connections between different aspects of mathematics. ... The pupils appear to find the maths that they are doing to be more challenging, but also more fun and interesting. There is frequently a cheer when I introduce a numeracy lesson!

This second extract from Trevor's writing, includes examples of the two themes present in many of the essays, namely changes to practice based on approaches to doing mathematics and affective factors. These are a small part of Trevor's writing, but other teachers, such as Elaine (below) considered them in more detail.

Changes to practice

Elaine started her assignment by outlining her approach to the problem 'Counting triangles' which started by asking the total number of triangles in an equilateral triangle with edge length four drawn on an isometric grid. A key algebraic aspect of Elaine's work was her search for number patterns and ultimately for a formula. There was a particular focus on square numbers and when Elaine started writing about connections to her practice she suggested that this new depth of understanding would impact positively on her teaching.

> These ideas made me reconsider the thinking process I went through as I was thinking about square numbers during my study of the number of triangles. I absolutely had to go back to the root of what a square number is. The impact of this has been to force me to challenge my own assumptions and this has given me a fuller, deeper, more meaningful understanding of what a square number is. I had to *deconstruct* to go to the *physical* root of square numbers. I have no doubt that this new depth of understanding will serve me well in my teaching and will have a positive impact children's learning.

Although Elaine acknowledges her new depth of knowledge, she writes in much more detail about other connections between her own investigation and her practice. Her account of her investigation made several references to the approaches she had taken, including those that suited her and those that did not. For example, in the early stages of working on the problem she showed enthusiasm for a 'lift the flap' approach using coloured paper, scissors and sticky tape, contrasting this to an approach based on looking at diagrams which others had used but which she described as 'not enough' for her. In later stages of the problem-solving process, she discovered with some amusement that "… there are actually 'how many triangles' facebook pages with hundreds of fans!" However she rejected this possible source of advice, in favour of working in person with an individual. When it came to making connections in the second part of her assignment, the focus for Elaine was on different approaches to working on mathematical problems and how the pupils she taught should be given the opportunity to approach mathematical problems in ways that suit them. She considers this below:

> In reflecting on my experience of trying to solve the triangle problem at the local meeting (in which I didn't have the time, equipment or freedom to explore the problem in the way I needed to) ... By exploring the triangle problem in a way which suited me, it supported my development, I learnt and it was challenging but thrilling! This has implications for the classroom. Children need to be given time to explore a problem in their own way and there needs to be an atmosphere in which children's thinking and ways of working are encouraged and valued ... this draws me to conclude that we absolutely must allow our children the time to explore problems in a way that suits them in order that they will **fully** understand.

Elaine's consideration of her own deeper understanding is reminiscent of the profound understanding of fundamental mathematics identified by Ma (1999) but she goes beyond this to draw clear messages about the practices she will employ in her classroom which relate directly to her own experience of working on the triangles investigation.

Enjoyment and motivation

Both Trevor and Elaine make some reference to enjoyment of mathematics. Trevor does this when he talks about his class, Elaine refers to her own pleasure when working in a way that suited her, although she also acknowledges the frustrations of working in a way that did not suit her. For some teachers, affective factors were a key part of their discussion about connections. One teacher, Phillip, talked explicitly about how he tried to make links to his practice and in doing so moved from connections about approaches to doing mathematics towards those related to motivation. Phillip had been looking at magic squares in his investigation, with a particular focus on identifying general formulae. Phillip is a confident mathematician and drew extensively on algebraic symbols and terminology in the first part of his work. In the second part of his assignment, he started to consider the connections between this experience and his role as a primary teacher.

> So what can I gain from this investigation that can positively influence my teaching? Initially, I thought that I could use this to help understand the process of problem solving in order to teach it more effectively. I often hear from colleagues that the using and applying aspect of the maths curriculum causes problems, as the children find it difficult to apply what they have learned. Typically this related to word problems…

Phillip moves on from this to acknowledge a mismatch between his experience of working on the magic squares problem and the way children are sometimes encouraged to tackle word problems. He briefly talks about what he calls ‘a more generalised understanding of problem-solving within mathematics’, but then moves on to discuss affective factors.

> This started taking me into a different direction. During the investigation I was highly self-motivated, I reviewed my work and thinking, I engaged in a high level of meta-cognition, I used and applied a range of knowledge and strategies and developed a problem solving approach subconsciously. So perhaps I need to focus on the use of problem solving as a valuable and enriching way of teaching mathematics and move away from the idea of teaching problem solving explicitly.

Although Phillip explicitly mentions motivation and engagement, he also considers knowledge and approaches to doing mathematics; hence the three strands we identified are closely interwoven in his account. Phillip’s initial feelings about doing mathematics were positive, but some teachers were more apprehensive, as illustrated by our final example, Karen.

Karen starts her account by acknowledging negative feelings about algebra and this is part of her rationale for embarking on an investigation that draws on algebraic notation. Karen elected to work on ‘Tring squares’ an investigation that explores the total of sequences of digits in growing squares.

> I have found that returning to undertake mathematics at a level that was going to be a personal challenge has recreated feelings towards mathematics that I used to experience at school. Whist at school, if a piece of mathematics was linked to algebra, I would develop this whole body experience of sweats, rising panic, surges of adrenaline, to the point that I would frequently drop my pencil and find any avoidance strategy to avoid completing the mathematics.

Karen provides more detail on these memories and feelings, but then indicates her determination to tackle this issue.

> It wasn’t until I sat down in our first local meeting to share the task of Tring Squares that I had ever admitted this mathematics anxiety to myself or anybody else. Therefore I decided that it was time to challenge this anxiety by unpicking what algebra I could do, to learn new methods to solve problems and to improve my teaching as a result of my new subject matter knowledge.

In the sections following, Karen tackles the Tring squares investigation and also works explicitly on her own understanding of aspects of algebra such as use of symbols. Later she reflects on the usefulness of her experience in her teaching. As part of this discussion, she picks up again on the idea of maths anxiety.

> I now have an understanding of what maths anxiety is which has resulted in me challenging myself to ensure that I don’t pass this onto the future children that I teach … but provide them with strategies to cope whenever they become stuck or they reach the panic stage when faced with something new or challenging.

Karen was not the only teacher who acknowledged some negative emotions as part of her description of how she went about the mathematical task. The encouraging thing was that teachers doing this, like Karen, were able to draw positive messages for their teaching from their own negative emotions, often focussing on what they wanted to avoid as far as their pupils’ experiences were concerned. Our findings about the importance of emotions in relation to doing mathematics, are in line with the work of Evans (2000) based on his study of adults. He points to the importance of the roles of affect and emotion in learning and suggests that they cannot be separated from cognitive engagement during learning activities.

Other aspects of mathematics

Having started to categorise the types of connections teachers made, our next step was to consider whether this categorisation still appears to be valid for those writing about aspects of mathematics beyond algebra. Our findings suggest a similar pattern in essays dealing with other aspects of mathematics. For example, a teacher exploring divisibility rules wrote about depth of understanding and about engagement with the investigation and approaches to getting stuck. As well as acknowledging knowing and feeling, this teacher drew on experience of doing mathematics to draw a range of messages for his own practice which included focussing on the mathematics as well as the patterns, focussing more on the journey and less on the answer and allowing children the space to make mistakes.

Another teacher acknowledged how she had deepened her understanding of arrays and multiplicative reasoning as a result of working on this aspect and at the same time experienced frustrations and developed mathematical resilience, linked to the 'willingness to struggle' identified by Johnston-Wilder and Lee (2008). The latter was done partly by the way she worked with others and this led her to consider the place of talk in her mathematics classroom and a consideration of what is needed for meaningful talk to take place.

A teacher who explored operations with negative numbers was led to considering the difference between the operation of subtraction and the meaning of a negative number, as well as developing her use of conjectures. She also identified perseverance as important to her in this work. Her connections to her practice included reviewing the meaning and usefulness of informal jottings and supporting children to use mathematical language as well as the importance of building positive attitudes to mathematics and resilience.

Summary and discussion

In making connections between their experience of doing mathematics and their practice in primary schools, the teachers did show evidence of developing and drawing on their understanding of mathematics. Sometimes this was about meeting new ideas, but often it was about developing a deeper understanding of aspects of mathematics they were already aware of. However, they also made other connections concerning approaches to doing mathematics and feelings about mathematics. All three are put forward by teachers as impacting on their practice and in many cases the three are written about in a closely linked way, giving further evidence of connections. These findings are in line with those of Barton and Paterson (2009), who noted that the teachers in their project drew parallels with the way they engaged in mathematical activities as part of their own learning and the ways they encouraged their classes to work subsequently. Our findings are consistent with the ideas advanced by Davis and Renert (2014) in their consideration of ways teachers have worked together to develop their mathematical understanding and how this has impacted on their practice.

Our findings suggest that if teachers are given the opportunity to engage in mathematical activities themselves, they are likely to make connections between this and their practice that includes a consideration of how the children they work with are enabled to do mathematics. With continuation of the MaST programme currently uncertain (Walker et al., 2013) and with the mathematics curriculum in England undergoing changes, it is to be hoped that opportunities for doing mathematics will still form part of professional development for primary teachers and that they will have enough freedom in implementing the new curriculum to make connections which encourage the children they teach to work mathematically.

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Classroom environment variables and mathematics achievement of junior secondary school students in Cross River State, Nigeria

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The study was designed to investigate the influence of classroom environment variables on mathematics achievement of junior secondary schools students in Cross River State, Nigeria. It was a survey research involving 1200 Junior Secondary 2 (JS2) students from 48 secondary schools in Cross River State. A valid and reliable achievement test and questionnaire were used for data collection, while multiple regression analysis technique was used to analyse the data. The research findings indicated that: there is a significant individual and combined prediction effect of the classroom environment variables on students' mathematics achievement; twenty-seven (27) out of the thirty-six (36) paths in the hypothesised recursive model are significant at .05 probability level. Based on the findings, it was concluded that classroom environment variables significantly influence students' mathematics achievement in secondary school. It was recommended among others that teachers and school counsellors should educate the students on the need to establish a good relationship with teachers and themselves in the classroom in order to improve their performance. Also, classrooms should be furnished and equipped to enhance effective teaching and learning.

Keywords: classroom, environment, mathematics, achievement

Introduction

Education in any country is a tool for national development. Investment in it therefore, is considered relevant both to individual nations and to individuals and their families. For individuals to function effectively in the society they must be educated. Education is seen by many people as the key to future success and mathematics as a major tool to turn this key in order to unlock the door to development (Meremikwu, 2008; Ayodele, 2004). Thus, the place of mathematics in education and development is central. This is because scientific and technological developments of the centuries have depended to a larger extent on mathematical development. Nigeria is regarded as an under-developed nation because of its backwardness in scientific achievements as well as in technological developments (Meremikwu, 2008).

The role of mathematics in the field of science and technology is enormous and cannot easily be over-looked. It is essentially a dynamic science which serves as the underlying knowledge for science and technology. Because of the importance of mathematics, the Federal Government of Nigeria made it a compulsory subject at the primary and secondary levels of education and a basic requirement for admission into all levels of education. A strong background in mathematics is therefore critical for many jobs and career opportunities in today's increasingly technological society.

The Federal Government of Nigeria in its National Policy on Education (Federal Government of Nigeria, 2004) set out some specific objectives to be achieved in the school system. To achieve this, a conducive classroom learning environment, apart from the background of the learner, should be nurtured and utilised by erudite teachers by invigorating the respective organs of cognitive, affective and psychomotor domains. For a teacher with a good knowledge of the subject matter in a good conducive learning environment, with students having a good background, it is expected that the intellectual achievement of every child would be above average. Within the classroom learning environment variables which have significant influence on students' academic achievement include time teacher spends in the classroom; physical layout of classroom; classroom climate; teacher's motivation of students; instructional material utilisation; classroom management skills; teacher-student classroom interaction; student-student classroom interaction and many more (Igiri, 2006; Mgbechi, 2006). Udonwa (2001) and Rasser (1993) are of the opinion that teacher classroom management skills influence academic performance. Essien (2004) found that teacher motivation of students influence students' performance. Battistich, Schaps and Wilson (2004) and Akpan (2002) see student-student interaction as one of the variables that influence students' performance. Jones (2004); Nnaka and Anaekwe (2000) in their studies found that teacher-student interaction do not influence performance. Barkley (1998); Stallings and Kaskowitz (1994) and Frazer (1994) are of the opinion that the physical layout of classroom, time teacher spent in classroom and classroom climate respectively influence students' performance.

Educational researches have also been carried out using static or stable variables such as intelligence, teachers' and students' characteristics such as age, sex and socio-economic status (Halpern, 1992). Others use variables that are part of the teaching – learning process such as classroom environment variables, instructional materials, instructional method (Meremikwu, 2008).

However, these studies only examined the influence and/or a combination of some of the classroom environment variables that is, teacher-student classroom interaction, student-student classroom interaction, learning facilities, physical layout of classroom, classroom climate, and classroom management skills just to mention a few on students' academic performance. Most of these studies thus use statistical techniques like t-test, Analysis of Variance ANOVA and correlation for data analysis. For instance, Ukashia (2010) used ANOVA to study the influence of classroom learning environment variables like class size, teacher-pupil relationship, peer group influence, learning facilities, teachers' leadership style and pupils' attitude towards the school. Also, Mgbechi (2006) used ANOVA and independent t-test statistical techniques to study the influence of classroom interaction variables like teacher-student interaction, student gender, teachers' interaction pattern, teacher gender and subject matter on interaction patterns.

Some of these studies did not provide empirical evidence of the influence of the classroom environment variables and the dependent variable (students' performance) especially when the classroom variables are taken together or combined. It is in order to fill this gap that provoked this study to ascertain the interaction of classroom environment variables (Time Teacher Spends in the Classroom TSI; Physical Layout of Classroom PLC; Classroom Climate CC; Teacher's Motivation of Students TMS; Instructional Material Utilisation IMU; classroom management skills CMS; Teacher-Student Classroom Interaction TSI; Student-Student Classroom Interaction SSI) and their effects on students' mathematics performance. This has made it imperative to study classroom environment variables and students' mathematics achievement in schools in Cross River State in order to make appropriate suggestions and recommendations based on the finding of the study. It is hoped that from this, inputs could be made to improve students' mathematics achievement.

Purpose of the study

The main purpose of this study is to determine the relationship between classroom environment variables and students' mathematics achievement. Specifically, the study was designed to: (a) examine the extent to which the following classroom environment variables (Time utilised by teacher; Physical layout of classroom; Classroom climate; Teacher's motivation of students; Instructional material utilisation; Classroom management skills; Teacher-student classroom interaction and Student-student classroom interaction) do individually and collectively predict students' mathematics achievement; and (b) examine the significant pathways in the path model through which the classroom environment variables determine students' mathematics achievement.

Hypotheses:

The following hypotheses were tested in the study:

1. There are no significant individual and combined prediction effects of the variables of the classroom environment variables (Time utilized by teacher; Physical layout of classroom; Classroom climate; Teacher's motivation of students; Instructional material utilisation; Classroom management skills; Teacher-student classroom interaction and Student-student classroom interaction) on students' mathematics achievement.
2. There are no significant pathways in the path model through which the classroom environment variables predict students' mathematics achievement.

Methodology

The design adopted for this study was survey research. Multi-stage sampling techniques were adopted for this study. The techniques include stratified random sampling and simple random sampling. The basic criterion for stratification was on the basis of educational zones and government areas. The state Ministry of Education has already stratified the state into three educational zones and the study was carried out in these three zones. A simple random sampling technique was used to select four Local Government Areas LGAs per zone, four schools per LGA and 25 students per school. The calculation of the sample size was based on the (Yamane, 1967: 886) formula: $N/(1+N(e^2))$ which gave a minimum sample size of 399.98 plus 200% of the minimum sample size resulting in 1200 students. The sample for the study consisted of 1200 Junior Secondary 2 (JS2) students randomly drawn from a total population of 18 573 (JS2) students in the study area.

Two research instruments were used for data collection, these were: a 48-item four point Likert-type questionnaire and a 40-item Mathematics Achievement Test (MAT). To achieve the validity of instruments, the items of the questionnaire were face validated by two experts in test and measurement, while a specification table was constructed for the content validity of the test and item analysis was done to show the item difficulty and discrimination. The reliability coefficient estimate of the questionnaire using Cronbach Alpha after a pilot test (since the derived scores were continuous) ranged from 0.54 for Instructional Materials Utilisation (IMU) to 0.75 for Classroom Climate (CC). While the reliability estimate of the Mathematics Achievement Test using the K-R 20 method (since the derived scores were dichotomous) was 0.51. A reliability coefficient of 0.50 is good enough for any instrument to be suitable for use (Nenty, 1985).

Results

Hypothesis one

Table 1 is in two parts; the upper part shows the combined effect of the eight classroom environment variables (teacher-student classroom interaction, classroom management skills, teacher motivation of students, physical layout of classroom, time utilised by teacher, instructional material utilisation, student-student classroom interaction and classroom climate) in predicting students' mathematics achievement, which yielded a coefficient of determination (R^2) of .368. The result shows that the eight classroom environment variables could account for up to 36.8% of students' mathematics achievement. The mean scores of the variables of the study ranged from 17.53 to 24.87 and their standard deviations ranged from 2.71 to 4.25. The low values of the standard deviations show the homogeneity of the response opinions of the respondents. The result shows a multiple R of .606, coefficient of determination R^2 of .368, adjusted R^2 of .363 and an F-ratio of 83.536 with 8;1149 degrees of freedom. It also produced an equation shown below:

MAT = 16.110 + .053TSI + .187CMS + .495TMS + .396PLC +.243IMU + .184TSC + .067SSI + 094CC

Table 1: Summary of multiple regression analysis of the individual and combined effects of the eight classroom environment variables to the prediction of students' mathematics achievement

Multiple R	=	.606
Multiple R – square (R^2)	=	.368
Adjusted multiple R-square	=	.363
Standard error of estimate	=	3.175

Source of Variation	sum of square	df	mean square	F- ratio
Regression	6738.862	8	842.358	83.536*
Residual	11586.223	1149	10.084	
Total	18325.085	1157		

Variable	Unstandardised regression weight (b)	Standard error of regression weight	Beta weight (β)	t-value	sig t
TSI	.053	.031	.040	1.711	.087NS
CMS	.187	.036	.127	5.155	.000**
TMS	.495	.026	.309	18.939	.000**
PLC	.396	.023	.225	17.014	.000**
IMU	.243	.040	.101	6.119	.000**
TUT	.184	.025	.106	7.348	.000**
SSI	.067	.027	.059	2.475	.013**
CC	.094	.036	.079	2.573	.010**

*Significant at .05 level (for F-ratio); ** P < .05; NS =Not significant at .05 levels.

MAT=16.110+ .053TSI+ .187CMS + .495TMS + .396PLC+ .243IMU+ .184TSC + .067SSI + 094CC

Looking at the individual contributions of the predictor variables, it is observed from the result that the standardised regression coefficients (β-weights) range from .040 to .309. The unstandardised coefficients range from .053 to .495. The errors of the estimates range from .023 to .040, while the t-ratios ranges from 7.348 to 18.939. However, the β-weights show the strength of each predictor variable (Table 1). The analysis of these strengths shows that classroom management skills (β-weight of .127), teacher motivation of students (β-weight of .309), physical layout of classroom (β-weight of .225), instructional materials utilisation (β -weight of .101), time utilised by teacher (β -weight of .106), student-student classroom interaction (β-weight of .059) and classroom climate (β-weight of .079) are significant predictors of students' mathematics achievement. Only teacher-student classroom interaction (β-weight of .040) is not a significant predictor of students' mathematics achievement.

The prediction equation with these eight predictor variables is also shown. From this equation, it could be observed that the variables predict students' mathematics achievement positively. Therefore, the null hypothesis which states that "there is no significant individual and combined prediction effect of the variables of the classroom environment variables on students' mathematics achievement" is rejected.

Hypothesis Two

To determine the path coefficients in the hypothesised model, the zero-order correlations and beta weights (β) for each path were generated using the enter method of the multiple regression analysis technique. The result shows that the least path coefficient is p_{76} (β = -.217) while the strongest path is p_{93} (β = .309). The result also reveals that 27 out of the 36 paths in the hypothesised recursive model are significant at .05 probability level. Therefore, the null hypothesis is rejected for 27 paths out of 36 possible pathways this is because the paths whose β-weight or coefficients are significant at .05 probability level (β-weights greater than .05) were retained, while the others were trimmed to produce the more parsimonious (over-identified) model.

Summary of findings

From the results of data analyses, the following can be deduced:

i. There is a significant individual and combined prediction effect of the classroom environment variables on students' mathematics achievement.
ii. Twenty-seven (27) out of the thirty-six (36) paths in the hypothesised recursive model are significant at .05 probability level.

Discussion

The result of the testing of hypothesis one, which is the individual and combined contributions made by the variables in predicting students' mathematics achievement, has this to say. For the individual contributions, the t-value associated with each of the eight variables of the study reveals that seven (classroom management skills, teacher motivation of students, physical layout of classroom, instructional materials utilisation, time teacher spends in classroom, student-student classroom interaction and classroom climate) out of the eight variables were significant at .05 level. This means that only these seven variables contributed significantly to the prediction of students' mathematics achievement at .05 level.

A closer look at these contributions shows that teacher motivation of student made the most significant contribution (t = 18.939) followed by physical layout of classroom (t = 17.014), time teacher spent in classroom (t = 7.348), instructional materials utilisation (t = 6.119), teacher classroom management skills (t = 5.155), classroom climate (t = 2.573) and finally student-student classroom interaction (t = 2.475|) to the prediction of students' mathematics achievement.

The result that teacher classroom management skills (t = 5.155, p< .05) was a significant predictor of students' mathematics achievement is in support of the findings of Udonwa (2001), Rasser (1993). The result that teacher motivation of students is a significant predictor of students' mathematics achievement (t = 18.939, p< .05) is consistent with Essien (2004), who studied teachers' variables and secondary school students' academic performance in social studies Cross River State with a sample of 1000 students and teachers and found that there is a significant relationship between teachers' motivation and students' performance. The physical layout of the classroom as a significant predictor of students' mathematics achievement (t= 17.014, p<.05) is consistent with Barkley (1998) who is of the view that closed classroom architecture (i.e. four walls and a door) is more conducive for learning than an open classroom design since this presents considerably less auditory and visual distractions that impair the concentration of students.

The result that the time a teacher spends in the classroom is a significant predictor of students' mathematics achievement (t = 7.348, p<.05) is consistent with the findings of Stallings et al. (1994) who found that effective time use in the classroom could raise students' achievement. In the study teachers and students were observed and students were tested for learning gains in some subjects taught.

The findings of the study show that student-student classroom interaction is a significant predictor of students' academic achievements (t = 2.475, p<.05). For student-student interaction, this finding is consistent with the studies of Akpan (2002), Battistich et al. (2004) among others who all agree that there is a significant relationship between student-student classroom interaction and students' academic achievement.

The findings show that classroom climate is a significant predictor of students' mathematic achievement (t= 2.573, p<.05). This agrees with Frazer (1994) who examined the results of 40 past studies that showed collectively, that classroom climate and classroom environment factors are positively linked to valued students' achievement and affective outcome. The results also show that teacher-student interaction (t = 1.711) is not a significant predictor of students' mathematics achievement which is contrary to the earlier findings of Jones (2004), Nnaka and Anaekwe (2000).

The result of findings of the combined effects of the eight classroom environment variables on students' mathematics achievement as shown in the upper part of Table 1 showed a significant effect. That is, the eight classroom environment variables (teacher-student classroom interaction, classroom management skills, teacher motivation of students, physical layout of classroom, time spent in classroom factor, instructional material utilisation, student-student classroom interaction and classroom climate) when taken collectively are effective predictors of students' mathematics achievement among the respondents in the study.

Conclusion

The results of the study show that classroom environment variables have a relationship with students' mathematics achievement in junior secondary schools. This suggests the importance of classroom environment as a major component of variations in the teaching-learning behaviour. Also, this is an indication that the classroom environment created by the teacher has a positive effect on students' achievement in mathematics.

Recommendations

Based on the findings of the study, the following recommendations are made:

i. As the teacher related classroom environment variables appear to be significant predictors of students' mathematics achievement. Teachers and school counsellors need to educate students on the value of establishing a good relationship with teachers and one another in order to improve their performance.

ii. Mathematics teachers can enhance students' performance in mathematics. Therefore, they should endeavour to provide a classroom environment conducive to learning.

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Modes of reasoning in the mathematics classroom: a comparative investigation

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This paper attempts to map a range of modes of mathematical reasoning employed in classrooms from Germany, Hong Kong and the United States taught by experienced teachers locally judged to be competent. Reasoning here is used as an umbrella term for modes of justification within a range of strategies that aim at making discursively available some elements of mathematical practice. The significance of this analysis consists in the attempt of describing modes of reasoning in a way that accommodates the diversity of mathematical topics, achievement levels, curriculum traditions and culturally sanctioned modes of interaction, rather than in the outcome of the comparison itself.

Keywords: Reasoning, justification, comparison, discursive analysis

Introduction

In mathematics education, as well as in philosophy and history of mathematics, a range of notions has been employed to capture different modes of mathematical reasoning. Mathematical reasoning (judged by the standards of Greek mathematics) has been characterised as being based on clear and concise language and appreciation of logical inference for deriving conclusions, which is directed towards extinguishing subjective elements from judgments. Often a distinction is being made between the context of discovery and the context of justification in mathematical activity, and the relevance of non-deductive modes for the context of discovery, such as inductive and abductive reasoning, computational and visual evidence, evidence from measurements, and other heuristics, has been pointed out. Further, explanations have been distinguished from proofs. While some proofs at the same time function as explanations, the latter are often associated with the context of discovery, where they might motivate new definitions. Hence, motivating definitions (or axioms) is described as an activity of justifying that differs from providing a mathematical proof. Further, mostly within mathematics education, a distinction has been made between argumentation and proof, while the latter is seen as a special version of the first with restrictions on legitimate warrants, expression and type of inferences (Boero, Douek and Ferrari, 2008; Pedemonte, 2008).

Curriculum frameworks usually include statements about some forms of mathematical reasoning, which students are expected to acquire. Related learning expectations pertain to a range of activities, such as scrutinising and justifying results of operations, explaining and motivating an approach to a problem, vindicating conjectures, verifying hypotheses, or justifying the validity of statements by local deductions and proofs. These activities are associated with developing students' understanding, as all are meant to include acts of interpretation and elaboration of meanings, which by most educators are ranked high and often contrasted with operationalised mathematical activities.

The task of the teacher in a particular classroom is to initiate students into what counts as mathematical practice, including its discursive and non-discursive elements. Providing reasons in the course of expositions ('explaining') is used by teachers as a pedagogic strategy for making discursively available to students some features of the entities and operations they are supposed to learn. As the principles for constructing mathematical arguments, or what is accepted as such in a particular classroom, cannot be made fully discursively explicit, students are expected to learn how to reason mathematically through participation and engagement. Hence, giving reasons as a pedagogic strategy aims at achieving both, modelling some forms of mathematical argument as well as rendering discursively available some principles of mathematical practice.

When and how students are expected to engage in reasoning is subject to the (emerging) norms for mathematical activities, which in the classroom intermingle with other social norms (cf. Bauersfeld, 1980; Yackel and Cobb, 1996). The exploration of reasoning activities in a range of classrooms from different countries promises to reveal how these might be shaped by culturally sanctioned forms of interaction, role-related asymmetries and pedagogical principles. The data used in this paper are from the Learner's Perspective Study and were recorded some years ago. The significance of this re-analysis consists in the methodological challenge of describing modes of reasoning in a way that accommodates the diversity of mathematical topics, achievement levels, curriculum traditions, rather than in the outcome of the comparison itself.

In this paper the focus is on students' contributions rather than on the teachers' explanations. The classrooms studied differed in average students' achievement, and it could be expected that achievement differences would be associated with some differences in students' ways of reasoning.

Methodology

In order to construct an empirically based language for describing the modes of reasoning, 'reasoning episodes' were identified in altogether 60 lessons (ten lessons from two classrooms in each country/region). An episode qualified as involving reasoning, if a 'reasoning exchange' was a part of the conversation. The person who provides a reason might interpret something as being not evident, doubtful or disputable (*prophylactic reasoning*), or is requested by another person (who interprets something as not evident, doubtful or disputable) to give a reason (*reasoning on request*). The episodes include the moves that prompted the reasoning, if any, and the 'closure' (e.g. signs of agreement or acceptance) and represent units of conversation with thematic coherence. The attempt to increase evidence or acceptance has to be visible for the other participants in the conversation, which can be seen by their reaction. These 'reasoning episodes' were identified in the transcripts and if needed the videos were consulted.

Student reasoning and prompts for uttering reasons

In the course of activities aimed at making discursively available things that are otherwise tacitly assumed or done (e.g. calculating, solving problems, producing graphs, drawing geometrical shapes) occasionally interpretations and elaborations of mathematical meanings, that is, *reasons*, were provided. As has been pointed out above, on the side of the teacher this is a pedagogic strategy. As far as the students are concerned, they did not frequently utter reasons for what they were saying or doing.
The table below shows how often students in each classroom (in 10 consecutive lessons) provided some mathematics-related reasons, either on request by their

teacher or on their own initiative. As not all conversations between students were captured in the video, especially during seatwork when several groups of students talked to each other, the count only includes reasoning episodes during whole class interaction. In all classrooms except the one from German School 1 (G1), students' explanations were more frequently produced on teachers' requests than prophylactically. As *argumentation* entails justification of an issue at stake by all involved, students cannot be expected to engage in argumentation with their teachers. However, there was a long episode where this occurred in G1, which accounts for the relatively high number of student-initiated reasoning episodes in this classroom. The students had been asked to 'prove' simple binomial expansions by means of a geometrical interpretation and present the outcome to their peers. The teacher happened to be unprepared for one group's line of argument and hence an 'a-didactical situation' (Brousseau, 1986) emerged.

Classroom, no. of students, achievement level	On teacher's request	On own initiative*
German School 1 (G1), 27, average-high	23	24
German School 3 (G3), 12, low-average	26	16
Hong Kong School 1 (HK1), 35, high- average	9	4
Hong Kong School 3 (HK3), 39, high	3	5
United States School 1 (US1), 29, low	82	10
United States School 2 (US2), 33, high	13	12

Table 1: Number of student reasoning episodes in 10 consecutive lessons from each classroom.

The high number of teacher-initiated student reasoning in US1 appeared to be due to an established norm that students 'explain their thinking'. In the other classrooms, teachers' requests for reasons, such as through asking, "Why is it …?" or, "How do you know …?" in most cases were to be taken as an indication that a student had produced something wrong. For example, many of the teacher's requests to students for backing up their solution procedures in G3 occurred in situations when the students' lack of familiarity with some basic arithmetic operations became obvious and the teacher wanted to elicit students' erroneous strategies. In contrast, the teachers in the Hong Kong classrooms did not have to struggle with a lack of students' fluency in arithmetic or did not find it worthwhile to spend much time with eliciting students' strategies when computational errors were made. The relatively low number of student reasoning episodes in the Hong Kong classrooms also reflects the norms for interaction, especially the more controlled turn taking mechanism. Student self-initiated reasoning often included the backup of a claim that another student's (and in some cases the teacher's) solution procedure was faulty, or cases when students thought a task posed by the teacher cannot be solved. Due to differences in how the hierarchy between teacher and students was constituted, such claims were more often made in the German classrooms, most commonly in G3 that also had the weakest regulated turn taking mechanism, which could be adhered because of the low number of students.

Identifying modes of reasoning across topics

In the six classrooms, much of the time the participants engaged in carrying out mathematical operations. In these activities, if the operations were of a local nature (such as solving a particular task), teachers (or occasionally students) provided an account of how the solution was achieved, sometimes while writing on the board, that is, they were *documenting* what they are doing. Reasons in these *documenting* activities might be provided by means of reference to the generalised procedures that are applied in the particular case.

When working with a new form of presentation (e.g. graph of a function, or an algebraic expression), the activities often consisted in *naming* and *defining*. Defining also included attempts of generalising graphical and symbolic expressions from particular cases. Providing reasons in a defining activity could entail showing the fruitfulness of definitions through making methodological judgments about alternatives (Tappenden, 2005). In US1 there were indeed instances when the teacher (more or less successfully) attempted to achieve this.

Making discursively explicit a generalised operation (e.g. for solving simultaneous linear equations) entails *explicating* a set of steps to be followed in a range of similar cases. Providing grounds in explicating activities included stating the condition under which the procedure is valid or attempting to motivate single steps by means of a more general heuristic.

When dealing with generalised expressions (‘formulae’) or generalised diagrams (e.g. a geometric shape without any specific properties), the activity of making discursively available the mathematical meanings and interpreting their relations (after setting up interpretations through defining) was used as a form of *proofing*.

The examples below provide instances of these activities. They are only from episodes, in which students were involved in some form of reasoning, and not when the teacher gave explanations. The headings of the short extracts indicate the topics of the episodes.

Defining

US1 - Exponents

T Now does that quite explain, though, how three to the zero is one?

S Yes

T How so [Name of student]?

S Cos it's three to the zero - it's the same as three to the - to the negative zero

T Say that one more time…please.

S Okay… um

T Little louder

S Three to the zero. It's almost the same as um ... three to the negative zero.

Before this episode, they had gone over 3^2 and 3^{-2}, 3^3 and 3^{-3}, and 3^4 and 3^{-4} to demonstrate a reciprocal relationship, and the student obviously built on this in his argument. The teacher in this classroom frequently asked the students to elaborate on the meanings of symbolic expressions and to give reasons (e.g. of the definition of ‘fraction’ or of the meaning of ‘A divided by 0’). As the students had no access to the mathematical principles, these attempts were in many cases not very successful in terms of reaching agreement on a motivation for a definition. (Had they been derived, the ‘definition’ would constitute a local theorem.). In US2, discussing definitions was also a common activity, however with less student involvement in terms of eliciting reasons from them. The outcomes of these ‘definitions’ often were used as a starting point for the sorts of activities that were practiced later in the lesson. The teachers from the other classrooms either did not motivate definitions or they provided some motivations themselves.

Documenting

This activity was the most common in all classrooms during whole class interaction with student involvement. While practicing operations through solving sets of similar tasks, students occasionally stated what they were doing ('explaining the steps'), in particular when presenting a solution to the whole class. As outlined in the previous section, in cases of disfluencies in the smooth flow of proceeding, when some doubts were uttered, or when errors became visible, the teachers requested a justification. Further, teachers asked for reasons, when they wanted to alert students to a common source of errors, as in the following example:

<u>G1 – Expanding algebraic expressions</u>

T Right… in the first problem there was something to pay attention to… why?

S Er… that there's a minus sign in front of the first pair of brackets so that it is turned round then

The teacher's request for a reason here is used as a strategy for pointing out particular pitfalls when executing the operation of expanding expressions. On request, the student is stating the condition for applying the operation as well attempting to state the procedure.

<u>G3 - Finding parameters of linear equations from graphs</u>

T Five is the point of intersection that MX in front doesn't exist you see. Why is there no MX in our equation now in that last example?

S Because it's zero

T Because M equals zero but our point of intersection does exist

Here the reason includes an explication of an element of a general procedure for reading off parameters from graphs. This was intended to provide an introduction to developing such a general procedure, rather than practising it.

<u>G3 – Calculating surface area and volume of various solid figures</u>

T So what kind of unit of measurement comes after this?

S Decimeters

S Decimeters second

T Right…square centimeters…why just square centimeters?

S Because we're dealing with a surface

This instance is from a lesson, in which the students struggled with choosing the appropriate units for volume and surface area, and the teacher frequently asked for warranting their choices. The student might refer to the general 'rule', 'Area is measured in square units, volume is measured in cubic units'.

<u>US1 - Powers and the order of operations</u>

T Is the final solution to this going to be positive or negative? Raise your hand. Is it going to be positive or negative? Final solution if we calculated an answer ... is it going to be positive or negative?

S It's gonna be a negative.

T It's gonna be a negative? Why - why do you say that?

S Because it's four times four is sixteen and then it's a positive sixteen and then a positive times a negative equals a negative.

T A positive times a negative, okay. Did everybody hear what he was saying?

Before this episode, they had introduced the notions $12 = 12^1$, $(-4)(-4)(-4)$ as $(–4)^3$ and $3^4 = (3)(3)(3)(3)$; the teacher preferred the use of '( )' for denoting multiplication.

Explicating

There were not many instances of students' involvement in discussing generalised operations. Only in the Hong Kong classrooms, general methods for solving systems of linear equations were introduced during the lessons that were recorded.

<u>HK3 - Solving linear equations in two unknowns using elimination</u>

S1 [to S2] It's the same no matter addition or subtraction… aren't they all addition? I used subtraction…this one is subtraction…this one C is subtraction

S2 (?) negative sign…then followed by a negative two and it'll become positive two...

S1 Of course not…we do not need to follow…

S2 If there is a negative sign when I multiply... do I need to multiply negative two?

S1 No…if you subtract a smaller value from a larger one you can use either addition or subtraction…you can choose among them…

This conversation can be seen as being about a general rather than a local procedure, as S1 asked about what 'all' of the examples have in common.

<u>HK1 - Solving linear equations in two unknowns using substitution</u>

T Why? Try to explain that…why do we choose the first one instead of the second one…why? [T points to a student to answer the question][S stands up]

T Why do we choose the first equation instead of the second one?

S It is easier [S asks his neighbor before answering the question]

This example can be read as pointing to a heuristic of choosing the variable for substitution. Heuristics are generalised procedures, even if they are not fully realised in discourse.

Proofing

<u>G1 - Evaluating binomial products by geometrical representations</u>

S1 Right…so there's the problem, so here this big this white square this is erm here A squared… because because quantity A so A to the power of two and then we've got this small square here so

S2 this is B times B so B to the power of two

S1 Exactly and now we want to erm take away here… now we take away from A squared…

S2 You take away the small area from the large area

S1 Exactly… and then this here should be the result and so we've already solved it (?)

S3 Push the board a little bit

S1 (?) and here the result is a little complicated now

In this lesson, several students discussed over a long period of time their proofs for formulae for three binomial products (for $(a + b)^2$, $(a - b)^2$ and $(a + b)(a - b)$, respectively), which they presented at the board without much teacher intervention. The short episode shows how they backed up their claims by reference to the diagram, which they had set up by naming and defining lengths and areas.

<u>US2 – Linear and direct linear relation</u>

T Direct variation…okay…lets put that down…direct variation…um… is this direct variation folks? [Points to another graph on white board].

S No.

T No. Why not?

S Because it doesn't pass through the origin-

T It does not pass through the origin. Okay? //Everybody got it?

In US2 the activities entailed more often some form of proofing than in other classrooms, usually through referring to graphs or diagrams. However, as can be seen from the number of 'student reasoning episodes' in the ten lessons, the activity did not often involve students.

Further observations and conclusions

As to similarities and differences, the analysis showed that *documenting* and *explicating* constitute common modes of activities that can be described across topics, and that these were very common in all classrooms. This is not to suggest that the same amount of time was devoted to these activities, as the lesson structures significantly differed in the classrooms. The lessons in US2 appeared to be closer than those in US1 to the lesson pattern reported by Stigler and Hiebert (1999) as typical for US classrooms, but how the activities were enacted does not match that description. Neither did the lessons in the German classrooms show the pattern reported as typical for German classrooms (Clarke et al., 2007), but the lessons in HK3 resembled that 'German' pattern. As there were quite long periods of individual seatwork in HK3, the number of students' reasoning episodes in whole class interaction is necessarily low.

With respect to the type of mathematical tasks, the two Hong Kong classrooms were more similar to each other than any other pair of classrooms, which is due to the fact that the topics in both classrooms were largely the same. Further, due to the differences in turn taking mechanisms, the amount of 'filtering' of students' contributions in whole class discussions varied, with least 'filtering' in US1 and most in HK1 and in G1. The extent to which the students shaped or acquiesced to the teacher's pre-defined lesson structure varied according to the strength of the teacher's control of the forms of interaction that constituted the structure. In this respect, G3 and HK3 showed some similarities, as in both classrooms deviations occurred.

Based on these observations, the amount of student reasoning in whole class interaction seemed to be a function of the turn taking mechanisms in interactional routines, rather than of authority relationships in general. In all classrooms except US1, and to some extent in US2, where 'explaining one's thinking' was part of the classroom norm, the asymmetry between prophylactic reasoning and reasoning on request was reminiscent of the same asymmetry in warranting norms of conduct.

These norms usually are not warranted prophylactically, but are likely to be substantiated when being contravened (e.g. when excusing oneself).

The framework for describing different modes of mathematical activities and the types of reasons provided for their justification seems to be viable across mathematical topics. But it does not include some activities that do not belong to the esoteric domain of school mathematics, such as mathematical modelling or activities that include evidence from measurement. The activities of *documenting*, *explicating* and *proofing* can be related to elements of Dowling's (2013: 16) three-dimensional scheme of the "modality of general and local esoteric domain apparatus", in which these activities would entail moves from non-discursive to discursive semiotic modes. However, this is not the case for what above has been described as *defining* by means of generalising particular instances of expressions. In addition, there are other pathways for moving from local examples to general cases, as for example by means of establishing general procedures from local ones, which would entail formalising. Hence, the framework needs to be further developed.

As only episodes with student involvement in whole class interaction were included, the modes of activities and reasoning observed cannot be taken as a signature of any of the classrooms. For this purpose, teacher talk would need to be analysed. A question to be asked is whether different modes operate in low- and high-achieving classrooms, or in teacher interaction with low- and high-achieving students in the same classroom.

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A patchwork of professional development: one teacher's experiences over a school year

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It is well recognised that professional development research often struggles to demonstrate that changes in a teacher's practice are as a result of a professional development initiative (e.g. Guskey, 2007). One reason is that teachers are influenced by a 'patchwork' of learning opportunities and it is sometimes impossible to pick out how, and to what extent, each opportunity may have contributed to these changes.

The research reported here was a joint effort between a mathematics teacher and a researcher. The paper, which draws on 'research conversations' between the authors, explores what counts as professional development for this teacher and describes his patchwork of professional development in terms of the processes in which he engages within his professional practice: exploring, experimenting and reflecting; mainly within the context of teaching but also more widely. We argue that critical reflection is crucially important within professional development, but so too is appropriate action to carry exploring and experimenting through to real development.

Keywords: professional development, reflection, research, mathematics, teaching

Introduction

The literature related to professional development for teachers of mathematics concentrates largely on initiatives of professional development, usually from the perspective of those who have set up the initiative (Desimone, Porter, Garet, Yoon and Birman, 2002; Goodall, Day, Lindsay, Muijs and Harris, 2005). There are also studies that examine more informal professional development, but again these usually focus on learning opportunities or activities (e.g. Jaworski, 2006; Kazemi and Franke, 2003). There is less literature related to the overall professional development experience, which goes beyond focusing on the initiative and looks at the teacher perspective.

The majority of research attempts to evaluate the effectiveness of initiatives of professional development, usually by gathering a) the views of teachers taking part or b) evidence of some changes in practice (hopefully related to the aims of the initiative). (e.g. Goldsmith, Doerr and Lewis, 2013). It is rare, however, to consider the effectiveness of one initiative in relationship to the overall experiences of the individual teacher or the development of an individual teacher against their whole patchwork of professional development experiences.

In accord with Bolam (2007), our philosophical perspective puts the experience of the individual teacher centre stage. For us, it makes little sense to evaluate the influence of a professional development initiative on a teacher without taking into account his/her teaching context, prior learning and previous and on-going experience, which would include this and other past and present professional development experience.

This paper is a jointly authored by a researcher (Marie) and a teacher (John) and is motivated by this philosophical perspective. The paper is concerned with John's professional development over the period of a year and considers what counts as professional development for him and what, in particular, works for him.

John Larsen: an introduction

John is a teacher of mathematics in a secondary school (ages 11 to 18) in Nottingham in England. He has taught mathematics for twenty years with a recent weighting towards older (A-level) students. His over-arching ambition is to be a consistently excellent teacher and he believes that he can always improve his practice.

John has always been enthusiastic about pursuing his own learning he takes active responsibility for developing his practice, and is committed to putting time and effort into doing so. For example, he is a member of five professional organisations, including the Association of Teachers of Mathematics.

What we did

Between October 2012 and July 2013, John took part in a research project run by a team at the University of Nottingham. John was asked to select six Formative Assessment problem solving lessons, which include detailed guidance notes, written by the research team. The overall aim of the project was to understand how these lessons are used in authentic classroom situations in order to inform future design and development work. It also aimed to investigate whether, and to what extent, reading and following the guidance and teaching the lessons might lead to professional learning. It became clear that, as this was one of many of John's activities that might lead to professional learning, unravelling a causal relationship would be difficult, if not impossible. We were interested, however, in understanding the patchwork of professional development for an individual teacher and to understand when and how professional learning takes place, but without attempting to find causal relationships between specific activities and learning.

Our research is underpinned by a commitment to *joint* understanding; in which both our voices are equally represented. In a deliberate and genuine attempt to reach a level of equality, we decided to opt for a 'research conversations' approach (adapted from the learning conversations of Gudeman and Riviera (1995)). Using this approach involved beginning with John's list of recent experiences that could count as professional development, and then to iteratively analyse our joint understandings by developing emerging themes and related critical questions.

In order to develop the analysis, the first concern was to understand what counts as professional development, in terms of *the sorts of activity* that can be considered as 'professional development activity'. The second, related, concern relates to *the sorts of outcomes* that can be seen as evidence that professional development took place.

A patchwork of activity

The question about sorts of activity that can be seen as professional development is addressed in the literature (e.g. ACME, 2002; Avalos, 2011; Joubert and Sutherland, 2009; Muijs and Lindsay, 2008; Ofsted, 2006). John takes part in a wide range of activities, which include more 'formal' activity such as training courses and school in-service training (INSET) days and less formal activity such as reading, browsing websites and following developments using social media. In addition, John teaches and as Eraut (2004) suggests, professional learning frequently occurs as a by-product of working.

In a series of research conversations, we attempted to determine which of John's activities counts as professional development. It became clear that none of the activities above counts as professional development *per se* for John. Instead, it appeared, we would need to frame our analysis in terms of the processes in which the teacher engages, for example, teaching, attending a workshop or reading. Our research conversations revealed that for John, these processes involve exploring, experimenting (enquiring) and reflecting either alone or with others. It is worth noting that, although the processes of enquiry and reflection are identified in the literature as key to effective professional development (Joubert and Sutherland, 2009), they emerged through conversation as John decided what counts for him; evidence, we suggest, of the authentic (thoughtful) teacher voice. Interestingly 'exploring' is less evident in the literature.

This section draws together the various aspects of John's professional development activity, with a focus on the academic year 2012-2013. The section is structured around the processes in which John engages as outlined above.

Exploring

In terms of exploring, John reads books, journals and articles (e.g. from *Mathematics Teaching* and *Educational Leadership*) widely, covering topics including pop psychology, cognitive psychology, neuroscience, business, education and mathematics. Recent examples are *Mindset* by Carol Dweck, *Professional Capital* by Andy Hargreaves and Michael Fullan, *Why Don't Students Like School?* by Daniel Willingham, *Teach Like a Champion* by Doug Lemov and *The Art of Problem Posing* by Stephen Brown and Marion Walter. These books range from general cognitive psychology, through system change to teaching and then mathematics teaching.

John also reads social media. He follows a range of tweeters, many of whom make posts relevant to his professional life (e.g. the DfE, researchED2013 and Dan Meyer) and he sometimes uses feeds such as #mathsed.

One of his key interests is in reading about initiatives in mathematics education and particularly about ideas for presenting tasks and interacting with students. He frequently finds references to resources for tasks on his various twitter feeds, the majority of which provide a link directly to the resource. He follows links and explores the resources they point to. In contrast to this somewhat ad-hoc exploration of tasks, he also visits web sites that provide banks of resources for use by teachers of mathematics; mainly *Nrich* (http://nrich.maths.org/) and the Mathematics Assessment Project (MAP) website (http://map.mathshell.org/materials/index.php). He comments that these web sites provide a 'wealth' of ideas and resources but that, like any large repository, to become familiar with what they offer takes some effort.

John attends seminars at the University of Nottingham, which expose him to some of the latest research in mathematics education. For example, during 2013 he attended a seminar related to the use of digital technologies in mathematics education, a seminar on enquiry methods in the teaching of mathematics and science and a workshop for developers on creating software to support collaboration and problem solving in mathematics lessons. He says that these seminars are valuable to him not only because they allow him to explore ideas but because he is able to hear contributions from a variety of attendees and make contributions by representing the 'teacher voice'.

Taking part in research can also been seen as exploring. As described above, John took part in a research project with the University of Nottingham. Through this he explored new approaches to formative assessment and to teaching problem solving.

Finally, John explores ideas by participating in workshops and courses. In the academic year over which this study took place, he attended three workshops run jointly by *Nrich* and the *Primas* European project (http://www.primas-project.eu/en/index.do), he followed a Massive Open Online Course (MOOC) from Brown University on applications of linear algebra and on iTunes-U he accessed lectures on Linear Algebra from MIT. He also attended a *MathsJam* weekend where attendees share five minute presentations on interesting, surprising, fun, useful or useless topics in mathematics. More recently, John attended the *ResearchED* conference in London which explored the use of evidence in education and ways to improve communication between teachers, researchers and policy makers.

Overall, John likes exploring; he likes to increase his knowledge of developments in education, and he likes finding out about new ideas, both for use in the classroom and for general professional knowledge. He says, however, that the results of much of his finding out and exploring are not used, and that without some systematic cataloguing of his new knowledge much of it disappears. On the other hand, he often finds it provides him with new ideas to try out in his practice.

Experimenting

Within teaching, John tries out different pedagogical approaches, some of which can be seen to 'tweak' his existing practice and others of which may represent bigger changes. For example, within the *Primas* project, he taught rich investigative tasks that he might not otherwise have done, between workshops. As another example, taking part in the research project with the University of Nottingham meant that he was committed to teaching at least six of the formative assessment problem solving lessons, which involved some changes to his usual practice.

Aspects of the *Primas* tasks and the research lessons possibly represent the most significant experimentation in John's practice over the period of this study. Other new ideas he tried out include the use of *Desmos*, a browser-based html5 graphing calculator, which was recommended on Twitter. He experimented with using it in the classroom and now uses it regularly, both as a teaching tool and as an iPad App for investigation by students. The Lemov book (see above) introduced him to a range of techniques such as 'cold calling', 'right is right' and 'no opt out' which see the teacher choose which students answer questions, rather than hands up, hold out for a high standard of contribution from the students and return to students who may, initially, be unable or unwilling to answer and ensure those students finish a sequence of questioning with a positive contribution.

In terms of questioning within the classroom, he has experimented with ideas from, for example, John Mason, Brown and Walters, Prestage and Perks and Dan Meyer. He has used a strategy known as 'and another, and another, and another' in which the teacher prompting students for multiple examples of a concept in mathematics encourages them to test the boundaries of what characterises an acceptable example. Brown and Walters' work helps the teacher and students generate a wider range of questions and possibilities and Prestage and Perks and Dan Meyer encourage rewriting existing questions to make them more engaging and profound.

John also experiments outside the classroom. He believes in a structured, shared approach to teacher improvement and experiments with ways to develop such a network with his colleagues. For example, he tries out different approaches to sharing new ideas with colleagues in the staffroom and he has developed mechanisms through which colleagues are able to share their experiences related to the departmental scheme of work through online channels. One aspect of John's professional work is the maintenance of the school's online learning resources. He constantly experiments with ways to curate these resources.

Experimenting is an important aspect of John's practice. He believes that, when he encounters ideas that may be useful in his practice it is important to try them out using a systematic approach, rather than 'just trying things'. Where the ideas to try out resonate with John's own beliefs, experimenting with them represents less of a challenge, but there are occasions where John has tried out methods and approaches about which he is more sceptical and this presents more of a challenge. For example, for some lessons within the MAP research project, he followed the teacher guidance closely, asking the students to present their work on posters to encourage collaboration while already being confident that the students could have effective discussions while writing on file paper.

Reflecting

The term 'reflection' is used widely within the literature related to professional development (Jay and Johnson, 2002; Harford and MacRuairc, 2008; Schon and DeSanctis, 1986; Bengtsson, 1995). Distinctions are made between different kinds of reflection e.g. descriptive reflection, comparative and critical reflection (Jay and Johnson, 2002) and reflection-in-action, reflection-on-action and reflection-for-action (García, Sánchez, and Escudero, 2006). The notion of reflection is clearly complex but it is generally agreed that reflection, post-hoc, is a key process within professional development. As García et al. (2006) state: "Reflection-on-action is an essential component of the learning process that constitutes professional training" (p. 2).

In the section on experimenting, above, the word 'systematic' was used to describe John's approach. For him, systematic approaches involve cycles of careful planning, trying things out and reflecting on the experience. Planning will involve, considering what to look out for (mostly in the classroom) and effective ways to assess students' work. For example, in experimenting with a new approach to collaborative working in the classroom, John would need to consider what improved collaboration might look like. Systematic approaches to experimentation would also involve gathering evidence and, finally, using the evidence to inform useful reflection.

It seems that, for John, a critical and reflective stance increases the likelihood of successful change; so 'just trying out the latest ideas' would probably not count as professional development but trying out the latest ideas and reflecting critically on the experience almost certainly would. John suggests that critical reflection is a driving process in professional development irrespective of any other processes involved.

The individual at centre stage: what works well?

This section is concerned with considering the sorts of outcomes that can be seen as evidence that professional development took place and what, in particular seems to be important in encouraging the development. Commonly, professional development is evaluated using a number of criteria, which include teacher learning, changes in teachers' classroom practice and improved student learning (Guskey and Yoon, 2009; Muijs and Lindsay, 2008). Our focus, however, is on the individual teacher and his/her learning within the entire patchwork of learning activity and it would seem sensible to take into account his/her priorities and learning or development goals.

John's priorities in terms of professional development are almost always directly related to his current or future classroom practice, within the context of whole school or departmental policies and national imperatives. He describes his personal learning goals in terms of his attitudes and approaches.

In terms of John's priorities, it seems that the patchwork of activity John selected was effective. For example, within the mathematics departmental policy of improving collaboration between students in the teaching and learning of problem solving, he took part in the research project based at the University of Nottingham, which required him to adopt new teaching approaches. As he said, "I was interested to see if this project would help me make that [collaboration] better".

However, he also participates in a range of other activities that could be seen as professional development even though they do not relate directly to his priority area of classroom practice. He keeps up with developments and research in mathematics education, for example by attending seminars at the University of Nottingham, following social media such as Twitter and reading.

John's learning goals are articulated in terms of attitudes and approaches. For example it is important to him to remain motivated both through having expectations from other people of professional development activity and through becoming energised by things he find himself. He provides evidence that he has remained motivated and this suggests that, for him, his patchwork of activity has been effective. He describes how he feels energised through various exploring activities such as attending seminars and through taking part in courses such as *Primas*.

He also questions, however, whether teacher learning counts as professional development even if there is no improvement in student outcomes. For example, he explains that he may have learnt something that improves his efficiency as a teacher, for example introducing a system of using folders for the students' work which makes his life easier but cannot be seen as improving the students' learning in mathematics.

We have claimed above that there was some evidence of professional learning. Here we draw out aspects of John's patchwork of activities that seem to work well for him and may therefore be valued by other teachers.

As discussed earlier, one of John's goals is to remain motivated. It seems that taking part in activities that expect or require experimentation and reflection, such as the *Primas* course or the research project at the University of Nottingham works well for him. He says that sometimes even though he has found out about new ideas or approaches, he needs a 'push' to experiment with them in the classroom; with the *Primas* workshop days there was an expectation that participants would teach a rich investigative task between the workshop days and then share their experiences with others during the workshops.

John's overarching priority is to provide the best learning opportunities for students. His exploring activities seem to be important in giving him fresh ideas; equally important are experimenting and reflecting on the outcomes. He is interested in sustained change, and suggests that changes that he has maintained are those that are 'easy' for him and the students, such as the use of *Desmos*. He explains that *Desmos* is easy to use both technically and within normal classroom practice.

While John recognises the importance of reflective processes, he suggests that he values support in achieving deeper levels of reflection. To a large extent, taking part in the research project with the University of Nottingham supported his reflective activity; partly because John was 'pushed' to reflect in some detail and because the evidence that was used as a basis for the research questions was explicit.

Concluding comments

This paper has sketched something of John's patchwork of professional development, providing examples of the wide range of activities in which he engages, analysing the processes which are important to him, giving evidence of his development and suggesting what works well for him.

However, in writing the paper and engaging in research conversations, further points related to John's professional development have emerged. First, John's original conceptualisation of what counts as professional development has expanded to include creating his own resources and putting existing knowledge into practice.

John has a particular interest in resources, and has good knowledge of sources of tasks for use in the classroom as well as other resources such as videos. John also creates his own resources such as 'mini-tasks' to provide students with practice in specific skills and Apps for use on an iPad. For John creating resources involves careful analysis of the mathematical learning he intends the students will achieve by using the task or App he has created, and this, he suggests, leads to his learning.

In terms of putting knowledge into practice, he remarks that 'sometimes all that is needed is effort'. He says,

> Just get on with it and plan sequences of lessons. Sometimes professional development has nothing to do with learning new knowledge, it is more a case of putting existing knowledge to use.

The second point to emerge is that, for John, what really matters is that his professional development makes a difference in the classroom. He recognises a tendency in himself to engage in 'professional development for its own sake'. As he explains,

> Reading books, attending seminars and conferences, watching videos and indulging in social media can become ends in themselves; what seems like motivation in the teacher often doesn't lead to changes for the students.

A third point to emerge is that the work involved in writing this paper, which includes the careful and deep analysis of his own learning, can also be seen as professional development for John. This emphasises the importance of reflection in professional development.

We conclude with a final observation. John may be unusual in the extent of what he does, but the complexity of his professional development patchwork probably is not. The lesson, perhaps, is that when we speak of a teacher's professional development, we should recognise that an individual teacher's learning is likely to be influenced by far more than a single initiative. This may seem obvious, but as the introduction to this paper pointed out, it is something the literature frequently seems to overlook.

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Supporting Students' Probabilistic Reasoning Through the Use of Technology and Dialogic Talk

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Research has shown that pupils and many adults have intuitions about probability that are often at odds with accepted probability theory. Drawing on the literature on probabilistic reasoning, effective pedagogical approaches and the use of technology tools, our aim is to examine the relationship between students' talk together, their use of *TinkerPlots* software and the development of their reasoning about uncertain outcomes. In this paper we report on findings from the first iteration of a design study conducted in an afterschool club for Year 7 students in Exeter. More specifically we describe the trajectory of two students making conjectures about the fairness of some games involving combined events, testing and revising their initial theories based on simulation data. Our analysis shows that these students' use of dialogic talk in combination with the technology leads to a shift from intuitive reasoning to probabilistic reasoning.

Keywords: secondary, probability, technology, dialogic talk.

Introduction

The teaching and learning of probability is a difficult task since formal probability theory can be counterintuitive for students. Earlier studies conducted by psychologists and mathematics educators have identified and documented difficulties students encounter when making judgment about an uncertain event.

The research of Kahneman and Tversky (1972) suggests that students often use intuitive strategies when judging the likelihood of uncertain events. One such intuitive strategy is called the representativeness heuristic and refers to evaluating the probability of an uncertain event based on the degree to which it represents some essential features of its parent population. For example, when flipping a coin six times, students often consider the sequence TTTHHH more likely to happen than either HHHHHH or TTTTTT, because it has an equal number of heads and tails and, thus, is more representative of the expected 50-50 distribution. Another study by Tversky and Kahneman (1982) also reveals a misapplication of the law of large numbers to small samples, that is the tendency to regard small samples as highly representative of the population from which they are sampled and to use them as a basis for inference and generalizations.

Equiprobability bias is another intuitive approach to solving probability tasks. Students with this bias tend to view all possible outcomes of random events as equally likely (Lecoutre, 1992). In the example of rolling a five and a six and rolling two sixes from a pair of dice, Lecoutre describes how students come to the equiprobable responses. For instance, a majority of students believe that likelihood of getting a six and a five is equal to that of getting two sixes because "random events should be equiprobable by nature" or "it is a matter of chance" (p. 561).

Students come to classrooms with these intuitive ideas and strategies and we need to challenge them to facilitate the development of more formal ideas. In this paper we examine what mechanisms of dialogic talk combined with the use of technology tools support students' probabilistic reasoning in games of chance. In particular, we are interested in the following research questions:

1. How do *TinkerPlots* tools and peer-to-peer dialogic interactions foster students' reasoning about probability events?
2. What are the mechanisms that lead to a shift in students' probabilistic reasoning?

Dialogic talk

Dialogic talk is a way of talking together that helps groups solve problems creatively. It is very close to what Mercer and others call 'Exploratory Talk', talk in which partners engage critically but constructively with each other's ideas (Mercer, 2013). Calling this productive way of talking 'dialogic' puts the focus more on the quality of the relationships than on the explicit verbal reasoning. In dialogic talk there is openness to the other and to otherness in general such that individuals are able to listen and to change their minds. The general dialogic mechanisms that have been observed in groups leading to success in creative problem solving are: opening a space of dialogue (e.g. asking open questions); deepening the space of dialogue (e.g. questioning assumptions); expanding the space of dialogue (e.g. introducing new voices and perspectives); seeing from the perspective of a specific other (e.g. listening to and taking on board the comments of a colleague, this includes the mechanisms of 'appropriation' and 'co-construction' noted by Mercer (2013) for Exploratory Talk); seeing from the perspective of a generalized cultural other (e.g. invoking the perspective of the absent addressee or audience for the product or referring to the generalized other of the community) and finally evidence of reflection through taking an outside perspective (e.g. genuinely asking why something is happening without any presuppositions, looking at it in a new and unexpected way) (Wegerif, 2013).

Technology use in teaching probability

A traditional approach to teaching probability focusing solely on mathematical computations and procedures is not sufficient to help students develop probabilistic thinking and reasoning needed in making judgments in real life. Learners need additional resources to develop probabilistic ideas in a more conceptual way (see Watson, Jones and Pratt, 2013). The computer technologies with the speed, dynamic visualisations, and capability to carry out laborious manipulations give students opportunity to focus more on their thinking and making generalisations and abstractions about data and chance.

With the advances in digital technologies and emphasis on the use of ICT in school mathematics, software tools with features for exploratory data analysis and probability simulations have become increasingly valuable in supporting children's learning. One such software is *TinkerPlots 2.0* (Konold and Miller, 2011), a data analysis tool with simulation capabilities (see Figure 1). It is a distinct computer program compared to other graphing or spreadsheet programs as it builds on the intuitive knowledge learners have about data representations and analysis. Students actually construct their own graphs when progressively organising their data by ordering, stacking, and separating. It also includes a variety of tools, such as dividers and reference lines, to intuitively analyse data in making inferences. One of the new features in *version 2* is the probability simulation tool that expands its focus from data to incorporate probability. With the Sampler tool, students can build their own chance models using a variety of devices (i.e., mixer, spinner, bars, stacks, curve, counter) that can be filled with different elements to sample from. They can connect series of devices to create a sequence of independent or dependent events. This tool then allows students to collect outcomes and carry out a large number of trials quite quickly.

During the field testing of the development version of *TinkerPlots 2.0* Konold and Kazak (2008) engaged students in developing integrated set of statistical and probabilistic ideas. The findings indicate that *TinkerPlots* environment facilitates students' visual reasoning via dynamic graphs where the results accumulate as they are generated by the Sampler. Through observing the simulation data from multiple trials coupled with sketching graphs (drawing only the overall shape and relative heights of stacks), students can explore the fit between the expected distribution based on the sample space and the empirical data. As a result of these observations they begin to perceive data as a combination of signal and noise (by signal we mean a stable shape in the distribution whereas noise is the variability around it due to chance).

Description of the study

The study is part of a larger design experiment (StatsTalk Project) investigating the relationship between students' talk together, their use of ICT tools and their development of conceptual understanding of key concepts in data handling and probability. The data reported in this paper come from the first iteration of this design experiment conducted with five 11-12-year-old students voluntarily participating in an afterschool club at a private school in Exeter.

Procedure and tasks

Each class in the afterschool program, co-taught by the authors, met for an hour, once a week over the period of eleven weeks. Students worked in groups of two or three and each group's work around a computer was videotaped.

Through the sequence of tasks designed for the purpose of the project, students were engaged in analysing data sets, such as reaction times and backpack weights, data modelling tasks, and chance activities using *TinkerPlots 2.0* software. In the eighth session of the afterschool program students began to investigate chance events. The focus of our paper will be on this task, called The Chips Game.

In addition to the use of *TinkerPlots* as an ICT tool to explore data and chance in the study, we introduced certain ground rules to promote a dialogic way of talking in joint activity (Dawes, Mercer and Wegerif, 2000). To facilitate effective talk students were expected to (1) make sure that each person has an opportunity to contribute ideas, (2) ask each other why, listen to the explanation, and try to understand, (3) ask others what they think, (4) consider alternative ideas or methods, and (5) try to reach an agreement before they do anything on the computer.

The chips game

Following the discussion of a fair method or a fair game, we asked students whether they thought the following games were fair or not:

> There are two bags containing game chips of two colours-red and blue. To play the game, you will randomly select a chip from each bag. If the two chips are the same colour, you will win. If they are different colour, teacher will win.
>
> Game 1 - Bag one: 3 red chips, 1 blue chip; Bag two: 2 red chips, 2 blue chips
>
> Game 2 - Bag one: 3 red chips, 1 blue chip; Bag two: 1 red chip, 3 blue chips
>
> Game 3 - Bag one: 2 red chips, 2 blue chips; Bag two: 2 red chips, 2 blue chips
>
> Game 4 - Bag one: 4 red chips; Bag two: 2 red chips, 2 blue chips

The task involved iterations of making predictions, testing initial theories by collecting simulated data from models built in *TinkerPlots*, and making conclusions.

Students' reasoning about probability in the context of fair game

We now focus on an analysis of the episode where two 11-year-old boys, Chris and Jacob (pseudonyms), investigated their initial predictions about the fairness of the games described in the previous section, by building a model of each game in *TinkerPlots* and collecting data.

Making Initial Predictions

We presented the Chips Game to the students and asked them to make a guess about the fairness of each of the four games listed on the board. They made a prediction for Game 2 first and worked on it before they moved to the next game listed in Table 1.

Table 1. Students' joint initial predictions and their reasoning about the fairness of the games.

Games	Predictions	Explanations
Game 2 Bag one: 3 red chips, 1 blue chip Bag two: 1 red chip, 3 blue chips	"fair"	Jacob: …both bags equals to four chips a piece. Teacher: Right Jacob: So half of them are eq.(.) are more (.) fifty fifty chance of winning (.) And now go to the same as for game 3 because both bags have two same colours of chips which will also equal up to just similar results.
Game 3 Bag one: 2 red chips, 2 blue chips Bag two: 2 red chips, 2 blue chips	"fair"	
Game 4 Bag one: 4 red chips Bag two: 2 red chips, 2 blue chips	"not fair"	Chris: Bag two is fair. Bag one is all red chips. It is impossible to pick out a blue chip from there. It is all made up of red chips.
Game 1 Bag one: 3 red chips, 1 blue chip Bag two: 2 red chips, 2 blue chips	"definitely not fair"	Jacob: I agree. It is totally unfair because in the first bag you get ¾ chance of getting red and ¼ chance of getting blue. And the second bag you get a 50-50 chance.

As can be seen in students' responses in Table 1, they failed in their first predictions about the fairness of all games, but Game 3. Their initial reasoning about the outcomes of the games relied on their attention to the contents of each bag. They expected that Game 2 would be fair because there were four red chips and four blue chips in total when the two bags were combined. They also inferred that Game 3 was fair too for the same reason. Even though their prediction about Game 3 was correct, the reasoning behind it was problematic. They focused on the total number of red or blue chips rather than the combined outcomes, mixed and same colour. They expected that the symmetry in the combined bags would generalize to the combined outcomes.

In Game 4, both students were initially quite certain that the game was unfair because of the first bag with 100% red chips. Focusing on the single events in each bag, i.e., bag two is being fair because of equal number of red and blue chips while bag one contains only red chips and no blue ones, Chris stated that the game was certainly not fair. In the same line of reasoning, they expected Game 1, which is slightly different than Game 4, to be unfair too even after they explored the previous situation and saw that they were wrong about it.

Testing Conjectures in TinkerPlots

To further test their initial predictions, students used *TinkerPlots 2.0* to build a model of the game situation. Even though they had bags and chips available to play the game, they chose to skip this part after the first exploration (Game 2). After the predictions for Game 3, they moved to the computer and used *TinkerPlots* to quickly generate and analyse large samples together.

Figure 1 shows a model for Game 2 that students constructed in *TinkerPlots*. When they built the Sampler (upper left), they used two spinners, one with 75% blue and 25% red and the other with 25% blue and 75% red sections, representing the proportion of red and blue chips in each bag. Each spinner spins once to execute a trial of randomly drawing a chip from each bag. Repeat number is set to 1000 (students' choice). The results table to the right of the spinners displays the repetitions as they occur in the Sampler. The plot at the upper right shows the percentage of four possible outcomes in the sample of 1000. In the graph at the bottom, the individual outcomes, 'blue,blue' and 'red,red' then 'blue,red' and 'red,blue', are combined into a bin by dragging one into the other to display the percentage of "the same colour" and "the mixed colour" outcomes respectively.

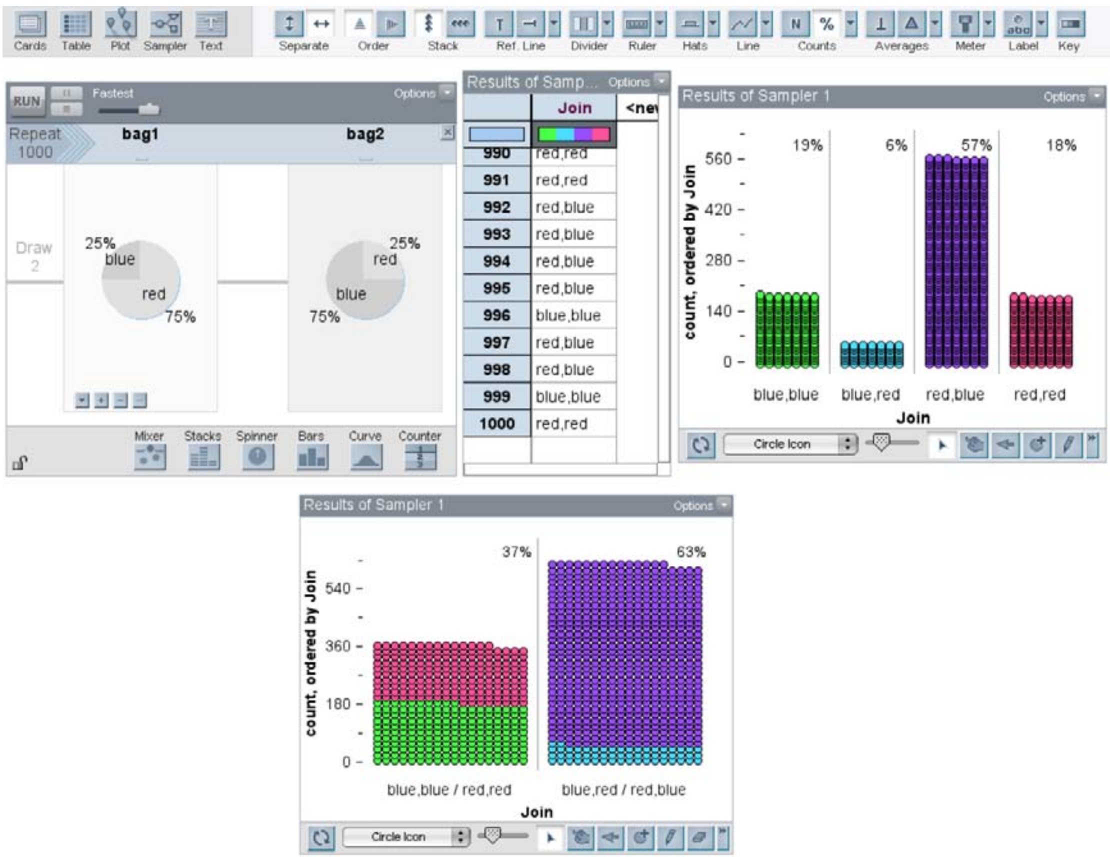

Figure 1. Computer model of Game 2 built by Chris and Jacob.

Seeing the results from a sample of 1000, Chris and Jacob realised that their initial prediction was obviously incorrect. Drawing large samples gave students more confidence in their conclusions based on the simulated data and helped them gain a new insight about the likelihood of outcomes in the game through the spinner model they built in the Sampler. For instance, to explain the data they observed in the computer simulation Chris further said, "I see what this is. There is so much for 'red,blue' [*pointing to the highest stack in the graph at the top*] it is because there is [*pointing to the first spinner on the screen*] that huge amount of red to choose from here, and there is loads of blue to choose from [*pointing to the second spinner*]."

Dialogic Talk around the Computer

After modeling game 4 in *TinkerPlots* and running 1000 repetitions of the game similarly to the example in Figure 1, the students completely changed their view of the situation again. In Kazak, Wegerif and Fujita (2013) we described the following exchange between the teacher [Sibel] and the pupils which happened right before they ran their model. When Jacob said "I am actually debating now" this implied that he was now aware of new voices and perspectives and beginning to *see the situation from more than one point of view*.

Sibel: Okay and you think that you guys will win most of the time, huh?

Chris: I think we will actually win most of the time.

Jacob: Actually, I am actually debating now [*while he presses the run button to collect 1000 data*]

The contradicting results from the simulated data led them to re-examine the chances in the game looking at their model.

Jacob: Oh yes, it is 50-50 because oh yeah!

Chris: Jeez, we got an entire army on our side!

Jacob: No, no, Chris you don't get it. The first one [*pointing to the first spinner on the screen*] you always get 100% red

Chris: Exactly

Jacob: Then the next

Chris: Then the next one you could get [*pointing to the second spinner on the screen*]

Jacob: It's a 50-50 chance of getting the same [*he is laughing and almost speechless*]

Chris: I don't get it!

Jacob: So basically the first time you will get a red, next time you got a 50-50 chance of getting the same or something different [*he is covering his face with his hands and laughing*]

Chris: I don't get this at all. Why are you laughing? Jacob, why are you laughing? Just calm down.

In Kazak, Wegerif and Fujita (2013), we argued that Jacob seemed to have a dialogue with an absent witness or addressee (Wegerif, 2013) when thinking aloud about why the outcomes turned out to be even. This relationship with an outside point of view enabled him to change his understanding. Chris's 'I don't get this at all' remark, suggested that he could not make sense of Jacob's utterances because he did not share in Jacob's new insight. Later, the exchange between the two pupils continued with Jacob's explanation of his insight addressed to Chris in a way that proved persuasive:

Jacob: [*now talking to Chris*] First one you will definitely get a red, so the next one you would get either a red or a blue. So basically you can either get 50% you will get red

Chris: Red, yeah. So it is

Jacob: 50% you will get blue

Chris: So it is 50-50.

Concluding remarks

In our study as well as others (i.e., Vahey, Enyedy and Gifford, 2000) the notion of fairness provided a motivating and productive area of inquiry for students investigating probability in computer-based activities. The Chips Game task involved students' active engagement in predicting, generating a large number of data through computer simulations, and interpreting the resulting outcomes in comparison to their initial predictions about the fairness of various games.

In chance situations, like the games involving compound events described above, students' probabilistic intuitions are not always reliable. In all but one game, their intuitive ideas led them to an incorrect judgment about the fairness of the game. When asked to give an explanation for their predictions, they seemed to focus either on the symmetry in the total number of red and blue chips in the bags (e.g., games 2 and 3) or comparing the likelihood of simple events (e.g., the chance of getting a red chip) in each bag (e.g., games 1 and 4). These findings are consistent with previous research in that the shift from simple events to compound events is difficult for students (Watson, 2005). One reason for this is that dealing with compound events entails considering all possible outcomes of an experiment and counting the number of ways that a combined event can occur. It is challenging for students to learn the need to consider the sample space in such situations; instead their thinking is often led by intuitive judgments (Konold and Kazak, 2008).

TinkerPlots played a significant role in fostering students' reasoning about chance in these games. Its capacity for simulations allowed students to construct models of the probability activities that they were given. These models meant that they could run many simulated iterations of each game thereby gathering far more data than they could have gained manually. The flexibility and speed of *TinkerPlots* enabled them to test and revise their initial predictions. The findings from this activity prompted them to rethink their early assumptions. Especially important to this process of prompting a re-think was the way in which the model they had to build in *TinkerPlots* showed that the process had two stages. Seeing the two stages of the game offered the new way of framing the problem that they needed to understand it.

The role of dialogic talk that emerged through the testing of initial predictions with *TinkerPlots* simulations was also noticeable in students' articulation of their thinking including their half-baked ideas (Kazak, Wegerif and Fujita, 2013). The exclamation "I don't get it!" by Chris was indicative of a certain humility and trust. This then led Jacob to explain his insight into the two-stage structure of the problem with reference to the model on the computer screen. The dialogic education approach that we had used here had as a key social norm or 'ground rule' that making mistakes and showing that you do not understand is OK. An atmosphere of trust was promoted in the class and this facilitated the admission of a lack of understanding and the asking of open clarifying questions. Without trust this important step might not occur.

In addition to the role of dialogic talk in helping each other understand and think, Jacob's initial switch in perspective can also be understood as dialogic. This switch in perspective is the outcome of an invisible dialogue with a projected external addressee or 'witness'. He is able to change to his mind because he looks at the problem again from the perspective of the projected external addressee. This dialogic mechanism at work is taking an outside perspective (Wegerif, 2013).

Overall, a combination of dialogic talk and the use of *TinkerPlots* helped students revise their initial conjectures and explanations as they reasoned about the chances in each game. However, it is worth noting that their understanding in a particular game is likely to be fragile. After the investigation of Game 4, they seemed to develop an insight about the likelihood of combined outcomes in the game but they were unable to use it in the next game, which required a very similar reasoning. This suggests that development of underlying probabilistic understandings is rather slow and we should begin teaching these ideas as early as possible and revisit them regularly in the secondary school years (Konold and Kazak, 2008).

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Networking theories of society and cognitive science: An analytical approach to the social in school mathematics

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Debate about the interplay between social and individual aspects of mathematics teaching and learning remains at the cutting edge of theoretical understanding of mathematics education research. In trying to make sense of the insights of these divergent perspectives I ask: How is it that social reality exists? What are the merits and limitations of considering the students in our classrooms as only collections of individual minds, in contrast with perspectives that posit the primacy of the social in determining the identity of mathematics learners? Can each be accorded its relative legitimacy in a rigorous and rational manner? Recent developments in analytical social theory may have the potential to address this issue productively. This paper covers the conflict between social-constructivist and socio-cultural perspectives in the literature and the critical role of inter-subjectivity in communicating mathematics through interaction. The paper concludes by drawing on Searle's notion of collective intentionality to address the networking and complementary use of theories based in cognitive science and critical theory and the interplay of the individual and social in school mathematics.

Keywords: collaborative learning; critical theory; cognitive science; social reality; intersubjectivity; collective intentionality

Introduction

This article is a theoretical discussion about elements of analytical frameworks, situated in the context of research on small group interactions in mixed ability year seven mathematics classes in England adopting elements of 'complex instruction'. The analysis was focused on what Sfard (1998) might call the interplay between acquisitionist and participationist metaphors for learning. In the course of this research, I developed an analytical framework based in Habermas' (1987) Theory of Communicative Action (TCA). Consideration of sociological perspectives and cognitive aspects of the research on small group interaction led me to question how social reality exists from an analytical perspective using Searle's recent works on social theory (Searle, 2010; Searle, 1997).

The importance of 'intersubjectivity' as a concept in mathematics education is a sign of a significant shift in the historical conceptualisation of the knowing subject and thus also in the nature of knowledge. Signs of the change can be seen in the shift of the mathematics education literature from conceptualisations initially based more or less solely in psychological theories towards theories that attempt to re-conceptualise the field using social and socio-cultural ideas (Lerman, 2006). This shift has been driven by awareness that there are complex issues at play in mathematics education that cannot be accounted for using solely psychological perspectives. Intersubjectivity becomes an important concept in socially oriented theories that seek to understand the nature of learning and teaching mathematics. In seeking a more technical, empirically based, understanding of the role intersubjectivity plays in the interactive constitution of knowledge in school mathematics teaching and learning, Habermas's theory of communicative action was found to be analytically productive (Kent, 2012).

I will describe the research context and aspects of analysis; discuss multiple theoretical perspectives on interaction, communication and mathematics education; discuss relationships between Habermas' TCA and Searle's Speech Act Theory; and make an argument for the critical potential and productivity of coordinating critical theories and cognitive theories using Searle's Theory of Social Reality.

The context: analysis of small group interactions in year 7 classes adopting equitable approaches to mathematics teaching in England

The case study focused on student interactions in the context of particular non-traditional mathematics pedagogy, complex instruction, in mixed-ability year seven classes. Three mixed ability year seven mathematics classes at three different school sites took part in the case study research. Participants included three teachers, three teaching assistants and four classes of students. Data included: audio recordings of preliminary interviews with participating teachers, data from a summer professional development workshop on complex instruction in mathematics education, video data from classroom observations and data from professional development workshops held at the schools prior to the case study. Teachers and researchers worked collaboratively to design tasks. Qualitative data was collected on student interactions around tasks using participant observation, and video recordings of whole class and small group interactions. Researcher memos supplemented video observations of the lessons. During the course of the case study, regular reflective interviews with teachers took place after every lesson observation focusing on the teaching of the planned tasks.

Analytical approach, codes and model of intersubjective small group interactions

The plan for data analysis was to adapt the constant comparative method as a rigorous way to address the iterative dialectical approach in Bassey's (1999) model for case study research. The integrated approach adopted combined the microanalysis techniques of open coding and constant comparison with the iterative analysis of Bassey's case study method. Codes and categories were developed out of the constant comparative method and open coding. These were brought into relation to one another theoretically through the development of a model of intersubjective small group interactions. The codes are heavily influenced by Habermas' TCA and the model is an attempt to articulate the dynamic relationships between different utterances and speech acts within the episodes of interaction (Kent, 2012). These tools were then used to develop analytical themes including coordination and communication; power in small group interactions; communicative breakdown; and the preconditions for communicative action in collaborative mathematics learning. The models and theories developed through the iterations of analysis as ideas generated were taken back to the data to be tested and refined. The analytical challenge of this endeavour is part of the motivation for this paper, which seeks to explore how social, communicative, and individual cognition may be related to one another.

Intersubjectivity and mathematics education: socio-cultural, situated and interactionist views

Intersubjectivity is raised by Lerman (1996) explicitly in an article that challenged the theorising of social constructivists (as based too much in radical constructivism and therefore incoherent with the primacy of the social that Lerman related to concepts of intersubjectivity) and in the same paper outlined several features of intersubjectivity in mathematics education. In particular, three features regarding intersubjectivity in mathematics education: the role of intersubjectivity in the consititution of subjectivity; cognition as situated in practices; and the nature of mathematics as cultural knowledge having a real but non-deterministic force.

The first is the idea that subjectivity is constituted through social practices and thus can be said to come into existence through intersubjective processes. This idea resonates with ideas from situated theory, which see identity as a process of coming to belong to a community of practice. Lerman uses an interpretation of Vygotskyian theories of enculturation, internalisation, and the zone of proximal development to address these questions. The second feature that Lerman identifies is the notion of cognition as situated in practices. Sfard and others (Sfard, 2008; Sfard and Kieran, 2001) have argued that cognition ought to be thought of as a process of communication[3].

The final feature that Lerman highlights is the idea of mathematics as cultural knowledge. In this situation Lerman again uses ideas founded in Vygotsky of the pre-existing social structures that are the force behind the development in participants of knowledge that is part of the cultural tradition of mathematics[4][5]. Lerman(1996) argues for the separation of cognitive traditions and sociocultural traditions, stating that it is incoherent to assert that a radical constructivism can be a primary foundation from which to understand the functioning of social processes as having force in the development of knowledge and subjectivity.

However, not all authors take such a strong position. Bauersfeld (1994) makes a case for interactionist perspectives as a middle way. The interactionist perspective as articulated by Bauersfeld seems at first to be quite close to the theories of communicative action as articulated by Habermas[6]. Bauersfeld, (1994), outlines the core convictions of the interactionists highlighting a series of interrelated conceptualisations of the nature and roles of: learning; meaning making; 'languaging'; knowing or remembering; 'mathematising'; internal representations; visualisations and embodiments; and teaching. This taxonomy represents a compromise between cognitive and social approaches emphasising which parts are about internal subjectivity and which are about social conditions and interaction.

In addition to these perspectives there is a literature of discourse in mathematics education that has developed over the last 10 to 20 years. Ryve (2011) undertakes a fairly comprehensive analysis of this emerging body of research. Ryve

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[3] This claim is pointing to the strong relationship between the conditions of satisfaction of intentional mental states and the conditions of satisfaction of communicative utterances (Searle 2010).

[4] It is interesting to note that he sees this function as non-deterministic yet having a real force that has a more or less direct impact on the development of knowledge in participants.

[5] It should be noted that this is not necessarily a transmission concept but is related to the former idea of cognition as situated in practices.

[6] However it should be noted that the literature on interactionist perspectives does not make reference to Habermas' theory of communicative action (TCA). Rather there is articulation and reference to speech act theory and pragmatic semantics and other (primarily sociological) traditions that overlap with the interests of elements of Habermas' reconstructive approach to theorising social science.

addresses a number of questions in relation to this literature including: how and to what extent are the articles theoretically conceptualised; what data are used and how are the data analysed; and to what extent do the articles relate to or build upon one another? In his analysis Ryve notes that conceptual clarity of many of the studies is weak and the cumulative development of theoretical approaches is uncommon.

Ryve asserts that the priority should be on developing the sophistication of the theoretical perspectives that have already been developed rather than introducing further approaches from other fields. Ryve's analysis indicates that general features of theoretical development such as defining keywords, building on the work of others, and clearly positioning the article in epistemological perspectives are of great importance for future studies in mathematics education.

Lave and Wenger (1991) developed the ideas of 'situated learning' in the early nineties and they have gained widespread influence as a productive theoretical perspective for the analysis of learning and teaching in mathematics education (Greeno et al., 1997; Boaler, 1999; Boaler, 2000; Boaler and Greeno, 2000; Kumpulainen, 1999). The theoretical positions they articulated conceptualised communities of practice as the location of identity formation and learning; the concept 'legitimate peripheral participation' as a process of coming to belong to such communities; and learning as the process of forming an identity in the context of a community of practice (Lave and Wenger, 1991). Further, Cobb et al. (2000) also raise the importance of situated theory to the ideas of the interactionists, arguing for a pragmatic use of situated theory that rejects a purely psychological point of view in the attempt to formulate theory that can have practical application in the improvement of mathematics education practices by taking into account theories that address the social aspects of learning.

The issue of intersubjectivity in mathematics education and its relation to communication and learning has been well recognised in the literature. Interactionist, situated, and socio-cultural perspectives of mathematics education provide a useful framework for the analysis of the complex dynamics of classroom practice. These ideas seek to go beyond the limitations of psychological interpretations of learning, teaching, and understanding as an experience of the individual by considering the linguistic and social contexts in which meaning is constituted. Habermas' TCA (1987) and Searle's philosophy of social reality (1997, 2010) can inform theoretical frameworks focused on intersubjectivity and the associated constellations of cognitive, linguistic and social factors in mathematics education.

Habermas and Searle: Frankfurt School Critical Theory and a Theory of Social Reality based in Speech Act Theory

In Habermas' TCA, 'communicative action' is the coordination of action by multiple goal-oriented actors through a process of cooperative interpretation or 'intersubjective understanding' (Habermas, 1987). For Habermas meaning and understanding are inextricably linked such that understanding the meaning of an utterance implies: "1) The recognition of its literal meaning; 2) The assessment of the speaker's intentions; 3) Knowledge of the reasons which could be adduced to justify the utterance and its content and; 4) Acceptance of those reasons and hence the utterance" (Finlayson, 2005). Communicative action, like other models of action, has as its purpose the achievement of the goals of the actors involved. However, in communicative action this is achieved through understanding (as defined by Habermas' theory of pragmatic meaning). This is a key insight for realising the usefulness of communicative action as a theoretical lens for evaluating interactions in the context of complex instruction.

Theories about the give and take of validity claims in pursuit of shared understanding and common goals can be related to ideas about cognition, language and society. Searle (1997, 2010) develops an analytical philosophy of social reality based in philosophies of mind and language, seeking to establish a perspective of the constitution of social reality. Searle argues that in order to understand social reality concepts such as 'intentionality' (the property of minds to be about things) and the ideas of speech act theory must be built upon with conceptions of collective intentionality, assignment of function, status functions, social institutions, and institutional facts. This is all done in the context of a weak realist position that seeks to avoid positivist reductions while maintaining a materialist position with regards to both social and physical reality.

Some main features of this theory include: a novel conception of the 'collective intentionality' and the role it plays in the establishment of language and institutional facts; the 'assignment of function' which is developed with recourse to the idea of collective intentionality in the context of analytical philosophies of mind and speech act theory and establishes the anatomy of social reality in an iterative and self-reinforcing manner. Collective intentionality is the idea that we have an innate capacity to have 'we-mental states' not just 'I-mental states'. Intentional states in this sense refer to the philosophical concept of intentionality as opposed to the common usage. The common usage of intentionality is a subset of intentional states. Intentional states are mental states that are about things. Essentially it is the property of thinking to be 'about' something. Thus desires, beliefs, knowledge, and even potentially perception or aspects thereof are all intentional states. Imagining a triangle on the surface of a sphere could be considered an intentional state. Thus collective intentionality involves mental states such that two people are more or less seamlessly coordinated on and through the same mental object. An example might be, "We are going for a walk."

This serves as the basis for Searle's description of the origin and anatomy of language. Searle asserts that the capacity for collective intentionality serves as the basis for the 'assignment of function' that is characterised generally as 'X counts as Y in context C'. This is the basic anatomy of language that builds the foundation up from non-linguistic coordination. Searle asserts that the rest of social reality is built up iteratively using the assignment of function. Searle asserts that this theory of social reality addresses a pressing need in philosophy about how to reconcile knowledge from the physical sciences with knowledge in the social sciences[7].

Networking Theories of Cognition and Critical Theory in order to deepen analysis of small group interaction and generate new avenues for investigation

Concepts of intersubjectivity based in Habermas' TCA can be coherently used alongside insights from other theoretical approaches, and in particular cognitive approaches. Similar attempts have been made by the interactionist researchers, and the profusion of theoretical points of view in the analysis of mathematics education has led to a situation wherein the knowledge being brought to bear in research often reflects insights from multiple perspectives. However, as is made clear by the issues raised by Lerman, Bussi, and Ryve, this can be a problematic undertaking and it is

[7] This explains Searle's articulation and emphasis of a weak realist position.

therefore essential to be clear theoretically and methodologically as to how this attempt will be made.

Bussi (1994) states that there is a need to ‘look for’ conceptual tools to deal with complementarity and the use of multiple theories, while Cobb (2006) raises two important points in relation to the use of multiple theories in mathematics education. First, in consideration of how various theoretical perspectives “…orient the types of questions asked and knowledge produced…” he suggests that the dichotomy between activity being viewed as primarily individual or primarily social in character fails to recognise the problematic nature of what is meant by the individual. Cobb suggests that, instead of positioning perspectives into these dichotomous categories, it makes more sense to compare and contrast the different characterisations and theoretical treatments of individuals[8].

Cobb’s second point is that the use of multiple theories, and the relative validity of each, should be dealt with in a pragmatic fashion. This is not to say one should use whatever works in a non-reflective manner; rather it is based in Dewey’s account of pragmatic justification such that a theory’s ability to provide insight into empirical situations is a key factor in determining their truth. Using these ideas of complementarity and pragmatic justification one can see the development of models based in TCA as productive while Searle’s ideas can be seen as having potential to contribute to networking of sociological, critical and cognitive theories of mathematics education[9].

Conclusions

By beginning to consider how the social realities of school mathematics might exist from the perspective of Searle’s theory, researchers may be able to recognise and attend to technical features of the institutions and institutional facts of mathematics education and their relation to the cognitive development of mathematics learners in a coordinated fashion. Cognitive perspectives may be complemented by and deepen sociological perspectives (and in particular critical theory) in a reciprocal fashion.

Consideration of how sociological and cognitive aspects of mathematics education can be coordinated productively using Searle’s theory of social reality that can strengthen the impact of research insights on mathematics learning and teaching. The actuality addressed by such by a critical approach to cognition and social reality in mathematics education research is the inequitable distribution of power, opportunity, and knowledge and the existence of conflict in mathematics classrooms. The critical context is the continuing reproduction of inequitable outcomes in England, especially in mathematics and sciences (Boaler, Altendorf and Kent, 2011) and the normative imperative of educating students of mathematics in a manner conducive to participation in a democracy (Boaler, 2008). The possible can be addressed using critical theories through the identification of principles conducive to more equitable communicative interactions. The use of Searle’s theory can help explain how concepts of intersubjectivity and communicative action can be brought

[8] An important point as a large part of Habermas’s project is a shift in focus away from philosophies of consciousness that consider rationality to be located primarily within the structures of the conscious subject, towards a treatment of rationality as inherent in the intersubjective features of communication.

[9] The validity of such conceptual tools would be primarily in their pragmatic application of deepening insight and understanding towards the development of new courses of action that may be useful in achieving the common goals of the mathematics education community, such as they are. This position is in line with the methodological arguments based in Habermas’ (1987) philosophy, and focused on the rigorous use of multiple theoretical sources in the development of knowledge in the social sciences.

into relation with cognitivist and psychological theories without reducing one to the other. Addressing the question of how social reality exists using Searle's ideas may allow mathematics education research to address links between cognitive science, critical social theory and mathematics education to achieve deeper understanding about the relationship between institutional reality of school mathematics and the development of mathematical cognition.

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The use of alternative double number lines as models of ratio tasks and as models for ratio relations and scaling

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In this paper we draw on ICCAMS project materials that used the double number line (DNL) to develop secondary school students' understanding of multiplicative reasoning. In particular, we look at the use of a DNL, and its alternative version, as a *model of* ratio tasks, as a *model for* developing an understanding of ratio relations, and finally (but only briefly) as a *model for* developing the notion of multiplication as scaling.

Keywords: double number line, ratio, scaling, multiplicative reasoning

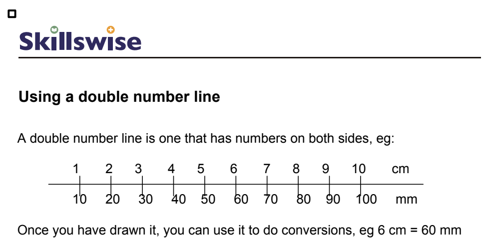
Introduction

Increasing Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS) was a 4½-year research project funded by the Economic and Social Research Council in the UK. Phase 1 consisted of a cross-sectional survey of 11-14 years olds' understandings of algebra and multiplicative reasoning, and their attitudes to mathematics. Phase 2 was a collaborative research study with a group of teachers that aimed to improve students' attainment and attitudes in these two areas (Brown, Hodgen and Küchemann, 2012). This included a design research element (Cobb, Confrey, diSessa, Lehrer and Schauble, 2003) that investigated how cognitive tools influenced student learning. In Phase 3 the work was implemented on a larger scale.

In Phase 2, we developed tasks involving the double number line (DNL). In this paper we discuss some of the insights gained from this. The DNL enables students to develop their understanding - it is more than just a neat tool for solving ratio tasks and is a more subtle and complex model than many curriculum authors suggest.

It is relatively easy to find advocates for the DNL, especially from researchers in the Dutch *Realistic Mathematics Education* (RME) tradition (e.g., van den Heuvel-Panhuizen, 2001). However, substantive research papers on the DNL are rare. We have found some interesting studies (e.g., Moss and Case, 1999; Misailidou and Williams, 2003; Corina, Zhao, Cobb and McClain, 2004; Orrill and Brown, 2012), but often the DNL plays only a small part in the research or the tasks used are not particularly well designed or implemented.

The Double Number Line (DNL) is beginning to appear quite widely in school mathematics curriculum materials, especially those influenced (directly or indirectly) by RME. Materials in the English language that stand out are the *Mathematics in Context* (MiC) project (developed in collaboration with the Wisconsin Center for Educational Research, University of Wisconsin-Madison and the Freudenthal Institute), and a UK project based on this, *Making Sense of Maths*. The DNL can also be found in homespun materials published on the internet, such as this extract (right) from a worksheet on the BBC's Skillswise website. Note here that the DNL is poorly articulated – for example, the zero marks are missing - and the approach is very procedural. Such limitations are not uncommon in materials



involving the DNL. The Common Core State Standards (which have been adopted by the majority of states in the USA) include this reference to the DNL:

> CCSS.Math.Content.6RP.A.3 Use ratio and rate reasoning to solve real-world and mathematical problems, e.g. by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

The mathematics standards for New Zealand also refer to the DNL, though surprisingly perhaps it does not appear in the September 2013 English National Curriculum 'programmes of study', nor in the NCTM Standards in the USA. Interestingly, though, NCTM has a 'representation standard', which is separated into these three components:

> Instructional programs from prekindergarten through grade 12 should enable all students to
>
> • create and use representations to organize, record, and communicate mathematical ideas;
> • select, apply, and translate among mathematical representations to solve problems;
> • use representations to model and interpret physical, social, and mathematical phenomena.

The third component chimes (to some extent) with the RME notion of creating a 'model of' a situation (e.g., Gravemeijer, 1999). RME argues that one should start by introducing students to accessible, perhaps 'real-life', situations which they are able to model in a natural way, and then, over time, students use these models in their own right to develop and formalise mathematical ideas (through 'vertical mathematising'). The models shift from being 'models of' a situation to being 'models for' mathematical ideas.

As mentioned above, most curriculum materials seem to focus on the second NCTM component, i.e. where a representation (or 'model of') is used directly as a device that helps students solve problems. So for example, in the MiC book *Models You Can Count On* (Abels, Wijers, Pligge and Hedges, 2006) students are told, "Learning how to use a double number line will help you make precise calculations effortlessly" (p. 43). In the teacher's version of the book (Webb, Hedges, Abels and Pahla, 2006), it is stated

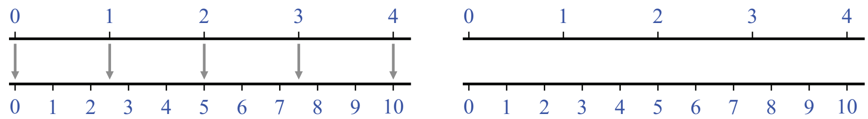
> The operations that students use on a double number line are similar to the operations they learned ... when they used a ratio table. Instead of a double number line, a ratio table could also be used. However, a double number line gives visual support: the numbers are ordered. Note that a double number line can start at zero, but a ratio table cannot (p. 40B).

This statement is highly cryptic, yet it is not elaborated for the teacher-reader. As such, it is likely to convey a procedural view of the DNL: students use the DNL to perform operations. We get a hint about the nature of the DNL from the reference to 'visual support' - but, the suggestions that this means 'the numbers are ordered' and that the DNL 'can start at zero' are rather inadequate (see below). The idea that a ratio table cannot contain 0, 0 seems plain wrong.

For pragmatic reasons, we did not spend as much time as RME would advocate to allow the DNL model to 'emerge'. Rather, our focus was on the first NCTM component, i.e. on using the DNL as a 'model for' exploring mathematical ideas.

The DNL appears most commonly in curriculum materials as a fraction-, decimal- or percentage-bar, with the purpose of comparing fractions (e.g., Which is greater, $\frac{2}{5}$ or $\frac{3}{7}$?) or for finding equivalences between fractions, decimals and percentages. However, it also used more generally for situations involving ratio relations, such as conversions (e.g. of metres to feet on a map scale) and geometric enlargement.

The DNL is essentially a mapping diagram, but one in which the scales on the two, parallel, axes have been adjusted in such a way that the mapping arrows are all parallel. It is most commonly used to represent linear relations, i.e. relations of the form $f(x) = kx$. For this, the zeros on the two scales are aligned and the scales themselves are both linear, as in the example for $f(x) = 2.5x$ below. (The standard version, without the mapping arrows, is shown on the right.)



A linear relation $f(x) = kx$ has the properties $f(p+q) = f(p) + f(q)$ and $f(rp) = rf(p)$. This means that if we have a linear relation that maps 3 onto 7.5, say (as in the DNL above) and we want to find the image of, say, 4, we can do this not just by finding and applying the general multiplier ×2.5, but by using a *rated addition* method such as this:

> if the relation is linear ('in proportion') and 3 maps onto 7.5, then 3÷3 maps onto 7.5÷3, i.e. 1 maps onto 2.5; and then (3+1) maps onto (7.5+2.5), i.e. 4 maps onto 10.

The multiplier method can be said to operate *between* the lines, whereas rated addition operates *along* the lines. The rated addition approach might appear more cumbersome; however, it is often the basis for mental methods and allows us, at least to some degree, to adopt an informal approach using simple relations of our choosing. There is no choice about the between-lines multiplier - unless we are prepared to work 'outside' the given lines, by in effect creating an alternative DNL (we discuss this in depth later). There is considerable evidence, albeit indirect, to support the notion that working along the lines is often more accessible to students than the general multiplier approach. For example, Vergnaud (1983) has found that students are far more likely to establish a relation that is *within* a measure space (what he calls a *scalar* relation) than *between* measure spaces (a *function* relation). We found evidence to support this when we gave these two versions of the *Spicy Soup* item, below, to parallel (but non-representative) samples of mostly Year 8 students (N=77 and N=74 respectively). Notice that the numbers 33 and 25 had been changed round.

> Ant is making spicy soup for 11 people. He uses 33 ml of tabasco sauce. Bea is making the same soup for 25 people. How much tabasco sauce should she use?
>
> Ant is making spicy soup for 11 people. He uses 25 ml of tabasco sauce. Bea is making the same soup for 33 people. How much tabasco sauce should she use?

Both items can be said to involve the multiplicative relations ×3 and ×2.27 (approx). The version where the simpler relation is scalar (11 people and 33 people) was found to be much easier than the parallel version where this relation was functional (11 people and 33 ml), with facilities of 91% and 51% respectively.

In the DNL, each number line usually represents a single measure space. So where these measure spaces are different (e.g. £ and \$, metres and feet, people and sauce), it is likely, that students will work with relations along the lines (as this involves *within* measure space relations), rather than between them, unless, perhaps, the between-lines relation is a very simple multiplier.

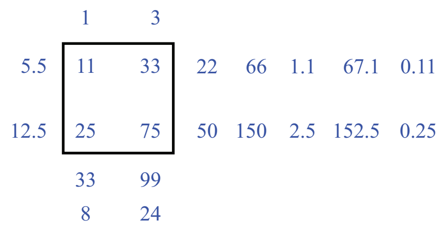
Our purpose in using the DNL was twofold – to explore the nature of ratio relations and to model a particular aspect of multiplication, namely *multiplication as scaling*. Our experience suggests that both uses can be enriching. However, they are far from unproblematic.

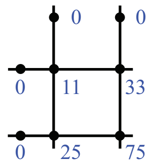
The use of alternative DNLs as models of ratio relationships

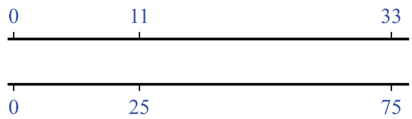
The use of the DNL to solve or analyse ratio tasks is not as straightforward as many curriculum materials seem to suggest. It is often possible to create *two* DNLs for a given task, and they can represent the situation in subtly different ways, or in ways that are hard to interpret.

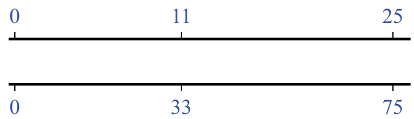
Imagine we have a table of numbers (right) where there is a ratio relation between the rows, i.e. $\frac{11}{25} = \frac{33}{75}$ (and hence between the columns, i.e. $\frac{11}{33} = \frac{25}{75}$. We can extend the rows with other numbers fitting the $\frac{11}{25}$ relation, and we can extend the columns with other numbers fitting the $\frac{11}{33}$ relation, e.g. like this (near right). And we can express this in a more general and coherent way using a horizontal DNL and a vertical DNL (far right). [The DNLs are drawn again (below), in the usual format.]

11	33
25	75



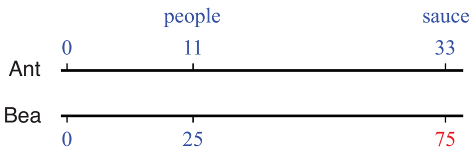


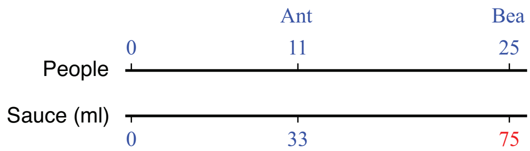




Now imagine that our original numbers arose from a 'real life' ratio context. What might the DNLs mean? Consider a recipe context, e.g. the *Spicy Soup* task discussed earlier and summarised in this ratio table (right). Here the second DNL (shown again, below right) seems to make perfect sense. One line represents numbers of people, the other ml. of sauce. We can easily create other, perfectly meaningful pairs of numbers on this DNL by 'skipping' along the lines, such as 11+11, 33+33 (22 people would need 66 ml) or 11÷3, 33÷3 (1 person needs 3 ml). However, on the first DNL (below, left) the lines seem to be hybrids, representing both people and sauce. It might appear that we can skip nicely from 11, 25 to 22, 50, say, to 33, 75, thereby solving the task, but what does a pair like 22, 50 mean? If it is 22 people and 50 people, how does this fit the story? To resolve this requires quite a high level of abstraction: students will need to blur the distinction between people and sauce, e.g. by thinking of the lines as simply representing 'quantity of ingredients' (if we're happy to accept people in our soup ...). Then 22, 50 could refer to, say, ounces of sugar for Ant's soup and Bea's soup, or respective numbers of tomatoes. However, operating on numbers along the line might still seem rather odd.

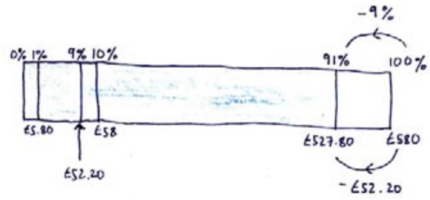
	people	sauce (ml)
Ant	11	33
Bea	25	75





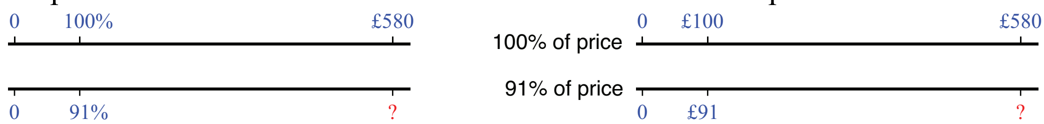
In some contexts it is easier to give a sensible meaning to both DNLs. The booklet *Fair Shares* (Dickinson, Dudzic, Eade, Gough and Hough, 2012: 16), from the RME-inspired series *Making Sense of Maths*, shows how a DNL can be used to find the cost of a £580 computer after a 9% reduction. The DNL is shown below (we have added a paired-down ratio table of the basic information). As can be seen, this

DNL works very well, since it allows students to solve the task using relatively simple moves along the lines.



100	580
91	?

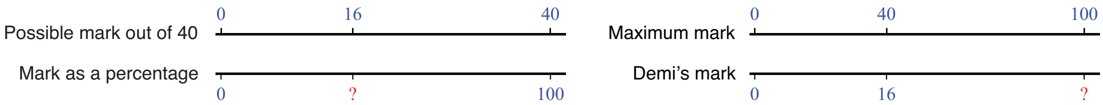
The alternative DNL would look like the one on the left, below. At first sight, this does not appear to work well: as with one of the *Spicy Soup* DNLs, the number lines seem to be hybrids, this time representing percentage (e.g. 100%) and price (e.g. £580) simultaneously. However, with this context it takes less abstraction to smooth this out, e.g. by letting all the numbers represent prices (below, right): the top line could then be thought of as showing the full price of various articles (be they computers or other objects), with the bottom line showing 91% of these prices. It then becomes possible to use a rated addition method along-the-lines in a quite meaningful way: if a £100 computer is reduced to £91, then a £600 computer would be reduced to 6×£91 = £546 and a £20 computer would be reduced to



£91÷5 = £18.20, and so a £580 computer would be reduced to £546 – £18.20 = £527.80.

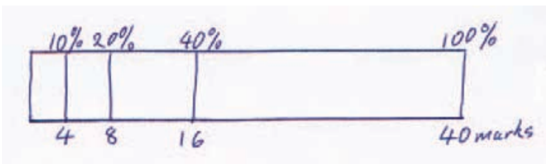
The booklet *Fair Shares* (p19) also includes a task about converting a test result into a percentage, in this case a mark of 16 out 40 achieved by a character 'Demi'. The task can be summarised by this ratio table (below, right). The booklet first tackles the task using a DNL. Again there are two possibilities: we can draw parallel lines through 16 and 40 and ? and 100, or through 16 and ? and 40 and 100. This time both DNLs work perfectly well (both are amenable to an along-the-lines approach) but they model the situation in

16	40
?	100



markedly different ways, as can be seen from the different labels we have given to the lines (below).

The booklet goes for the first version (flipped over), which is used in an along-



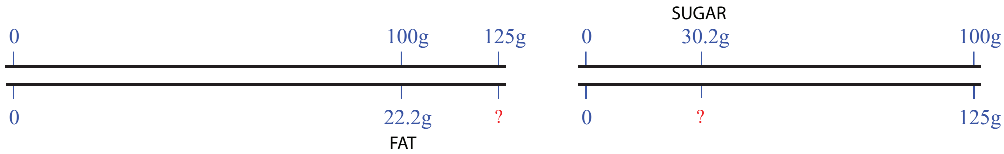
Mark	16	8	4	40
Out of	40	20	10	100

the-lines way (below, left) to arrive at the answer, 40%. The task is then solved again using an extended ratio table (below, right). However, this does not fit the first DNL. It is perhaps unfortunate that the answer, 40 (%), is the same as the total number of marks on the given test. As a consequence, we get the same pairs of corresponding numbers on the DNL as in the ratio table (16,40; 8,20; 4,10; 40,100). However, their meanings are very different. The DNL and ratio table do *not* correspond here - the ratio table matches our second DNL (above). A ratio table that matches the first DNL would look something like this (below, right).

Possible mark out of 40	40	4	16
Mark as a percentage	100	10	40

The first DNL (and corresponding ratio table) models what a range of possible marks on the 40-mark test would be as a percentage, whereas the second DNL (and corresponding ratio table) is taking Demi's specific test result of 16 out of 40 and modelling what her equivalent score would be if the total number of marks was different. Both DNLs (and corresponding ratio tables) are fairly easy to use in this task, i.e. they lend themselves to an informal, rated addition approach. However, students need to be able to switch between the two views of the task, which may not be easy, especially if the existence of two viewpoints is not acknowledged.

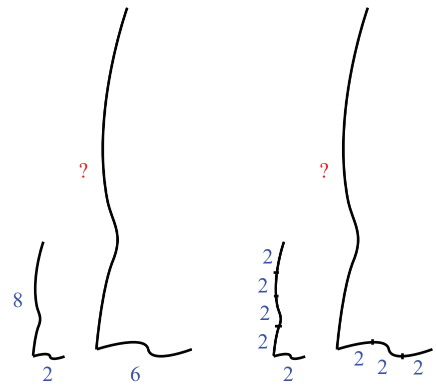
An early draft of the ICCAMS materials included a task about a 125g portion of cheesecake. Students were asked to estimate the amount of fat in the portion, on the basis of a 'nutrition table' which stated that there were 22.2g of fat per 100g of cheesecake. Students tended to solve this informally, using rated addition, in this kind of way: "An extra 25g will contain an extra 5g and-a-bit of fat, making about 28g in all". The obvious way to model this on a DNL would be as shown below, left. However, we wanted to look at other ingredients, e.g. sugar, of which there were 30.2g per 100g. We thus decided to present a DNL like the one below, right.



This latter DNL is very powerful, if it is perceived as expressing the fact that we can map *any* quantity in the 'per 100g' nutrition table onto the 125g portion of cheesecake, by using the single, general, between-the-lines multiplier ×1.25. However, this is far from intuitive and thus, as an early example of the DNL, it caused considerable confusion - among students, teachers on the project, and ourselves. In time, working through this confusion was an enriching experience - and it roundly demonstrated that the DNL is something other than a problem-free device for solving ratio tasks. However some teachers were put off the DNL, as has occurred in other studies (e.g., Orrill and Brown, 2012).

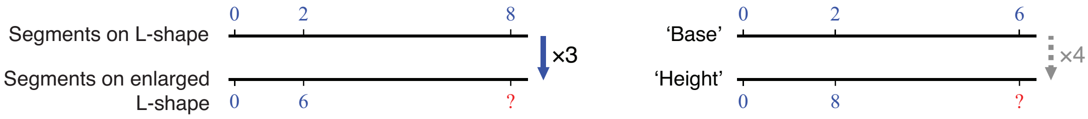
A vital context for a thorough understanding of ratio (although not featured in the *Fair Shares* booklet) is geometric enlargement. There is considerable evidence to suggest that this is a challenging context (e.g., Hart, 1981; Hodgen et al., 2012). A possible reason for this is that enlargement, especially of a curved 2-D shape, does not lend itself well to rated addition. However, in turn this suggests it might lend itself, *relatively well at least*, to using the general multiplicative relationship, which of course in this context is the scale factor.

Consider an L-shape with a curved 'base' of 2 units and a curved 'height' of 8 units and imagine it is enlarged (near right) such that the curved base is now 6 units. We can try to construct two kinds of rated addition arguments:



1. The original 'base' fits 3 times into the enlarged 'base', so the enlarged 'height' is 3×8 =24.
2. The original 'base' fits 4 times into the original 'height', so the enlarged 'height' is 4×6=24.

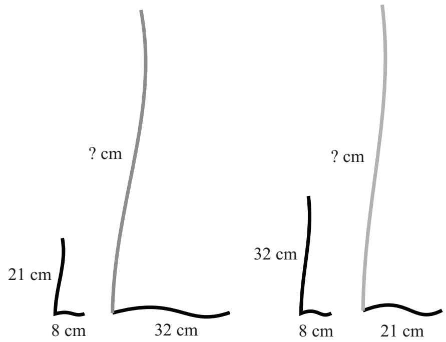
However, neither argument is entirely convincing. Because the segments are curved, the original 'base' clearly does *not* fit into the enlarged 'base' or into the original 'height' (above, far right). The bits are *different shapes*. The true relationship here is that the enlarged 'base' is *the same shape* as the original and that it is *3 times as large*. And, of course, this is a general rule that applies to the whole plane and, specifically, to any corresponding line segments on the original and enlarged shapes. As far as the two potential DNLs for this situation are concerned, this general relation is best expressed by the between-the-lines multiplier on the DNL below, left. (The DNL



below, right models the less compelling between-the-lines relation that for any scale factor the 'height' is 4 times the 'base'.)

We wrote earlier, in reference to Vergnaud's work, that a between-the-lines multiplier expresses a function relation when the lines represent different measure spaces, and that students tend to prefer scalar relations. In the present context, it can be argued that the lines represent the same measure space, so that the between-the-lines multiplier is scalar for an enlargement. Either way, we have evidence [below] that for an enlargement, students are more likely to relate elements *between* an object than *within* an object - in terms of the DNL above, left, this suggests they tend to prefer between-lines rather than along-lines relations.

[We gave parallel samples of mostly Year 8 students an *Enlarged-L* item where they were asked to find the length of the grey line in one or other of these diagrams (right), given that the two Ls were "exactly the same shape". Both items involve the relatively simple multiplier ×4. In the case of the near-right diagram, where this is a between-objects multiplier, the facility was 75% (N=73), whereas for the far-right diagram, where ×4 is a within-object multiplier, the facility was only 36% (N=74). Note also that both facilities are substantially lower than the corresponding *Spicy Soup* facilities of 91% and 51%.]



Multiplication as scaling

Young children tend to see multiplication additively, i.e. in terms of repeated addition. Even when multiplication involves non-whole numbers, it can be difficult to free oneself from an additive view: it is still possible to construe an expression like 2.3 × 3.7 as '2.3 lots of 3.7'. The area model might be helpful here (e.g, Barmby, Harries, Higgins and Suggate, 2009), though in the UK it tends to be introduced rather hastily and reduced to a rule (such as area = length × breadth) whose meaning students can quickly loose touch with. And area does not really banish an additive perspective: we can still think of the area of a 2.3 cm by 3.7 cm rectangle as being covered by 2.3 rows of 3.7 unit squares, or 3.7 columns of 2.3 unit squares.

A situation where an additive view can be more problematic is *scaling*, as in 'This pumpkin weighs 2.3 kg; that one weighs 3.7 times as much'. Here one could think of 3.7 lots of the smaller pumpkin as being equivalent to the larger pumpkin, but this is not the *same* as the larger pumpkin – it would win you no prizes ... The same thing arises in the case of geometric enlargement of the plane: an additive interpretation of the enlargement of a line segment, say, can give the correct total length, but the result is not congruent to the enlarged segment. This is particularly salient when a segment is curved, as with the L-shapes above. This suggests that geometric enlargement, despite being cognitively demanding, provides a vital context for developing the notion of multiplication as scaling.

In turn, an awareness of the notion of scaling should help students apprehend that the DNL provides models for ratio by means of between-the-lines as well as along-the-line relations and thus help students develop a more abstract, multiplicative understanding of ratio.

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From the physical classroom to the online classroom – providing tuition, revision and professional development in 16-19 education

Stephen Lee
Mathematics in Education and Industry

In recent years Mathematics in Education and Industry (MEI) has undertaken extensive work to develop techniques for utilising online classroom technologies to deliver tuition, revision and professional development in 16-19 education. Interaction using these technologies has involved tens of thousands of students and many hundreds of teachers.

In this paper a short background section on online learning will be presented, before discussion of how MEI has evolved its work to move from the 'physical' classroom for tuition, revision and professional development, to that of using online technology. An overview of feedback from online participants is then given. Techniques and strategies for utilising the technology are then discussed before final conclusions of the work are made.

Keywords: online, tuition, revision, CPD, 16-19.

Introduction

This discussion paper reports on how online classroom software has been used to support and, in some instances, replace the more traditional practice of teaching and learning via a 'physical' classroom. The online classroom has been used for tuition and revision of mathematics in 16-19 education within a national project, as well as for associated teacher professional development.

The work has been developed by Mathematics in Education and Industry (MEI) for their national Further Mathematics Support Programme (FMSP) project. The FMSP has existed since 2009 having followed on from the successful Further Mathematics Network that began in 2005. The primary aim of the FMSP is to give every student who could benefit from studying AS/A level Further Mathematics the opportunity to do so. The origins of the FMSP can be seen in Stripp (2010), though in the past few years the remit of the FMSP has continued to expand over and above those outlined for 2009. More recently Lee and Searle (2012) reported on an evaluation of the FMSP in respect of stimulating an increase in the uptake of Further Mathematics through a multifaceted approach.

In respect of the primary aim of the FMSP, data from the Joint Council for Qualifications shows that the uptake of A level Further Mathematics in England has increased from 5627 in 2005, to 13 821 in 2013. An even greater increase has been seen in AS level Further Mathematics entries, up from 5054 in 2005 to 22 601 in 2013.

What follows in this paper is a short background section on online learning that then leads into how MEI has evolved its work to move from the 'physical' classroom to that of using online technology. Feedback from online participants is then presented before techniques and strategies for utilising the technology are discussed. Final conclusions of the work are then made.

Background

Though there are numerous pieces of software that could be used to facilitate an online classroom, see Karabulut, and Correia (2008), the FMSP trialled Blackboard Collaborate's Elluminate software and subsequently took out a full license.

Elluminate is also the software used by the UK's largest distance-learning provider, the Open University. Mestel, Williams, Lowe and Arrowsmith (2011) detailed the main features of Elluminate, which include, but are not limited to:

> In addition to audio communication, there is a whiteboard with a wide selection of writing and drawing tools (including some very basic mathematical symbols), a text-chat area, a limited selection of emoticons and ticks/crosses for feedback, polling to facilitate whole class interaction, breakout rooms for small group discussions, webtours (taking the class on an internet journey), application and desktop sharing, file exchange, webcams and a recording facility. (2011: 1).

A screenshot of the Elluminate software once open and in use can be seen in Figure I. The main features of the software are also indicated.

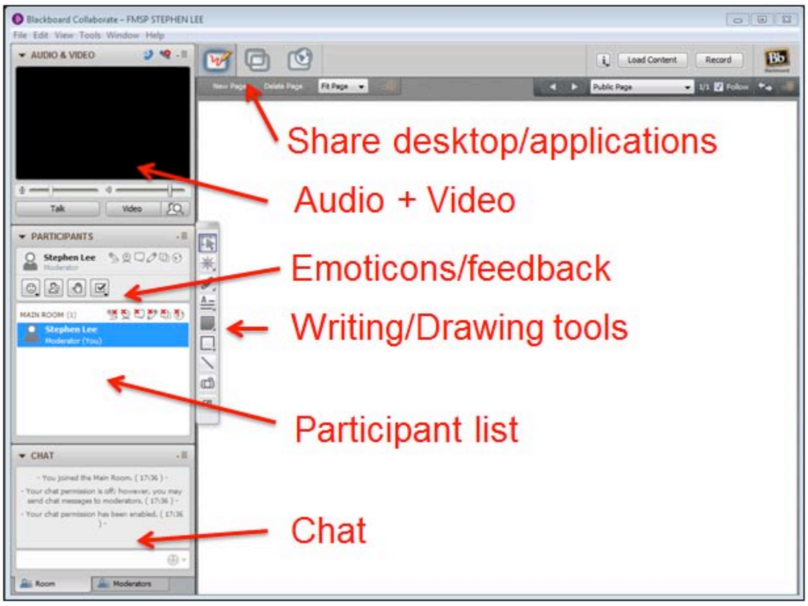

Figure I – Screenshot of Blackboard Collaborate's Elluminate software (version 12).

The 'physical classroom', where a lesson/lecture takes place both 'face-to-face' and 'live', and takes the form of "one person speaking, more or less continuously, to a group of people on a particular subject or theme", Fardon (2003: 701), has this widely accepted definition. The idea of an online classroom itself is also quite clear, but how one is used is less well defined. For example an 'e-lecture' was defined by Jadin, Gruber and Batinic (2012: 282) as "a media based lecture including (such things as) an audio or video recording, synchronised slides, table of contents and optional complementary information (e.g. external links)".

Trenholm, Alcock and Robinson (2012) undertook a comprehensive review of current research in the area of 'Mathematics lecturing in the digital age' and determined that an increasing amount of research has been done in the area in recent years. They also cite the widespread use of different terminology for this online teaching and learning: (in addition to e-lectures) "Other terms used are web-based, digital or online lecture, webcast, screencast, podcast or lecture case" (2012: 1).

The majority of this teaching/learning is occurring within Higher Education, though citations are also evident in Further Education to a lesser extent, e.g. Becta

(2005), Golden, McCrone, Walker and Rudd (2006). The work of the FMSP, specifically in respect to its tuition, lies somewhere between these two contexts – it takes place in a Further Education context and level, but the techniques are highly aligned to those practiced in Higher Education. This can be seen in subsequent sections.

From the physical classroom to the online classroom

Online technology, as described in the previous section, means that there are now more options than to simply deliver tuition, revision and professional development via a traditional 'physical' classroom. In this section detail is given of how MEI has developed its work to have online technologies complement, and in some cases replace, the more customary classroom approach.

Tuition

A key aspect of the work of the national FMSP is to be able to have the flexibility to meet the needs of a local situation. In respect of tuition of AS/A level Further Mathematics students, delivery options have been developed to include:

- Face-to-face tuition
- Live online tuition (LOT)
- A mixture of the two – Live interactive lectures for Further Mathematics (LIL FM)

The face-to-face tuition usually takes place either at a local school (possibly a student's own) or at a local university. Typically an external tutor delivers a weekly session to a small group of students. An extensive virtual learning environment of specific mathematics resources (called Integral) is made available to students and they are required to do an amount of self-study using Integral between sessions. Detail of the construction and pedagogy behind Integral was given by Lee and Browne (2011) and the precursor to Integral, the 'MEI Online Resources for Mathematics', by Button, Lee and Stripp (2008).

LOT is very similar in principle to the face-to-face tuition – students meet with an experienced tutor weekly and then use Integral to support their study – the only difference being that the tutoring takes place online, rather than in a physical classroom. Button and Lissaman (2011) and Lissaman, de Pomerai and Tripconey (2009) provided discussion of the pedagogy and practicalities of LOT.

LIL FM is a much more recent development and originates as a mechanism to also support teacher professional development. On a bi-weekly schedule, students participate in dedicated online 'lectures' one week, and have classes in their school with their own teacher on the alternate week. Materials are provided for the teacher, and students also have access to Integral. This is somewhat similar in style to what might be seen as standard university teaching, where a large class lecture is followed up with smaller problem/tutorial classes.

Revision

To aid students in their preparation for their mathematics AS/A level examinations the FMSP offers a national programme of 'physical' revision events across the country, as well as a comprehensive selection of live online revision sessions via Elluminate.

The physical days are either a half or full-day event covering a whole AS/A level module held at a university. In previous years, with over 100 such events taking place across the country attendance might vary from relatively small numbers to several hundred students, as well as a number of teachers, at each event. The online revision sessions are one off, live events taking place in the early evening for 90-120 minutes. In 2011-12 70 sessions took place attracting a handful of students for the more specialised higher modules, to many hundreds for the required Core modules.

Professional Development

It is worth acknowledging that there isn't one widely accepted absolute definition of what professional development (also known as teacher development) is (Evans, 2002). However, we adopt the general definition cited by OECD (2009: 49): "Professional development is defined as activities that develop an individual's skills, knowledge, expertise and other characteristics as a teacher."

In the same way that FMSP tuition models have developed to include wholly 'physical' and wholly 'online', to a combination of both, so too have the models for providing teacher professional development. These courses fall into one of three categories:

- Face-to-face events
- Live online professional development courses (LOPD)
- Extended 15 month professional development courses, including study days/events and online sessions, e.g. Teaching Further Mathematics (TFM), Teaching Advanced Mathematics (TAM)

The face-to-face courses are usually one-day in length, held at a university, or suitable school/college and aim to improve teachers' subject knowledge and pedagogy skills. The day courses centre on either AS/A level module(s) or specific areas of interest, such as ICT software/hardware to aid mathematical teaching and learning. Consideration of integrating ICT into classroom mathematics teaching is becoming an area of greater interest recently (Tripconey, de Pomerai and Lee. 2013).

LOPD courses are either five hours (short course) or 10 hours (long course) in duration, with teachers meeting live online in Elluminate during the early evening on a weekly basis. Details of the practicalities, advantages and disadvantages of such professional development were considered by de Pomerai and Tripconey (2011).

Extended courses are 14-15 months in duration and teachers attend several 'physical' study days as well as online sessions. They receive email and online support, including having access to specific course materials in Integral.

Participation and feedback from the online classrooms

With the emergence of online classrooms and their integration into several areas of MEI's work, it is useful to gain a sense of just how many people are interacting with this technology. This section will give indication of the number of participants via MEI's work, as well as an outline of feedback received.

Participation

As a national project FMSP provision is extensive and so too is the uptake. Over 350 students a year now receive tuition wholly or in part online. There are around 3000-5000 students in total attending the live online revision events and 5000-7000 views of the recordings of the sessions each year. Over 1000 'teacher days' of professional development is undertaken, with more than 450 teachers per year now participating in dedicated online courses (LOPD), or extended courses (TAM/TFM) that incorporate online professional development.

It is standard practice to request that students and teachers complete feedback after attendance at any FMSP event. With online provision this usually entails a short online survey. It is noted that this methodology is likely to induce a smaller response rate than say a paper based survey (Nulty, 2008), but in many instances (specifically the online revision events) it is one of the only ways to feasibly attempt to obtain feedback.

Feedback

At the time of writing comprehensive analysis is still being undertaken for events that occurred during the 2012-13 academic year, preliminary exploration indicates unanimously positive feedback in line with that seen in previous years.

Of the approximately 1800 feedback responses from the live online revision events between January 2011 to Summer 2013, 92.5% rated 'Excellent' or 'Good' for 'Quality of delivery' and 91.8% rated 'Excellent' or 'Good' for the 'Online classroom as a platform for delivering the session'. Relatively few found any aspects to be poor, with 92.6% of respondents indicating 'Yes' to 'Do you feel better prepared for your examination?' and 96.8% indicating 'Yes' to 'Would you recommend these sessions to other students?'.

Students who received tuition wholly or partly online were asked to complete an online survey about their experiences. The Likert scale of 'Poor/Adequate/Good/Excellent' produced an average response equivalent of between 'Good' and 'Excellent' for the questions with that scale.

A relatively high response rate to the online survey has been seen from teachers on the LOPD courses, with almost 60% of the approximately 200 teachers who undertook a LOPD course in 2012/13 having replied to date. Furthermore, 97.4% rated 'Excellent' or 'Good' the 'Online classroom as a platform for delivering the session', which is higher than the equivalent question for revision events detailed earlier.

Within the online survey, in addition to questions with answers placed on a Likert scale, there are some open questions about the tuition/revision/professional development. Many participants completed these and although it is impossible to reflect upon all of them in this paper, there are many positives identified by respondents, such as:

> Tutor makes the topic engaging and fun to learn. Presentations are detailed, rigorous and well-planned. Elluminate is an excellent platform upon which distance teaching/learning is done. (LOT)
>
> I find the online sessions really helpful as you can go over the recordings as many times as you want to make your understanding clear. (Revision)
>
> Spreading the content out over time gives you time to reflect on what they have taught you and put it into practice. At the same time you can use that experience and speak to the instructors about how it went. (LOPD)
>
> This format worked well for us allowing Further Mathematics to be introduced into college. (LIL FM)
>
> All the benefits of a CPD course without the hassle of missing time at school, or difficult and timewasting travel! (LOPD)

Emerging techniques and strategies when using online classrooms

In its most simple form when considering utilising online classroom software the question that should be asked is – what works best in the ‘physical’ classroom and how can this be transferred to the online classroom? Though there are some inherent limitations, careful consideration of the ‘tools’ available mean practice not necessarily available offline can also be used effectively online. In respect to this it is pertinent to note that more and more technology is being used in the physical classroom in schools and universities to engage students, e.g. interactive whiteboards (Higgins, Beauchamp and Miller, 2007) and electronic voting systems (King and Robinson, 2009).

The concept of being interactive with participants who are in the online classroom to engage them in the teaching and learning is a key technique that was commented upon in the feedback. Some techniques and strategies to enable this to happen within Elluminate are:

- Seek feedback/confirmation of understanding throughout a session via the emoticons/chatbox
- Run quick quizzes via polling, which is particularly helpful with large groups
- Ask participants to annotate the whiteboard, including using different colours and adding to pre-created templates
- Share applications that are used outside the online environment, e.g. Geogebra, Mathematica, Matlab
- Give participants time to think and to do mathematics. For example, use breakout rooms to create smaller groups or areas for individuals to work in
- Get participants to use the audio option, i.e. ask them to verbalise the mathematics and the tutor writes it down
- Put hyperlinks into the chatbox, (which makes them active links) to direct participants to other resources etc.
- Record sessions, so that they can be replayed by participants at their own pace

The choice of using one or several of these techniques should be determined in respect of the specific participants and the objective of each session, be it a small group of students receiving tuition, hundreds of students in a revision session or a group of teachers receiving professional development. This consideration is in agreement with Fuller’s “customised delivery of content” (2009: 88), which referred to both the content of the session and the means by which the interaction was initiated.

It is the case however that using online software can cause some issues for both those initiating the session(s) and those attending, these can include:

- Internet/firewall connection issues
- Sound issues
- Off-task communication, i.e. using instant messaging/chat
- Time to familiarise with previously unused software

Conclusion

As familiarity and expertise in online classroom software has increased within the FMSP over recent years, new and interesting teaching and learning strategies have been created, trailed and implemented. Such developments have enabled tuition, revision and professional development of mathematics in 16-19 education to be undertaken by students and teachers across the country, in situations where they wouldn't necessarily have been able to benefit. Even though there are some technical issues and limiting factors with online classroom software, the usefulness of the technology to facilitate teaching and learning opportunities much outweigh these in respect of the work of the FMSP.

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What makes a claim an acceptable mathematical argument in the secondary classroom? A preliminary analysis of teachers' warrants in the context of an Algebra Task

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The study we report builds on previous research conducted by Nardi, Biza and colleagues, which examined mathematics teachers' considerations of what makes a claim an acceptable mathematical argument in the secondary classroom. We identify teachers' considerations in their written responses to tasks and then in semi-structured interviews that probe these written responses. Here we present data from six teachers and one such task, a (GCSE-level) Algebra Task. The tasks we invite the teachers to engage with, of which the Algebra Task is one, are structured as follows: a mathematical problem that students are likely to encounter in typical secondary mathematics lessons; fictional student responses to the problem (grounded on student responses found by relevant research as typical); and, an invitation to teachers to solve the problem, consider the purposes of its use in the lesson, reflect on the student responses and describe the feedback they would provide to the students. So far, we have proposed a theoretical tool for analysing mathematics teachers' warrants for the preferences they express in their written responses and the interviews. The tool is based on our adaptation of Toulmin's model of argumentation in which we classify teachers' warrants according to pedagogical, epistemological and institutional considerations.

Keywords: teacher argumentation; warrants; Algebra; proof; Toulmin's model for argumentation.

Studying the practical rationality of mathematics teaching with Toulmin's model

The study we draw on here aims to refine typologies that describe teachers' knowledge and beliefs – such as Shulman's (1986; 1987) constructs of pedagogical content knowledge and Hill and Ball's (2004) mathematical knowledge for teaching – and explore how these knowledge and beliefs transform into pedagogical practice. Our aims resonate with those in works – such as Herbst and colleagues' (Herbst and Chazan, 2003; Miyakawa and Herbst, 2007) – that address the complex set of considerations that teachers take into account when they determine their actions. Of particular relevance to our analyses is what Herbst and Chazan (2007) call the practical rationality of teaching, "a network of dispositions activated in specific situations" (p. 13). To explore this, in the study we draw on here, we invite teachers' comments on classroom scenarios (Nardi, Biza and Zachariades, 2012; Biza, Nardi and Zachariades, 2007) that they are likely to experience in their lessons. We invite these comments first in writing and then in interview. We then analyse the arguments that teachers put forward using an adaptation of Toulmin's model of argumentation (1958) and Freeman's (2005) refinement of parts of the Toulmin model.

In what follows we describe this adaptation, and how it came to be, and illustrate its employment in a sample of recently collected data. We conclude with a brief discussion of how these recent analyses fit with other works and how they relate to where our subsequent research in this area is heading.

A classification of warrants in the arguments of mathematics teachers

Toulmin's (1958) model describes the structure and semantic content of an informal argument in terms of six basic types of statement, each of which plays a particular role but which are not necessarily all present in the utterance of an argument: the conclusion (C) is the statement of which the arguer wishes to convince; the data (D) are the foundations on which the argument is based; the warrant (W) is what justifies the connection between data and conclusion and is supported by the backing (B), which presents further evidence and justifications; the modal qualifier (Q) expresses degrees of confidence; and, finally, the rebuttal (R) consists of potential refutations of the conclusion.

Toulmin's model has been employed by researchers in mathematics education across educational levels and mainly to analyse student arguments. Most use reduced versions of Toulmin's model (CDWB or CDW) but recently researchers have argued in favour of employing the full model – see Nardi et al. (2012: 159-160) for examples of studies of both kinds. Of particular interest to us are works that elaborate the model by offering a classification of warrants, such as Freeman's (2005) and, within mathematics education, Inglis et al. (2007)'s classification of inductive, structural–intuitive and deductive warrants that underlie the *mathematical* arguments of their participants. Our analysis aims to discern, differentiate and discuss the range of influences (epistemological, pedagogical, curricular, professional and personal) on the arguments that *teachers* put forward when they elaborate the decisions they make in the course of a mathematics lesson.

Our adaptation of Freeman's (2005) classification of warrants is as follows:

- *a priori* warrant: resorting to a mathematical theorem or definition (*a priori–epistemological*) or to a pedagogical principle (*a priori–pedagogical*);
- *institutional* warrant*:* a justification of a pedagogical choice on the grounds of it being recommended or required by institutional policy, such as a national curriculum or a textbook (*institutional–curricular*) or that it reflects standard practices of the mathematics community (*institutional–epistemological*);
- *empirical* warrant: the citation of a frequent occurrence in the classroom (according to teaching experiences, *empirical–professional*) or resorting to personal learning experiences in mathematics (*empirical–personal*);
- *evaluative* warrant: a justification of a pedagogical choice on the grounds of a personally held view, value or belief.

The purpose of such an elaboration of the types of arguments teachers use is to demonstrate that the decisions teachers make do not have exclusively mathematical (epistemological) grounding. Their grounding is broader and includes a variety of other influences, most notably of a pedagogical, curricular, professional and personal nature. Acknowledging the breadth and scope of teachers' warrants implies the need to re-define our criteria for evaluating teachers' arguments in a pedagogical context and for exploring aforementioned practical rationality of teaching (Herbst and Chazan, 2007). We demonstrate this breadth and scope with samples of our preliminary analysis of recently collected data. We start with presenting the aims, methods and participants of our study.

Methodology

The six teachers who participated in this study were staff members in a single school, located in the East Midlands of England. The school was approached with an invitation to take part in the study because it had previously been involved in research projects with the University of Nottingham and Loughborough University. All members of the mathematics department were invited and the six teachers were self-selected. The school is an average-sized, mixed-gender, state-funded, secondary school serving the 14–19 age group. Students' results in GCSE examinations are higher than the national average. The proportion of students that could be described as disadvantaged – based on the number of students eligible for free school meals – was much lower than the national average. The school population is predominantly white British.

The mathematics department consisted of a newly appointed head of department, an assistant head of department, an advanced skills teacher and six mathematics teachers who were part- and full-time. Two of the teachers held senior leadership roles in the school, one of whom was the head of mathematics in the previous year and had been promoted recently.

The teachers who participated in the study were all female, and included: the new head of mathematics, Teacher P; Teacher S, an experienced teacher but also new to the school; Teacher R, a PGCE student; Teacher T, an Advanced Skills Teacher who had been teaching for five years; Teacher A, who was in her second year of teaching; and, Teacher M, the previous head of mathematics and now assistant principal who had twenty-five years' teaching experience. Each teacher was given £40 as a reward for participating in the project.

Anonymity, confidentiality and the right to withdraw were guaranteed to all participants who had to provide their consent in writing. Ethical approval for the study was awarded by the Research Ethics Committee of the School of Education and Lifelong Learning at UEA, which supported the study – see Acknowledgement.

Data-collection involved teachers completing a questionnaire in which they engaged with a Task – see next Section. The questionnaires were distributed and collected by the third author, Watson, who also carried out interviews with the teachers soon after. The interview protocol was designed to elaborate the teachers' written responses to the Task, with a particular focus on the warrants underlying the feedback they provided for each of the fictional student responses.

The interviews were audio recorded and transcribed verbatim. Initial analysis of the six datasets (each consisting of the Task script and interview transcript from each of the six teachers) was completed collectively by the team. During this first scrutiny of the data key issues and themes were identified. Each team member then carried out further independent analysis of two datasets steered by this initial identification of themes and deploying the theoretical approach developed in earlier studies by Nardi, Biza and colleagues (see Biza et al., 2012; Nardi et al., 2012). The preliminary analyses we present here are a first attempt to weave together these independent analyses.

A first observation that emerged from our initial scrutiny of the data concerned the relatively strong presence of pedagogical and institutional considerations in the teachers' justifications for what they appear to prioritise in their written and interview accounts. In the following, we sample from the six datasets in order to substantiate and elaborate this observation. First, however, we introduce the Task.

The Algebra Task

The Task that the six teachers engaged with consists of five questions (Figure 1). In question 1, teachers were asked to solve a mathematical problem from GCSE level algebra. Then, in the three parts of question 2 (2a, 2b and 2c), three fictional students' responses were offered to the same problem and teachers were asked to provide feedback to these responses. Finally, the teachers were asked to reflect on the aims of this problem (question 3); to comment on whether these fictional responses are likely to occur in their lessons (question 4); and, to offer any other comment on the Task (question 5).

Question 1
Write down your solution to the following problem.
Is the expression $x^2<x$ always true, sometimes true or never true? Justify your response.
Question 2
The teacher of a higher ability of year 10 class gave students the above question. Please provide feedback to the students who gave the following responses:

> **Question 2a**
> I tried several positive and negative figures and the outcome was never true:
> For $x= 0$, $0 < 0$ which is not true.
> For $x = 1$, $1 < 1$ which is not true.
> For $x = -1$, $1 < -1$ which is not true.
> For $x = 1.5$, $2.25 < 1.5$ which is not true.
> For $x = -1.5$, $2.25 < -1.5$ which is not true.
> So, this expression is never true.
> **Question 2b**
> This expression is *never* true because the square of a number is *always* bigger than the number.
> **Question 2c**
> If x is negative, x^2 is positive. So, $x^2>x$ is true, therefore, $x^2<x$ is not true.
> If $x = 0$, $0^2 = 0$. So, $x^2 <x$ is not true.
> If x is positive, x^2 equals x times x and this is always bigger than x.
> So, $x^2> x$ is true, therefore, $x^2<x$ is not true.
> As a result the expression is never true.

Question 3
In your view what is the aim of the problem featured?
Question 4
Are the student responses likely to be observed in your lessons? If so, which ones? If not, what responses would you expect?
Question 5
If you have any comments on the task, in relation to the main study we are intending to carry out, please write them here.

Figure 4: The Algebra Task

The idea for this Task originated in a popular activity for students on the *evaluation of validity of statements and generalisations* (Swan, 2006) in which students have to decide whether a statement is "'always', 'sometimes' or 'never' true [and] then justify their decisions with examples, counter-examples and explanations" (pp. 146-147). In resonance with this type of activities, in the Algebra Task teachers were given the algebraic expression $x^2<x$ and were asked to consider whether this expression is always, sometimes or never true. This expression is *sometimes true*: it is

true for any value of x between 0 and 1, and false for any other value of x. There is a range of approaches to justifying that the expression is *sometimes true*:

- Trial numbers for which the expression is not true (e.g. 0, 1, 1.5, etc.) and others in which it is true (e.g. $\frac{1}{4},\frac{1}{2}$ etc.). A trial of one number for each case is sufficient evidence.
- Make the graphs of $y=x^2$ and $y=x$ and observe that the graph of $y=x^2$ is always above the graph of $y=x$, except for the interval (0,1).
- Solve the algebraic inequality: $x^2<x$ or $x^2-x<0$ or $x(x-1)<0$ which is true only when x and x-1 have different signs. This is true only when $x>0$ and $x<1$.

The expression and the fact that it is not true for all integer values of x was a deliberate choice. Students often try specific numbers to validate the truth of an expression and ignore fractions and decimal numbers, especially those which are between zero and one. This imprecise practice is reflected in the fictional response 2a. Additionally, the expression $x^2<x$ challenges a common misconception that the square of a number is always greater than the number, reflected in the response 2b and in the third step of the response 2c. Response 2c, in comparison to 2b, offers a more elaborate explanation with a distinction of cases for different numbers (i.e. negative, zero and positive). We included this type of response to examine if teachers' feedback would be affected by the format of the fictional student's response. Would teachers, for example, prefer a more apparently formal response that distinguished cases for different types of number? We sample from the six teachers' responses in order to explore what the teachers prioritised and how they justified these priorities.

Data and analysis: the six teachers' pedagogical and institutional considerations

Here we focus on the six teachers' pedagogical and institutional considerations as evident in their intended feedback to the students in their written responses to the Task and their comments in the interviews.

Regarding the teachers' ***pedagogical considerations***, the teachers appear willing to engage the students, ask probing questions that feedback directly to the students and generally move in accordance with where the students are. For example, Teacher A asks the student in her response to Question 2a: "Are there any other types of numbers that you haven't considered?." When, in the interview, she was asked to elaborate her choice, she responded:

> I try and get the students to work out for themselves what to do, kind of leave them an open question so I've asked if there are any other types of numbers that you haven't considered. Now, I'm not sure if they would get that from that, but I couldn't think of another way. […] so I don't want to give them the answer because at first I was going to write "have you thought about numbers between zero and one?" But then that's just giving them the answer and I want them to think more deeply about...

Teacher P also acknowledges that the teacher needs to start from the student's point of view: "I think it's depending on the…because of the answer and how it is and how the students approach the problem, it's getting them to basically sort of move on from where they are." We see Teacher A's intention to avoid 'just giving them the answer' and prioritising the opportunity to 'think more deeply' – as well as Teacher P's willingness to 'move on from where they are'– as *a priori pedagogical* warrants for their prioritising of certain approaches to the mathematical problem in question.

In the interview Teacher S starts off too with the statement of a pedagogical principle, the value of assessment for learning:

> When you're giving feedback obviously you're not just going to say, "no, that's wrong," so this is what you should have done. So if you are giving feedback and using assessment for learning then you are going to ask questions that hopefully will make them think further.

In response to Question 2b she writes: "Could you please explain how you know this? Could you give me some examples to justify this statement?" Invited to consider her response in this question she proclaims that her plea for more explanation is, "Because I've got nothing to work on there, I can guess at what is going on in their head but I have nothing, almost nothing to feedback with." Transparency of the student's mathematical thinking (for pedagogical reasons) is her priority, "So at least I can start seeing what their thinking [is] and then move them on a bit further."

A sharp contrast between the teachers' pedagogical and epistemological priorities emerges on the occasion of discussing what Teacher S describes as the illustrative power of examples (similar to Teacher R who wants to help the students see how different examples can lead to different answers). It seems that identifying and demonstrating a range of examples is crucial in some teachers' feedback (pedagogical priority), even though (for mathematical reasons) they recommend coverage of all real numbers in their responses to Questions 2a and 2c.

However, this explicit prioritising of pedagogy is not the case for all participants. Teacher T, for example, – who, incidentally, had misunderstood the mathematical problem in the Task and realised the misunderstanding towards the end of the interview – writes in response to question 1: "never true because squaring a number is always bigger than the value of *x*." Under the influence of this common misconception she writes in response to question 2a: "Well done for trying positive and negative values of *x* and also decimals. Could you provide any further reason for your answer? Any proofs?" Algebraic misconception notwithstanding, Teacher T, overall, wants to see the students involved in particular mathematical practices such as exploring and generalising:

> I would want them to think about it, then generalise from that, like we were just saying generalise whenever you square a number it always ends up bigger, I want them to put a bit more meat into it instead of just trying a few examples for it.

In question 2b she requests a trial of specific cases of numbers: "Well done for providing a reason for your answer would you write this as proof. But is this true for positive and negative values of *x*? What about decimals?" We note her use of words such as 'reason', 'proof' and 'generalise'. But we also note that she wishes to see more exploration of the problem and is interested in encouraging students to investigate the problem through specific types of numbers in order to gain access to their way of thinking (reverting in this way to a prioritising of pedagogy, which is similar to Teacher A, Teacher R and Teacher S):

> No you see I don't know whether they have thought about it in their head and that's where they've come from, we as teachers always need to see things written down don't we but they might have thought about things in their head.

Regarding the teachers' ***institutional considerations***, we noted that constructs such as student Ability Level and Strength/Target policy – both terms reflecting a use of language that is common in curriculum and policy documents – are part of the discourse of the teachers quite strongly (explicitly or implicitly) and appear to influence feedback to the students.

Teacher M, the most senior participant in the study, often responds in ways that demonstrate solid compliance with the school's marking policy: "identifying the 'strength' in a student response and making the 'target' the student should aim for explicit." She tailors her pedagogical approach to this frame:

> […] well I would I do now follow the school marking policy, which is s stands for strength, t stands for target, and so I'm just in that mind-set now when I'm marking work and so try to pick out the strength whatever that might be and then obviously for me the more interesting thing is the target in what direction to, you know, lead them into as a responsible piece of homework. This particular one yeah it's sometimes difficult to pick out the strengths you know, like I've just already said we're more interested in the targets about you know about using that information to stretch them and to push them further forward and so.

Analogously, the level of guidance Teacher A aims to offer students is tailored to their perceived ability:

> …depends on your group doesn't it? I think like with your middle ability and lower ability they might need a bit more direction. But I wouldn't…I don't think I would feel confident just going in and giving them something to do straight away. I like to let them feel uncomfortable because I think that's good and then when they start to kind of lose the will to carry on it's kind of then step in and…

Concluding remarks

Across the six datasets a strong observation is that the teacher responses are more intensely characterised by a pedagogical discourse, and less a mathematical one. For example, although some of the teachers mentioned 'proof', we have little evidence of what they actually mean by this term. Some stress the importance of 'justification' in students' responses but it is not clear whether they mean proof / verification of the statement, being convinced of the truth of the statement or some explanation accompanying the answer to the mathematical problem.

Furthermore, we noted that the teachers appraise the fictional student response in question 2c because it is written in "quite a mathematical way" (Teacher A); or, it is a "more general statement" (Teacher P); or, because students "managed to express what they're thinking in a more clear way […] they put it in logical order" (Teacher R); or, they made a "really brave attempt [to] use inequality signs" (Teacher S). These were occasions where a glimpse into their epistemological priorities was possible.

Generally, however, the teachers' pedagogical aims (and their *a priori pedagogical* warrants thereof) are explicit and detailed in their responses, while their mathematical aims (and *a priori epistemological* warrants thereof) are not always clearly discernible. Even when teachers speak about the trial of *other numbers* in questions 2a and 2c – Teacher S, for example, seems to prioritise rigour when she writes "to prove it for all possibilities, not just a selection" – they do not cite a fully-fledged, mathematically acceptable process for *checking* all numbers. It seems that, for most, what is missing is simply a trial of a number between 0 and 1. And, in most cases, priorities ultimately revert to pedagogy: when, for example, Teacher R expresses her willingness to "link back" to "concrete mathematics", her previously stated epistemological priorities (for the need for clarity and transparency) emerge as rather dimmer than her clear and explicit pedagogical priorities.

The claim that emerges from this preliminary analysis of the six datasets and which we put forward in this paper is as follows: explicitly, in at least four of the six datasets, the teachers propose the use of examples in question 2b, even though they recommend coverage of all real numbers in questions 2a and 2c. There is a pattern in the teachers' stated pedagogical priority that relies on this perceived power of exemplification. Teacher P says that she does this because she wants to start from the point students are. Teacher A says that she will start with examples and then she will ask the students to try other numbers, if they still believe that the expression is never true. Teacher R wants to help the students see how different examples give different answers and Teacher S talks about the illustrative power of examples.

This pedagogical prioritising – and the *strength and explicitness of the a priori pedagogical warrants* on which it stands – is in some contrast with the *less explicit, briefer and less strongly warranted epistemological prioritising.* We credit the classification of warrants that underpins the analysis of these data with allowing us this type of insight and we aim to expand and deepen this insight further in subsequent phases of our analysis.

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Lesson study and Project Maths: A Professional Development Intervention for Mathematics Teachers Engaging in a New Curriculum

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Since 2010 there has been a phased introduction of a new post-primary mathematics curriculum in Ireland entitled ‘Project Maths’. This new curriculum places a greater emphasis on problem solving and on an investigative approach for students. This implies not only changes in the curriculum content, but also changes to teaching and learning approaches within the classroom. This research aims to provide teachers with a school-based professional development structure through which they can engage with the curriculum and attempt new teaching and learning strategies. This structure involves mathematics teachers engaging in lesson study as a professional development intervention and is investigated in two schools (phase 1 and phase 2 of Project Maths). Teachers engage in lesson study cycles repeated throughout the academic year and the research questions how effective an approach this may be in encouraging teachers to engage with and implement a new centralised mathematics curriculum. The research also investigates how effective an approach this may be in developing teachers’ pedagogical practices. In this paper, initial findings will be discussed from teacher research meetings and interviews.

Keywords: curriculum reform; lesson study; Project Maths; Ireland

Introduction

This research aims to employ a curricular framework for a community of mathematics teachers engaged in lesson study during a time of curriculum change (Charalambos and Philippou, 2010; Fernandez, Cannon, and Chokshi, 2003; Grossman, Wineburg and Woolworth, 2000; Lewis, 2006; Lewis, Perry and Hurd, 2009). In this research a community of mathematics teachers are engaged in the practice of lesson study, where they collaborate in an iterative cycle of planning, teaching, observing and reflecting on research lessons. The objective of this research is to evaluate this model as a sustainable, school-based professional development for mathematics teachers which may enhance their practices and thus positively impact student engagement and learning within the classroom. This paper will question how effective an approach this model may be in 1) cultivating the teaching and learning approaches envisaged by this new curriculum and in 2) engaging teachers more with the content and learning outcomes included therein. This study contributes to the literature by examining how lesson study can contribute positively to the implementation of a new mathematics curriculum and in establishing how lesson study may be implemented and sustained within an Irish post-primary setting.

Project Maths

In Ireland in 2010, the national phased roll-out of a new centralised mathematics post-primary curriculum, entitled Project Maths, began. Project Maths was designed to change not just what students learn about mathematics, but how they learn and how they are assessed (Jeffes et al., 2012) and represents a philosophical shift in Irish post-primary classrooms from a highly didactic approach with relatively little emphasis on problem solving (Lyons, Lynch, Close, Sheerin and Boland, 2003; Oldham, 2001) towards a dialogic, investigative, problem-focused approach to teaching and learning mathematics. The curriculum is based on a discovery learning approach where learners' active participation and sense-making is an important basis in designing the teaching and learning approaches experienced within the classroom (Lynch, 2011).

Implementing a new curriculum

Effective curriculum reform requires a change at the individual teacher level (Harris, 2003; Hopkins and Reynolds, 2001; Wallace and Priestley, 2011). In addition, the knowledge that teachers already hold, both content and pedagogical, will ultimately determine the shape and direction of the experienced curriculum (Van Driel, Bulte and Verloop, 2007; Van Driel and Verloop, 2002). However, when teachers are encouraged to modify curriculum materials supplied to them, they are more likely to implement them within their own classroom (Hanley and Torrance, 2011). It has also been found that the most significant learning for teachers occurs during teachers' processes of enacting and observing curriculum in the classroom (Remillard and Bryans, 2004). It is thus important in attempting to translate the intended curriculum to an enacted one that teachers are afforded opportunities to engage meaningfully with the curriculum, with curriculum materials, and to observe curriculum approaches being implemented.

Professional development

In order to change a teaching approach that may impact on teachers' beliefs, there is a need to move from the more traditional form of professional development to one that provides opportunity for reflection on practice (Fetters et al., 2002). Research increasingly points to the importance of considering the social dimension of learning for teachers. When teachers work collaboratively to create resources, it fosters a sense of ownership and leads to changes to classroom practice (Hindin, Morocco, Mott and Aguilar, 2007; Priestley, Miller, Barret and Wallace, 2011; Voogt et al., 2011) and teachers often use knowledge of their colleagues' teaching strategies to initiate changes in their own practices (Meirink, Meijer, Verloop and Bergen, 2009). Experimentation and reflection within the classroom is efficacious for lasting change (Remillard and Bryans, 2004; Wallace and Priestley, 2011). Teachers' content knowledge and pedagogical content knowledge (Shulman, 1986, 1987) affect their practices and this knowledge should be challenged and enhanced through teacher learning. In this research, it is hoped to implement a model of professional development that includes these opportunities noted above.

Lesson study

Lesson study is a teacher-oriented and teacher-directed practice where members of a group determine a particular goal for their teaching, observe and examine their practice through planning and conducting lessons (Fernandez, 2002; Lewis, Perry and Murata, 2006). The lesson study model encompasses many factors necessary for successful curriculum reform or policy initiatives: teachers collaborate with their colleagues; have opportunity to see curriculum enacted; are encouraged to modify curriculum materials; have opportunity to observe student engagement in a lesson;

and explicitly reflect on classroom practice. It is important to note that it is not the 'product' of the research lesson or lesson plan which is of import, but rather the process of teachers collaborating and conversing with one another on curriculum and pedagogy that is of worth.

Methodology

The investigation is conducted as a case study in two sites (Merriam, 2009; Yin, 2009) with two schools involved at different phases of the national curriculum roll-out: Crannog school where the mathematics department have been teaching the Project Maths curriculum since 2008; and Doone school where the new curriculum is currently introduced on a phased basis to differing year groups. Participating teachers taught students in all year-groups of varying abilities.

The research was undertaken during the 2012-2013 school year and data was generated during teacher meetings in both sites; through individual semi-structured interviews; and utilising field notes from teacher meetings. The researcher was present as a facilitator during the lesson study cycles and entered the research as a former mathematics teacher in a phase 1 school. All interviews and teacher meetings were transcribed, reviewed for correction and an initial coding system emerged relating to curriculum themes. All references to participating teachers are pseudonyms and initial findings are discussed below.

Framework of analysis

The revised mathematics syllabus encourages teaching and learning approaches which form a frame of analysis for this research. These approaches may be summarised as follows:

1. A greater focus on learners' relational understanding (Skemp, 1976) of mathematical concepts, building from the concrete to the abstract and from the informal to the formal (Lynch, 2011).
2. An approach to teaching and learning which gives prominence to developing learners' skills in communicating their mathematical understanding with others with teachers as facilitators of discussions (NCCA, 2008, 2012a, 2012b).
3. An approach to teaching and learning mathematics which encourages an investigative, problem-focused approach with emphasis on application in real-life settings and contexts where students become active participants in developing their mathematical knowledge and skills (Cosgrove, Perkins, Shiel, Fish and McGuinness, 2012; Lynch, 2011).
4. A move for teachers to use supplementary resources instead of a traditional over-reliance on textbooks as a curriculum source (Cosgrove et al., 2012).

Data and findings

Initial findings from the data suggest that participation in lesson study impacted on teachers' content knowledge and pedagogical content knowledge. The majority of teachers participating in this research also expressed a change in their thinking when planning research lessons with an increased focus on how students would express their ideas, engage with, and interact during the lesson but also, significantly, when planning their own lessons. While teachers did feel that the reform was being imposed upon them, in these schools teachers were enacting the new curriculum to the best of their abilities and were eager to engage in lesson study as a model which could assist them in engaging with more of the Project Maths approaches and resources. Findings within the frame of the new curriculum will now be discussed with explicit reference to how teachers changed their teaching and learning approaches and engaged with the curriculum.

Relational understanding

In Crannog, teachers explicitly referenced their wish that students understand why they were doing an activity as opposed to just 'how' to do that activity. In planning research lesson 2 in Crannog, Fiona and Peter discuss the lesson objective:

> **Fiona:** Are we going to lead them into just multiplying out brackets or are they never going to see that? Are they always going to see it by an area model?
>
> **Dave:** No, sorry, I would have thought they need to see this but I think this probably helps, just as a concrete tangible way of getting started or seeing *why* this result works, *why* this result makes sense, rather than just learning how to multiply stuff out, following procedure and not understanding….

As the lesson study cycles continued in Doone, their objectives also became more focused on students' understanding a concept before utilising an associated skill.

Teacher as a facilitator of learning

In Crannog, Eileen describes how her approach to teaching a topic has changed from one where she readily gave the answer to students, to one where she is challenging the students to do more while she acts as a facilitator of their questions. She acknowledges that her participation in the lesson study group has given her the confidence to allow students to discover an approach for themselves and in this following quote from her second interview, she describes employing a new way of factorising quadratics:

> **Eileen:** I would say I am a bit more willing to put them in pairs, in groups now just to see how it goes, just let them at it and even, I suppose, it was factorising this week - it's not saying much and seeing what they come up with themselves. I would say maybe this time last year I would have been like "do this".

Students working in groups

The introduction of group work into more classrooms specifically addresses the curriculum aim of encouraging students' communication of their mathematical understanding. While Crannog had experienced the new curriculum for three years prior to Doone, and while the teachers seemed to agree with the teaching and learning approaches in the curriculum, very few of the teachers had changed their practices to encourage students working in groups. By the end of the year, teachers in both schools had experimented with utilising group work and some teachers changed their classroom environments to permanently seat students in groups. Teachers were also allowing more time during the lessons for students to work with each other.

At the beginning of the year Owen in Doone was adamant that he did not like his students working in groups. However, after two lesson study cycles, one in which he taught the research lesson, his negative opinion of group work appears to have changed.

> **Owen:** I just think I'm more open-minded to the fact that they will learn. And the fact that I've seen it in practice and been able to be an observer, so standing in behind someone's class and say "this is fantastic" that helped big time. Standing in Lisa's room was great. Just to even see them even when they were off-task, to see what they were talking about and how can you get them back on task in a group work situation.

Martin, a very experienced and senior teacher in Crannog, acknowledged that he didn't use group work in his teaching as much as he could, but that observing the research lessons gave him the confidence to attempt this new practice.

> **Martin:** ...from sitting in a classroom observing how groups work it is easier to know what to expect, you know, when you go to do it yourself, but you observe how students learn...I suppose in terms of planning future lessons, it does help to maybe get sort of the timing right, the timing I mean of, if it's a lesson discovery, when to give out information, maybe also how to maybe get the kids interested... So I think in terms of timing of the lesson and in terms of expectations of feedback and that sort of thing…what I saw in those lessons is where the teacher couldn't see all groups at once, therefore there's a necessity to move around groups fairly frequently and that's a hard skill, do you know. Not to get bogged down in one group.

A focus on problem solving

While it was not an explicit goal of either of the two lesson study communities, both groups of teachers focused on encouraging students in developing problem solving skills. In Crannog, teachers regularly tried to contextualise problems to make them interesting to students and attempted to direct students as little as possible in finding a solution to a problem. In Doone, a clear focus on problem solving emerged in planning lessons 3 and 4.

> **Lisa:** ...I think again we need to stress the problem-solving technique, this, the context, what's important here, can you identify the right angled triangle and what information on the paper is going to help, what are we being asked to do, what maths do we know that'll help us do it?

In this school, teachers began to provide specific, relevant scaffolding for students' questions and developed a resource for all students to use in solving problems.

Textbook as a resource and not a curriculum source

Teachers felt that they had accessed the syllabus more due to lesson study and were more familiar with it from having to directly reference it when planning the research lesson. Teachers were also using the new curriculum as a primary teaching resource for planning lessons instead of referencing the textbook.

> **Michael:** I refer to the syllabus a whole lot more now.
>
> **Researcher**: Before you wouldn't have had as much?
>
> **Michael:** Maybe, maybe not. It depends on how I felt you know? I've always been conscious of the fact that there is a syllabus there but I find I always go to it, but now I might go to it and say okay this is what's expected, then I will think how would I go about it, you know what I mean? And then I might go and talk to somebody else and say this is what I was thinking, what did you do for this? Or how were you going to do?

While this research does not assess student learning or changes to student engagement in mathematics lessons due to these research lessons, Lisa, in Doone school, acknowledged in their final research meeting that lesson study benefitted both teachers and students:

> **Lisa**: Yeah. And it benefited…our audience, which is our kids. Like...we got a benefit but the kids are the ones who benefit from this.

Conclusion & Discussion

In this research, I claim that teachers modified their teaching practices to more reflect the Project Maths curriculum as a result of participating in lesson study. These changes to classroom practice supported the introduction of the new curriculum with implications that would, hopefully, extend beyond the participants' classrooms and also impact on student learning. While Crannog had longer experience of the new curriculum and had embraced the philosophies behind the changed approaches, teachers had not enacted these within their classrooms and required time to experiment with resources and approaches which had been introduced to them as part of Project Maths. In Doone, teachers were more willing to engage students in discussion and to scaffold students' learning during problem solving more as they engaged further in lesson study.

This study was limited to teachers in two schools who voluntarily participated in this research and were thus active in wanting to engage in the new Project Maths curriculum. As such, the findings may represent teachers who are already engaged in and aware of curriculum changes. However, all participating teachers would recommend lesson study as a form of professional development and wished to continue the intervention in their school. Teachers did note the lack of time available to engage in in-school professional development and it is perhaps worth noting how teaching contracts are constructed within the Republic of Ireland. Further questions are also raised by the research: how was teachers' pedagogical content knowledge and mathematical content knowledge enhanced through engagement in lesson study? How did these groups of teachers develop as communities? What features of such communities are necessary in order to develop and sustain such a model of professional development? Further research on these and other questions is important for understanding the advantages and restrictions of school based professional development for mathematics teachers.

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Revisiting school mathematics: A key opportunity for learning mathematics-for-teaching

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Learning mathematics-for-teaching (MfT) involves revisiting school mathematics and learning new mathematics. The notion of *revisiting* is operationalised and exemplified within pre-service secondary mathematics teacher education, specifically a financial mathematics course. A framework for MfT was developed consisting of nine interrelated aspects of teachers' mathematical knowledge, and provided an analytic tool for exploring opportunities to learn MfT within the course. These opportunities are exemplified through a revisiting task where pre-service teachers (PSTs) were required to make sense of learners' responses to a compound growth task. The learners' responses provided a springboard for learning other aspects of MfT of compound growth such as essential features, modelling and applications, and knowledge of context. The decompressed nature of the learners' responses opened opportunities for PSTs to reconsider their own knowledge of compound growth, of the mathematics that underpins the formula, the process to obtain the formula and thus the way in which the formula models compound growth in different contexts.

Keywords: revisiting, mathematics-for-teaching, teacher knowledge, compound interest

Introduction

There are compelling arguments that revisiting school mathematics is an important component of learning mathematics-for-teaching (MfT) (e.g. Cooney and Wiegel, 2003; Stacey, 2008; Zazkis, 2011). However, existing literature gives few suggestions as to how this revisiting might be done. Moreover, the calls for revisiting school mathematics tend to assume pre-service teachers (PSTs) had adequate opportunity to learn the mathematical content at school, and now need to deepen and/or "top up" their knowledge with particular attention to conceptual aspects. In South Africa many PSTs have not had the kinds of opportunities to learn mathematics at school that might be assumed in other parts of the world. So for many students revisiting school mathematics is much more than "topping up". In the most extreme cases, it may involve learning for the first time the mathematics they will teach in schools. In this paper we focus on PSTs who are preparing to teach Grades 8-12 (14-18 year olds).

Our paper is drawn from a larger study of teachers' mathematical knowledge for teaching involving 42 PSTs registered for a course on financial mathematics specifically designed for teachers, and taught by the first author. The construct of revisiting emerged through the analysis of the data, and thus reflected the interplay between the theoretical field and the empirical setting (Brown and Dowling, 1998). In this paper we consider revisiting school mathematics as a particular aspect of learning MfT. We elaborate the notion of revisiting and focus on the differences between revisiting school mathematics and learning school mathematics for the first time. The specific instances discussed below emerge from an analysis of analytic narrative vignettes (Erickson, 1986) constructed from classroom episodes in the course, and from PSTs' journal entries. We begin with a brief discussion of a framework for MfT.

A framework for Mathematics-for-teaching

In the larger study, a framework was developed to elaborate the notion of mathematics-for-teaching, drawing together elements of several existing frameworks and typologies for teachers' mathematical knowledge for teaching (e.g. Ball, Bass and Hill, 2004; Ball, Thames and Phelps, 2008; Even, 1990; Even, 1993; Ferrini-Mundy et al., 2006; Huillet, 2007; Kazima and Adler, 2006; Shulman, 1986, 1987; Watson, 2008). The framework consists of nine interrelated aspects of teachers' mathematical knowledge all of which contain elements of mathematics and teaching, but which can be broadly grouped into three clusters:

Aspects that are mainly mathematical: essential features, relationship to other mathematics, mathematical practices, modelling and applications;

Aspects that are mainly pedagogical: basic repertoire of key tasks and examples, different teaching sequences and approaches, explanations, learners' conceptions; and

Contextual knowledge of finance: financial concepts and conventions, socio-economic issues and financial literacy.

This framework provided an analytic tool for exploring opportunities to learn MfT in the course – both school mathematics such as compound interest, and new mathematics such as annuities.

Elaborating the notion of *revisiting*

We define *revisiting* as the re-learning of known mathematical content for the ultimate purpose of teaching. Revisiting assumes we cannot learn everything about an idea at once, but it is not simply repetition for mastery. The purpose of revisiting is to increase one's mathematical knowledge through connections that deepen and broaden knowledge of the concept, paying attention to mathematical aspects that are important for teaching. While such knowledge may be useful in many professions, it is essential for teaching. Zazkis (2011) explored the similarities and differences between learning and re-learning (or revisiting) mathematical content in the context of a mathematics course for pre-service elementary teachers. She argues that re-learning mathematics includes (1) reconstructing and restructuring; (2) shifting from operational to structural conceptions (Sfard, 1991), (3) expanding PSTs' concept image (Vinner and Tall, 1981) by extending the range of examples that PSTs encounter, and (4) connecting a known mathematical concept to a larger class of mathematical entities which may require a redefinition of the concept.

Four aspects of revisiting

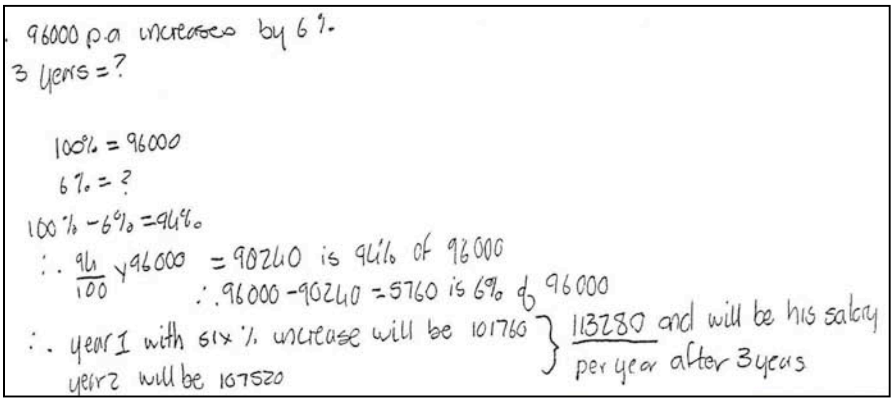
An important component of the larger study was to describe how revisiting was enacted in the course, and then to identify how it might differ from learning a piece of mathematics for the first time. By studying several classroom episodes of revisiting through the lens of the MfT framework, and identifying commonalities across the sessions, four inter-related aspects were identified: content, goals/purposes, task and activity, and resources. These aspects are illustrated with reference to a compound growth task that PSTs worked on in one of the sessions on school mathematics. However, first it is important to distinguish the original (school level) learner task from the revisiting task for PSTs. The learner task was a test question, typical of Grade 10 level:

> "A computer operator earns R96 000 a year. Her salary increases by 6% per year. What will her salary be after 3 years?"

The solution in the memo indicated learners were expected to apply the formula for compound growth and to substitute the given values. However, a closer reading of the question suggests that the wording is ambiguous and therefore open to different interpretations of the timeframes involved. For example, it is not clear whether the increase should be applied at the beginning or the end of the first year.

The revisiting task contained the test question (without memo) and responses of four learners. PSTs were required to analyse the learners' responses, to identify errors and to suggest ways of helping one particular learner. We now return to the four aspects of revisiting:

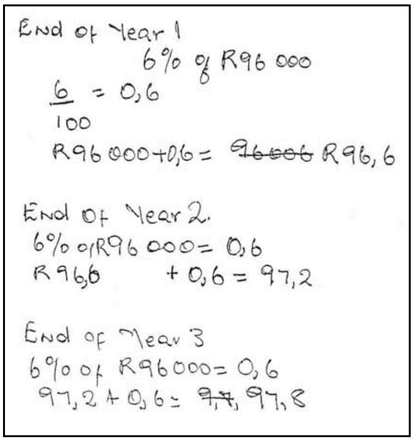
Content – Revisiting a piece of mathematics assumes the mathematical content has been encountered previously, that the PST has an overview of the content and is familiar with the terminology, notation and techniques, yet may display some typical misconceptions. In this task PSTs were familiar with these aspects of compound interest/growth. The learner responses had been carefully chosen – none of the learners had used the compound growth formula, and some responses did not involve compound growth. Furthermore the logic and strategies of some learners were not immediately obvious, and their responses included incorrect use of symbols and notation. For example, Learner 1 (figure 1) determined that 6% of 96 000 is R5 760 and added this to R96 000, giving R101 760 for the salary for year 1. The learner then added R5 760 twice more getting R107 520 for year 2 and R113 280 for year 3. Thus the learner modelled the increase at the beginning of the first year, and a simple growth scenario for the following two years since each salary-increase was calculated on the base amount of R96 000. This does not reflect the typical method of calculating salary-increases in the workplace. Note too the use of "100%" as a label rather than a quantity in the third line, and the inefficient strategy to determine 6% of 96 000


96000 p.a increases by 6%.
3 years = ?

100% = 96000
6% = ?
100% − 6% = 94%
∴ $\frac{94}{100} \times 96000$ = 90240 is 94% of 96000
∴ 96000 − 90240 = 5760 is 6% of 96000
∴ year 1 with six % increase will be 101760 } 113280 and will be his salary per year after 3 years
year 2 will be 107520

Figure 1. Learner 1's response

Learner 2 (figure 2) made several arithmetic errors, including a conversion of $\frac{6}{100}$ to 0.6, 'losing zeros' when R96 000 became R96, and adding percentages to amounts of money. This learner modelled a simple interest scenario by adding 0.6.


End of Year 1
6% of R96 000
6/100 = 0,6
R96 000+0,6 = ~~96006~~ R96,6

End of Year 2.
6% of R96 000 = 0,6
R96,6 + 0,6 = 97,2

End of Year 3
6% of R96000 = 0,6
97,2 + 0,6 = ~~97,7~~ 97,8

Figure 2. Learner 2's response

Thus we see how the learners' responses become the content of the revisiting task together with the original mathematics task.

Goals – The goals of revisiting are different from the initial encounter. Most obviously when the PSTs first studied the mathematics, they were learning it as school learners; now they are learning it as a requirement to become teachers, and then to teach it to others. One goal is to deepen and broaden their knowledge, for example making links between representations, and links to other aspects of mathematics. In this task they were required to make links between the compound growth formula and the multi-step approaches produced by learners. In terms of the MfT framework, the revisiting task explicitly required PSTs to engage with *learners' conceptions*. However, the deliberate selection of unanticipated learner responses was also intended to challenge students' own knowledge of compound growth and use of the formula. Therefore it was anticipated that PSTs would also have to engage with *essential features* and *modelling and applications.* For example, they may need to check whether the formula gives the annual salary earned during the third year or the starting salary for the fourth year.

Task and activity – We draw on the well-known distinction that a *task* is set by the teacher (or teacher educator), while *activity* refers to what students do in response to the task (Christiansen and Walther, 1986). Thus the task is a key element in framing the PST's mathematical activity. When revisiting mathematics, the task should require more of PSTs than it would of school learners. In this case the task involved dealing with learners' responses that in turn led to a deeper analysis of the question itself, and the mathematics compressed in the formula.

Resources – Revisiting involves the use of new or additional resources (tools, artefacts and knowledge). The learners' responses may be considered as resources that are brought to bear on the learner task. The PSTs introduced T_n notation as an additional resource to resolve a concern about timeframes. When they were first introduced to compound interest in Grade 10, they would not yet have been aware of this notation. Furthermore, some PSTs were not familiar with the details of annual salary-increase, and therefore were unable to draw on this assumed everyday knowledge as a resource for interpreting the learner task and the learners' responses.

What opportunities emerge for learning MfT of compound interest?

While it would be expected that the inclusion of learners' work would provide opportunity for PSTs to deal with learners' conceptions, the revisiting task also provided opportunity to engage with other, more mathematical, aspects of the MfT framework, each of which is discussed below.

Learners' conceptions – PSTs had to make sense of learners' responses, which involved recognising a range of different errors, including adding a constant amount each year rather than calculating 6% on the latest salary, and adding amounts of money to percentages. After completing the task, PSTs were required to write a journal entry to reflect on how the learners' responses might impact their future teaching of simple and compound growth. Several students noted that learners had different interpretations of the question and that these should be taken seriously in teaching. They also noticed that learners may not make use of efficient strategies, and that the question could be correctly answered without use of a formula. In her journal entry, Palesa commented "formulas are just the simple ways of getting the answers". I followed up on this comment in an interview with her mid-way through the course.

> Palesa: They kind of all had different approaches to the, the question. But it was different from how would I do it because I would think about the formula. If I want the interest it's *Prt* or what, but they did it in, they put their understanding in the thing like the maths understanding like if *x* of this is how much, and how will I find it? So it was not about the formula of compound interest or the simple interest. They just used their thinking and that's how they find it …

Palesa acknowledged the need to make sense of learners' strategies. She implied that the learners had to think carefully to produce their answers and that their responses should be taken seriously. The learners' responses required the PSTs to work between compressed and decompressed forms of mathematics, a key component of MfT (Ball et al., 2004; Zazkis, 2011).

Essential features – An essential feature (Even, 1990) of working with compound growth is the need for precision in references to time. This does not necessarily mean the task should be unambiguous about timeframes but rather that students should be precise in the ways they work with time. Revisiting compound growth provides opportunity to emphasise the importance of explicit and precise references to time. For example, during class discussion Sizwe raised a concern about the timeframes in the original question.

> Sizwe: … eish, they say, this, this computer operator earns ninety-six thousand per year, so what I don't understand is if you start counting from this year, two thousand and eight (2008) to two thousand and nine (2009), that is the first year, how much is the computer operator earning in that year and after three years, what is "after three years"? What, which year is that?

Sizwe's concern appeared to have been prompted by the learners' interpretations of timeframes which in turn led him to reconsider his assumptions about the timeframes in the question statement. This led to an extended discussion about whether the salary increase should be implemented at the beginning or end of the first year and the meaning of "after three years" – was it measured from the first salary increase or from "this year" (i.e. 2008)? During this discussion students introduced T_n notation to talk about time. Sizwe's difficulties stemmed from the fact that he did not distinguish between the beginning and end of a year, and he appeared to muddle discrete points in time with the intervals between those points. He was not assisted by Jenny who explicitly linked time to the terms of a sequence when she said "so, say term one *is* year zero, then term two *is* year one, term three *is* year two and term four *is* year three". Jenny's repeated use of "is" left the timeframe ambiguous although from her other contributions it was clear that she was referring to the end of each year. This instance of interaction between PSTs reflects how they introduced a new resource (T-notation) in order to negotiate their interpretation of the original task.

Knowledge of context – In the MfT framework this aspect was referred to as *contextual knowledge of finance*. However, in the context of this task, the broader label *knowledge of context* is more appropriate. The learner task assumes knowledge of the salary context. Firstly, it assumes that a salary-increase with a constant rate of increase can be modelled by the compound growth formula. Secondly, it assumes that salary is taken to be the amount earned in a twelve-month period (as opposed to cumulative earnings over several years). The responses from both the learners and the PSTs suggest that knowledge of the context of salary-increase cannot be taken for granted. It might, however, be argued that the learner task resembles a typical text book word problem and does not require knowledge of the context of salary-increase but simply the selection of the correct formula and appropriate substitution. Thus a learner who knows the relevant mathematical concepts and the genre of word problems may complete the learner task successfully with little knowledge of the salary context. However, teachers require knowledge of the salary-increase context, and they may need to draw on this knowledge in helping learners. It seems that knowledge of the salary context becomes even more important when dealing with multi-step approaches, and when unpacking how the compound growth formula models salary-increase. This became visible through the PSTs interactions around timeframes and their interpretations of learners' responses.

Modelling and applications – The learners' and PSTs' responses confirm that the original question contained some level of ambiguity with regard to timeframes. While the single solution in the memo suggests that the ambiguity was unintended, modelling tasks are typically designed with some degree of ambiguity. This requires one to make assumptions explicit as part of the modelling process. For example, one needs to be explicit about whether the 6% increase is assumed to take place at the beginning or the end of the year (which is effectively the salary for the third year). It can be shown that the compound growth formula models (annual) salary-increase, with a fixed rate of increase, *r*, for *n* years. Thus in the formula, $A = P(1+r)^2$, the amount *A* gives the new salary at the end of the second year, where the increase has been effected twice. We can view salary in two subtly different ways. Consider a salary of R96 000. This means that R96 000 is both the amount earned by the *end* of the year *and* the amount on which each month's salary is calculated from the *beginning* of the year. Put another way, it is the "whole" that is subdivided into 12 equal monthly portions. It is thus acted on from the beginning of the year, giving a monthly salary of R8 000. Thus 96 000 is a number linked to the end of a year but also applied throughout the year. Once again when working with decompressed forms of compound growth, one is forced to consider more deeply the relationship between the mathematics and the situation that is being modelled. These aspects are not in focus when learning to apply the compound interest formula in high school.

Conclusion

We have shown how a revisiting task that includes learners' responses to a school mathematics task provides a springboard for learning several aspects of MfT of compound growth such as essential features, modelling and applications, and knowledge of context. We do not claim that the PSTs in the study learned these aspects but we suggest that the revisiting task, and how it played out in the course, *provided the potential* for PSTs to learn aspects of MfT beyond the aspect of learners' conceptions. However, it is not the use of learners' work *per se* that opens up these opportunities. It is dependent on the *kinds* of learner work that is selected. The mathematical demands of making sense of learners' inefficient, multi-step and partially correct responses are far greater than checking whether a learner has substituted correctly into a formula. It is the decompressed nature of the learners' responses that opens opportunities for PSTs to reconsider their own knowledge of compound growth, of the mathematics that underpins the formula, the process to obtain the formula and thus the way in which the formula models compound growth in different contexts. This requires them to shift between operational and structural conceptions of compound interest. Furthermore, working with carefully-selected learner responses may prompt consideration of issues that would not arise when pre-service teachers simply work though school mathematics tasks to produce answers for themselves.

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Lesson study as a Zone of Professional Development in secondary mathematics ITE: From reflection to reflection-and-imagination

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We here add to the sparse literature on the use of Lesson Study (LS) in initial teacher education (ITE), reporting how LS can mediate development of reflective practice (RP). A cohort of 50 student-teachers, all secondary mathematics postgraduates, were involved in lesson study subgroups: planning, teaching-and-observing, analysing and reflecting on observations, and re-teaching. Each study group worked on their own LS in selected schools with teachers/school based mentors that had some previous experience of lesson study. We tutors/researchers/authors observed the planning, teaching and post-lesson analyses, and one of us interviewed selected participants. Two main findings were: (i) the significance of imagination for reflective practice, here prompted by the focus on improving and 're-teaching' the lesson; and (ii) the importance of the ITE-peer group and its relations with more powerful others (mentors and tutors) to development. We conclude that LS may complement and even 'lead' the development of reflective practices of student teachers, providing a Zone of Professional Development.

Keywords: lesson study, initial teacher education, secondary mathematics, reflective practice

Introduction

Preparing teachers to learn from teaching, through critical reflection on their practice, is one of the main skills to develop during initial teacher education (ITE) (Hiebert, Morris, Berk and Jensen, 2007). The capacity to learn from teaching has been described as a meta-competency, which fosters the development of many other teaching competencies (Collins and Karsenti, 2011). However, despite its relevance in teaching practice and, therefore, its centrality in teacher preparation programmes, there is a lack of explicit guidelines on how to teach reflective practice (RP) (Russell, 2005). Some propose teaching explicitly, directly, thoughtfully and patiently modeling their own use of reflection-in-action (Russell, 2005). Others underline the importance of building 'reflective communities' (Beattie, 1997), moving beyond individual activity to communities of inquiry that nurture teachers' practice.

Lesson study (LS) is a means of collaborative teacher professional development that originated in Japan (e.g. Fernandez and Yoshida 2004). In the last 10 years LS has become increasingly popular internationally, especially in the US (Hart, Alston and Murata, 2011). This has led to theoretical models (Lewis, Perry and Hurd, 2009) and adaptation in different contexts (Fernandez, Cannon and Chokshi, 2003). Although this has informed continual professional development (CPD) for in-service teachers, adaptations for ITE are relatively rare. This paper presents results from a LS experience, in the context of ITE for secondary mathematics teachers (PGCE) in the North West of England. The main goal of this study is to explore how the LS process can support students' development of reflective practice.

Background

Teaching could be described as a ‘messy’ and ‘ill-defined’ practice (Dewey, 1933; Schön, 1987). In this context, teachers’ capacity to think about what happened during a classroom lesson, why it happened, and what could be done next time to make it happen more successfully - commonly referred to as RP - is central (Osterman and Kottkamp, 2004). Schön (1983, 1987) has been one of the most influential writers on RP. He proposed (1983) that, in order to improve some aspect of practice (what he calls tacit knowledge or *knowing-in-action*), reflection is needed. He distinguished two components of RP: looking back after an event (*reflecting-on-practice*), and modification of immediate actions (*reflecting-in-practice*) (Roth-McDuffie, 2004).

Although there is agreement on the relevance of RP for teaching, it is still not clear how best to help novice teachers engage in productive reflection. Novice teachers are easily distracted by practical lower-order skills, such as ‘managing students’ behaviour', and have a tendency to describe rather than identifying reasons for success (analysing) (Parson and Stephenson, 2005). Some of the agreed conditions for productive RP in ITE are derived from conceptions of teachers’ learning as situated (e.g. Putman and Borko, 2000) and as a social practice (Hoffman-Kipp, Artiles and López-Torres, 2003). First, learning to reflect should be grounded in authentic practical activities (Putnam & Borko, 2000), as ITE students have limited previous experience. Combining school-based tasks with scaffolded reflection allows student-teachers to experience innovative classroom/school practices that avoid automatic enculturation into and reproduction of traditional norms of the teaching profession (Zeichner and Tabachnick, 1981).

Second, RP - as for any other higher order skills or competences - is best learnt collaboratively (Hoffman-Kipp et al., 2003), with ITE students benefiting from the assistance of others (peers and those more expert) in joint activity. This collaborative approach has being considered in ITE models that include peer collaboration, for instance, ‘paired teaching placements’ (Nokes et.al., 2008) and ‘critical partnerships’ (Parsons and Stephenson, 2005), and more horizontal collaboration between mentor and student, where they plan, teach and evaluate together (Charlies et.al., 2008).

Finally, bridging theory and practice (praxis) and focusing on pedagogical aspects must take precedence over pragmatics and everyday operations (Hoffman-Kipp et al., 2003). However, mentors may lack confidence when talking about pedagogy (Roth-McDuffie, 2004) and in the current performative culture operational aspects of teaching dominate school institutions (Williams, Ryan and Morgan, 2013).

LS provides a model that considers all of these conditions. It is pedagogically focused on designing, implementing and discussing an authentic lesson through a collaborative approach. In the LS cycle teachers collaboratively plan a lesson, observe and gather information about students' learning, and analyze, reflect and discuss these observations (Fernandez and Yoshida, 2004). Importantly in the model practiced here, they imagine and agree improvements or refinements and re-teach the hypothetically improved lesson.

One of the few examples of using lesson study in ITE is reported by Fernandez (2005; 2010) who led a group in the US exploring the use of LS combined with micro-teaching (MLS). In MLS, groups of student teachers teach a small group of classmates. They found that prospective teachers learnt through active engagement in a collaborative environment (Fernandez, 2010), found it easier to connect theory to practice (Fernandez and Robinson, 2006) and used a knowledgeable advisor’s formative feedback to develop a critical perspective (Fernandez et.al., 2003). In our model, however, the students meet with teachers and teach in classrooms: we anticipate important differences.

In real classrooms few examples were found in the preparation of elementary school teachers. Murata and Pothen (2011) found LS was a useful way of relating university-based and in-placement activities. Yu (2011) found that LS could help to avoid enculturation into traditional norms of the teaching profession (Zeichner and Tabachnick (1981)). However, some student-teachers experienced teaching the research-lesson as stressful (Murata and Pothen, 2011), and tutor guidance was needed to maintain focus on higher level skills and competences during the post-lesson discussion (Fernandez and Zilliox, 2011).

In the light of this evidence, we designed an intervention that allowed collaborative reflection and interaction between student-teachers, in-service teachers and university tutors, and was school-based. During the academic year of 2011/2012 we piloted a LS process with in-service teachers and student-teachers whilst on placement in a number of partnership schools through a small funded pilot (AGGS, 2012). In each school up to three LS cycles were completed. The pilot aimed to bring together a process of CPD and ITE and was focused on mathematical dialogue. Although this experience was very successful in terms of CPD and in-service-teacher engagement, student-teachers had a relatively peripheral involvement in the process. In many schools the student-teachers did not teach and were observers rather than active participants in the planning and discussion meetings. Following this experience we wanted to increase ITE student involvement whilst maintaining experienced teacher participation. The LS focus was again the use of dialogue to develop mathematical understanding, but the plan was: 1) The school selects a lesson topic (suitable for a dialogic approach); 2) A group of five student-teachers plan the lesson with tutor guidance; 3) The entire group (five student-teachers, school-teacher and university tutor or researcher) meet in school to discuss the planned lesson; 4) The school-teacher teaches the lesson, with the group observing students' learning; 5) The group discuss the lesson and modify the planning; 6) The school-teacher leads the new lesson with the group observing; 7) Final discussion of the entire group. In this paper we report results of this experience, focusing on two research questions: 1) How can a lesson study cycle mediate ITE reflective practice? and 2) What contexts favour/hinder the student-teacher reflective process?

Method

Implementation

The entire LS process happened during January 2013 in five schools and in one school in June. We worked with six partnership schools (some had been involved in the pilot), having different levels of involvement, especially of the classroom-teachers. In most of the schools (four) school-teachers' participation in the planning meeting was limited and they did not lead the teaching of both lessons of the cycle, which was done by the students or tutors. Whilst they gave feedback in the post-lesson discussion the teachers mostly did not contribute to decisions about changes to the lesson plan. In the other two schools, the teachers taught both lessons and were active participants in the planning and post-lesson discussions.

Data collection and analysis

For data collection purposes, we adopted a case study approach of the PGCE LS experience, comparing its particular contexts of application (schools) (Yin, 2011). Data collection was made during and after the lesson study cycles. The information gathered during the cycle was used in the post-lesson discussions, and as triangulating data for the research. It included two observation schedules and audio recordings of lessons and post-lesson discussions.

After the lesson study cycle was finished, student teachers and teachers who participated in the experience were interviewed, individually or in groups. Two focus groups comprising five student teachers, and five individual interviews with teachers and student-teachers, were conducted. The purpose of the semi-structured interviews was to gather perceptions of the lesson study experience. The interview protocol included questions about implementation, assessment and the extent to which lesson study was new or similar to other ITE experiences.

The analysis of the data included thematic analysis of interviews for eliciting how lesson study can mediate student-teachers' RP; and comparative sub-case studies for eliciting what contexts favour/hinder RP. For the thematic analysis, relevant extracts from the audio recordings were identified and transcribed. The data was analysed, coding interesting features and organising into potential themes (Guest, MacQueen and Namey, 2012).

In order to compare the different contexts we developed a case description of each sub-case (particularities of the implementation in each school). We used a holistic narrative approach (Stake, 1995), maintaining the coherence of each context. In addition we analysed evidence of developing RP in these different implementation contexts. An emerging priority/focus was the development of different roles in the post- and pre-lesson discussion.

Results

Mediation of Reflective Practice by the Lesson Study cycle

In the interviews, student-teachers said they found the collaborative planning difficult and not very effective. Most groups discussed what they might do and shared the planning work. Some had a group-leader, who wrote up the plan and collated the resources, often acting as the main contributor to the lesson. In addition to these difficulties, some groups failed to select tasks suitable for developing mathematical dialogue, which was meant to be the main objective. Student-teachers were more concerned about 'getting a lesson-plan together' rather than considering how to promote mathematical dialogue, As one student said afterwards:

> The activity wasn't good for discussion. It should have been more investigative to bring more discussion between the students. A lot of the things that we were teaching were either…it seemed more black and white, it was this or that… whereas, if we'd done an investigation it would have brought more discussion..

Despite these difficulties, working on the plan led to a sense of ownership. In all the interviews student-teachers talked about 'our' lesson, and 'our' changes to the plan. They valued being 'the only ones finalising the plan', in charge of bringing together their own evaluation with school-teachers' and tutors' feedback.

In contrast, all of the interviewed students said they valued and found easier the post-lesson discussion as a context for developing reflection about practice:

> I think reflecting with other people is useful, because, especially when you're teaching, you kind of have like blinkers on, (laughs) and you see one kind of thing, and if someone else is stood at the back or took another part of the lesson, what they see may be different to what you thought you did (...) And discussing as a team helps to process it more, discussing it.

The reflective process at first was focused on concrete and specific aspects of the lesson experience, using observation evidence and the audio recordings to help focus on classroom dialogue:

> We had opinions about things, but they were supported by things we noticed during the lesson, so we could convince everyone at the table, 'Look, we have ten critical points in this part of the lesson, we could make that activity longer to see if we can get more critical points'.
>
> And you also forget things, because when you're teaching, it all goes so fast, so that's why the people recording was good, because we could look at it and because it is written down, just like a little reminder.

Subsequently, student-teachers used these observations to improve the plan for re-teaching. The student-teachers' roles as observers and analysts was strongly influenced by the goal of improving the lesson. They used the information they gathered to 'think otherwise', an aspect of the experience they valued greatly. It was important to the process of reflection that the students were encouraged to imagine alternatives, and that the re-teaching of the lesson was the crucial motivation for this re-thinking. They commented that improving a lesson was a unique opportunity:

> I enjoyed it, because if you're in teacher training you never have the chance of teaching the same lesson 'til the next year. If I'm teaching one lesson today, that lesson I can't improve it 'til next year, when I get to teach it again. So it is good to teach a lesson once and then, here are the changes, we'll teach it again within two to three days to see the improvements. So I liked it, I liked that (...) I found it very, very powerful, to think about what worked, oh, yeah, I'll try that. Whereas, we reflect on doing something differently, but you can't teach that same lesson again 'til next year (...) and you won't remember what went wrong, unless you'd changed the resources straight away.

Although student-teachers recognised and mentioned dialogue as the main focus of LS, it wasn't always the main focus of their post-lesson discussion. For instance, when analysing recordings of post-lesson discussions dialogue was not mentioned until the school-teacher or the tutor prompted the student-teachers. The student-teachers noted that the tutor guided the discussion to the LS focus:

> There was a little bit of a disagreement... I think some people were of the impression that we were discussing teaching style, so recommending improvements for the teacher and how they could have led the lesson better or the task could have been improved (...) but XX [the tutor] kept coming back to dialogue and what that meant in the classroom.

Different context, different collaborative practices

Two main contexts of implementation were observed in this LS experience: schools where class-teachers taught both lessons of the cycle and schools where school-teachers had less involvement. In the former the student-teachers valued observing an experienced teacher and focussing on the effectiveness of the lesson:

> It was really nice to improve the lesson, and because we weren't teaching the lesson, we were just observing, we could pay a lot more attention to what was going on. And because when I do teach I don't usually pay attention to which part

> was weak or strong in the plan. But because that was all what we were doing, it was brilliant.

The student-teachers developed a more ambitious lesson plan than if they were to teach the lesson, thus facilitating reflection on planning and teaching separately:

> It took all the stress out of the lesson plan (...) so we were more ambitious in the lesson planning (...) And I really appreciated it because I got the opportunity to see an experienced teacher carry out my lesson plan. Because that had never happened before. Previously I worried that my lesson plan was let down by my teaching or that my lesson plans were letting down my teaching, one way or the other, because I knew they relied on each other. I just wasn't sure which one was the weak link.

In the schools where teachers had less involvement, student-teachers found it more difficult to reflect on the implementation of the lesson:

> I was teaching it, and when you're teaching you don't actually have much opportunity to realise what is happening, so just with listening to someone just reading the highlights of the class back to you it is nice, and to have the chance to discuss that with a group, I think it is very effective.

In these schools participation in post-lesson discussion was unequal. In one school the classroom-teacher dominated the discussion giving his opinion of the lesson and answering questions about his teaching practice. No decisions were made about modifications to the plan whilst this teacher was present.

Where the teacher was involved in the entire process and taught both lessons, the post-lesson discussion was more distributed. Both, student-teachers and school-teacher gave opinions, commented and elaborated on each others' contributions and asked for information about others' observations. The discussion ended with agreeing changes to the lesson plan and sharing responsibility for making those changes.

The interviewed teachers that did not teach the lesson attributed this to their limited involvement in the planning of the lesson:

> This time, (i.e. this year) again, I wasn't involved in the planning, I just sort of saw the plan before, in the lesson I was watching people, but I didn't have a role. I was just sort of supervising and giving sort of general feedback, on how the lesson went. Yes. So I didn't feel I was involved in the lesson study.

Even when the classroom-teacher did teach one of the lessons, the lack of involvement in planning did raise implementation challenges:

> If I'd thought about it more, if I'd had more time, I would have had sort of a better story to go with everything of what we were doing

However, the teachers that committed with the entire cycle saw their limited involvement in planning as a strength of the process. They valued the opportunity of trying different approaches and challenging 'their ways'.

> Well, as a teacher I think you get quite set in your ways and I think is nice that you're coming, and say "try this", and we may try it and it may actually work really well, so I'm quite happy to try it, even though it is a little bit out of my comfort zone - that what I usually do, I think it is quite nice to try it, and if it goes well, keep it for the second lesson, and if it doesn't, I may do what I normally do.

Conclusion

The structure of LS can contribute to the professional development of ITE students, particularly the role of imagination in their RP. Whilst collaboratively planning the initial lesson was difficult, it had a major role in developing ownership of the lesson.

Students' and tutors' comments suggest that, in order to develop a more productive reflection in this early stage of the LS cycle more guidance in reflecting/imagining from experienced teachers is needed. Then the students were committed to analysing and improving the lesson, taking responsibility for bringing together contributions from peers, teachers and tutors in re-imagining and re-engineering a hypothetical improvement. We argue that this role of imagination in reflecting on practice (after the lesson) is crucial to the students, and that the re-teaching in lesson study may play a vital developmental role in this (Schon, 1983).

Thus the post lesson analysis and imagination of improvements was instrumental in developing ITE students' RP, adding a new highly relevant dimension to their 'zone of professional development'. This explains and supports previous findings on the relevance of 'repeated cycles' in LS with ITE students (Fernandez, 2010).

Finally, the implementation of LS was mediated by the social relations between those involved. In this case student teachers, university tutors and in-service teachers worked together with a view to improving their practices. When classroom-teachers were involved in the entire process, the distribution of roles within the post-lesson discussion were more equal: student-teachers and school-teachers contributed opinions about the lesson and possible changes to the plan. Where the classroom-teacher was less involved, a power imbalance was evident and the student-teachers were not able to contribute their opinions of the lesson on equal terms. Following this we conclude that engaging the participants in equal terms, sharing their roles and experiences, helps in providing a collective experience of LS. This collective experience is what we suggest provides more adequate opportunities to learn, which we refer to as a Zone of Professional Development. The challenge for the team and a relevant area of future research is how best to facilitate this.

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Towards a model of professional development for mathematics teachers integrating new technology into their teaching practice

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This paper focuses on the professional development journeys of mathematics teachers when integrating new technology into their teaching practice. The research is part of a year-long innovative cooperative intervention in a cross-phase and cross-school setting with transcribed interviews, teacher meetings and on-line communication used for data collection. The Interconnected Model of Professional Growth was used to analyse these journeys and one teacher's journey is illustrated here. The analysis indicates that all four teachers' development journeys exhibited similar cycles of practical experimentation and reflection that led to long-term changes in teacher practice. A model for how mathematics teachers' integrate technology into their teaching is proposed.

Keywords: technology, professional development, mathematics teacher

Introduction

This paper focuses on the process of change experienced by mathematics teachers integrating technology into their teaching practice and is part of my PhD research, which investigated the professional development of mathematics teachers in a cross-phase and cross-school collaborative setting. The collaboration centred on teachers learning a new technology for teaching and learning mathematics and this paper focuses on the process of change experienced by the teachers in terms of their use of this new technology.

Historically, research has focused on how technology facilities and supports pupils' learning with little or no attention paid to the teacher (e.g. Pierce and Ball, 2009, and Williams et al., 2000). More recently, however, the complexities of integrating technology into teaching practice has been reflected in the literature and the focus is now not only the use of technology in learning mathematics, but also the *teacher use of technology* in teaching and learning mathematics (e.g. Lagrange and Erdogan, 2009). In addition, journals such as The International Journal for Technology in Mathematics Education (IJTME) and Educational Studies in Mathematics now regularly include the 'integration' aspect and reports from the teacher perspective. Of the 46 articles published in the eight IJTME issues during 1997 and 1998 only one incorporated a focus on the role of the teacher, compared to eleven of the 24 articles in the five issues from March 2008 to March 2009. This paper attempts to enrich the field of research into mathematics teacher use of technology by suggesting a model for teacher development when integrating technology into teaching practice.

Methodology

The wider research project was a longitudinal multiple case study of two pairs of teachers, where each teacher pair consisted of one primary school key stage 2 teacher and one secondary school key stage 3 teacher (see Rempe-Gillen, 2012). The teachers had ownership of their own professional development, with each teaching pair choosing the technology (Bowland Maths and Geogebra), the mathematics topic and the groups of pupils to focus on. Data was gathered in semi-structured interviews, teacher meetings and online communications, which were audio-recorded, transcribed and shared with the teachers. Data collection also included field notes and documents, and lessons were recorded for peer- and self-observation.

The area of research for the wider project, cross-phase and cross-school collaboration of mathematics teachers, is identified as an area for further research (NCETM, 2009) so a grounded approach was employed. This involved repeated readings of field notes, interview transcripts and meeting transcripts, with cycles of coding the data leading to emergent themes and teacher journeys of development. For each of the four teachers a journey of development was produced, which was a chronological list of activities and decisions that the teacher made during the project. The four journeys were then analysed using models of professional development. The next section summarises the teachers' journeys and these are then analysed using four models of development.

Results - the teachers' journeys of professional development

A summary of Kimberley's and Kirsten's journeys of development

The key stage 3 teacher, Kimberley, developed in relation to her learning Geogebra, from having only seen it some years previously, during her PGCE course, to confidently using it in her lessons.

Kimberley had been introduced to Geogebra during her PGCE course but she had never used it in her lessons. The training session on Geogebra, and discussions with the key stage 2 teacher, Kirsten, on topics that it could be used for, led her to decide to use it with her year 11 pupils. She did not feel the year 11 pupils were positive about Geogebra and they would have been more positive if they had been able to use Geogebra as well. This led her to consider that Geogebra was better used as a whole class pupil activity rather than a demonstration tool. However, she was unable to book an ICT room and, realising she would not be able to use Geogebra in an ICT room in future, she changed her goal, focusing on using it as a demonstration tool because she would have more opportunity to use it as a demonstration tool in the future. She used the internet to search for more information on Geogebra and how she might use it in her lessons. Kimberley then taught a lesson using Geogebra as a demonstration tool and after the lesson she reflected on her use of Geogebra. One example of her self-evaluation was that she considered Geogebra helpful in making certain tasks more efficient, e.g. drawing straight lines. She also considered how she could manipulate objects on the interactive whiteboard.

Kirsten's knowledge of Geogebra came from the training session, her discussions with Kimberley and her internet searches. These also informed how she would use Geogebra in her lessons. She then taught her lessons using Geogebra and then she considered how her pupils had used Geogebra, she considered their enjoyment in using it and their ability to complete the tasks she had designed. She compared Geogebra to other software that she used in lessons and her reflections on these aspects of the use of Geogebra informed her decision of whether she would use it again. She also reflected on her own performance, how she manipulated the objects on screen and the ease of using Geogebra compared to other software that she knew about.

A summary of Wendy's and Winona's journeys of development

Each of the Bowland Maths Case Studies provides lesson plans and resources that teachers can use in their lessons. Winona, the key stage 3 teacher, chose the *Alien Invasion* case study and planned to rewrite the lesson plans herself but she did not have time to do so, and so used the provided plans with her year 8 pupils. After the lessons she considered the behaviour of her pupils and said she preferred lessons where pupils moved around the classroom more. Because she saw this as an important feature of her lessons she considered discontinuing the use of Bowland Maths. Winona considered pupil feedback also important, and at the end of the final lesson in the project she asked pupils for feedback on their learning and their enjoyment of the lessons. Pupil feedback was positive and she found that the resources motivated pupils. Speaking to me immediately after the lesson and then with Wendy, the key stage 2 teacher, Winona reflected on the lessons and considered how she might alter her own practice in future. Her consideration included what she had written on the board and if any resources could be removed from the series of lessons. From her reflections on the lessons, together with Wendy's comments on the lessons, Winona re-assessed her view of the important outcomes of the lesson and adapted parts of the lessons, removing some sections where she had used technology, for when she taught the same lessons again to a group of primary school pupils. At the end of the eight lessons Winona decided to share the resources with a colleague and she planned to use Bowland Maths with other primary schools.

For Wendy, participation in the project and her observation of Winona's year 8 lessons showed her how Bowland Maths could be a useful learning activity for her primary school pupils. Wendy chose not to teach the primary school pupils' lessons but she was involved in the planning of them with Winona. In the end, Wendy taught the final primary school pupil lesson because Winona was absent. Wendy's lesson preparation involved working through the resources herself, from her pupils' perspective, and she too garnered feedback from her pupils. The feedback focused on the pupils' enjoyment and learning and this informed her value of the resource and her confidence in using it. Although initially reluctant to teach with the Bowland Maths resources at the start of the project, after the project ended Wendy did use Bowland Maths again with primary school pupils.

Modelling the Process of Development

Understanding what constitutes professional development and what exactly is being developed can lead us to models of professional development. Day (1999), in giving his definition of professional development lists "knowledge, skills and emotional intelligence" (p. 4), Evans' (2011) model of professional development categorises attitudinal, behavioural and intellectual change and, in mathematics teacher use of technology. Monaghan (2004) highlights how research focuses on 'beliefs' and 'knowledge'. He notes the lack of focus on what he terms the 'whole experience', which suggests that for a mathematics teacher to develop from one who does not use technology in the teaching and learning of mathematics to one who does use technology we should consider both the internal - un-observable - changes (such as beliefs, attitude and emotion) and external - observable - changes (e.g. behavioural and skills). In this research I used Evans' categories of intellectual, attitudinal and behavioural change.

A number of models for professional development exist in the literature and four models were applied to the teachers in the study, in conjunction with Evans' three categories to describe the changes that occurred in the process of development. Each model demonstrated a possible sequence of developmental change that may have been expected to occur but none of the models were able to model the process of all four teachers.

Model 1

One example of a possible sequence is intellectual change after learning how to use the new software, leading to attitudinal change motivating the teacher to use the technology in her classroom (behavioural change). This order of change is in line with an implicit model of teacher change following an in-service development activity, illustrated in figure 1:

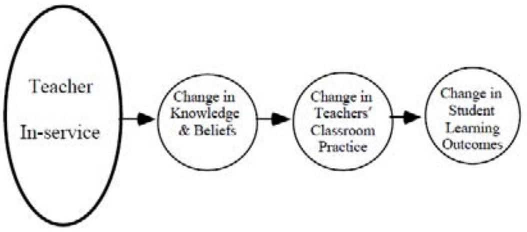

Figure 1: An implicit model of purpose of teacher professional development (Clarke and Hollingsworth, 2002: 949)

Here in-service CPD activities lead to change in teachers' knowledge and beliefs. This in turn leads to teachers changing their classroom practice that results in a change in student outcomes. An example of this model occurred in the case of Kirsten: she learned how to use Geogebra in the training session and then she designed a task for her pupils, evaluating the lesson by considering her pupils' learning and feedback. However, this model does not demonstrate the change process experienced by Winona, since her knowledge and beliefs changed after changes to her classroom practice.

Model 2

Another example of a possible sequence is an intellectual change after learning how to use the new software, leading to the teacher using the technology in their classroom (behavioural change) and an attitudinal change when the teacher evaluates the lesson. This order of change is in line with Guskey's model of teacher change, illustrated in figure 2:

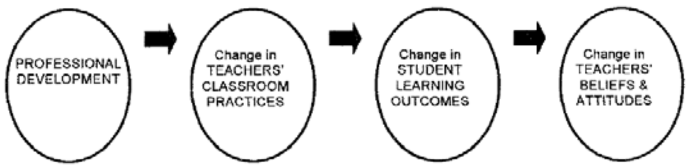

Figure 2: A model of teacher change (Guskey, 2002: 383)

Here CPD activities lead to change in teachers' classroom practices that in turn lead to changes in student outcomes, which influence teachers' beliefs and attitudes. Winona exemplified this model of change; she used the prepared Bowland Maths lesson plans and evaluated the lessons by considering her pupils' learning and feedback, which influenced her continued use of Bowland Maths.

Despite these two models being evident in the case of two of the teachers, neither model incorporates consideration of a sequence of professional growth that does not include behavioural change. An example of this would be if the teacher considered the software inappropriate to use. In this case, there may still be intellectual and attitudinal change but no behavioural change. Kimberley exemplifies a teacher in this position because she wanted to design Geogebra activities where her pupils used Geogebra (intellectual and attitudinal change), but, since she was unable to access an ICT room, her behaviour did not change.

Model 3

A third potential model of teacher change is exemplified by the teacher having already identified a deficiency in her teaching, participating in the project as a means to improve her teaching. This would correlate with a deficit model of teacher development where the first stage involves indentifying an aspect of the teacher's practice for improvement. This model of teacher change would be in line with Bell and Gilbert's (1996) model, where development occurs in three domains: social, professional and personal, with each of these having three stages. There is evidence that some of these stages occurred for some of the teachers, such as dealing with constraints (Kimberley encountered difficulty with access to ICT), empowerment (Wendy's use of Bowland Maths) and development of classroom practice (Winona's continued use of Bowland Maths). However, there is no evidence to suggest that the teachers participated in the research because they felt an aspect of their teaching was a problem (personal development 1), nor did they see working in isolation as a problem (social development 1).

Although elements of each of these models are evident in the development of the four teachers, none of the sequences reflects the development of all four teachers. If we consider only the elements of the models, without the constraint of a causal linear process, then we can begin to see a model that would allow for multiple pathways between elements, modelling the development of all four teachers. Moreover, it is evident that the teachers' development could involve change in some elements on more than one occasion. One example of this cyclic nature of development is Winona. Her initial use of Bowland Maths, based on her pupils' lack of movement around the classroom, led to her negative attitude towards Bowland Maths; however, feedback from her students led to a change in her knowledge. This then led to change in her classroom practice since she decided to use Bowland Maths again. Consideration for the cyclic nature of development is the basis of Clarke and Hollingsworth's (2002) Interconnected Model of Teacher Professional Growth, and this was the fourth model utilised to analyse the teachers' development.

Model 4

The Interconnected Model of Teacher Professional Growth (IMPG) (Clarke and Hollingsworth, 2002) is an analytic tool for the change process. Developed from empirical research, it is used in the development and design of courses (e.g. Coenders et al., 2010) and the analysis of professional development of science teachers (e.g. Justi and Van Driel, 2006) and mathematics teachers (e.g. Witterholt et al., 2012). The model situates professional change in one of four domains within a change environment: an external domain (sources of information or stimulus), a personal domain (teacher knowledge, beliefs and attitude), a domain of practice (professional experimentation), and a domain of consequence (salient outcomes). Clarke and Hollingsworth acknowledge that these domains are "analogous (but not identical)" (p. 950) to those of Guskey's model, however, where Guskey's model is a linear path through the domains in a set order, Clarke and Hollingsworth's model encompasses the complexity of the change process. Changes are mediated through en-action and reflection, where en-action is "the putting into action of a new idea or a new belief or a newly encountered practice" and they cite Dewey (1910) in their use of reflection as "active, persistent and careful consideration" (Clarke and Hollingsworth, 2002: 953-954).

Clarke and Hollingsworth consider salient outcomes as individual to the teacher and this was evident with these four teachers since they found different aspects of their lessons important. For example, Winona preferred her pupils to be out of their chairs and moving around the classroom during lessons so she considered not using the same lesson again with another group of pupils. However, her paired teacher, Wendy, after observing the same lessons, was very positive and eager to teach the same lesson to another group of pupils. The two teachers had different views of the lesson because they each had different views on the important aspects of a lesson.

In contrast to the previous three models, the IMPG was employed to model all four teachers' development. For conciseness I restate only Kimberley's journey to illustrate the model (in figure 3) and I refer to Rempe-Gillen (2012) for further analysis of the other three teachers' journeys utilising the IMPG.

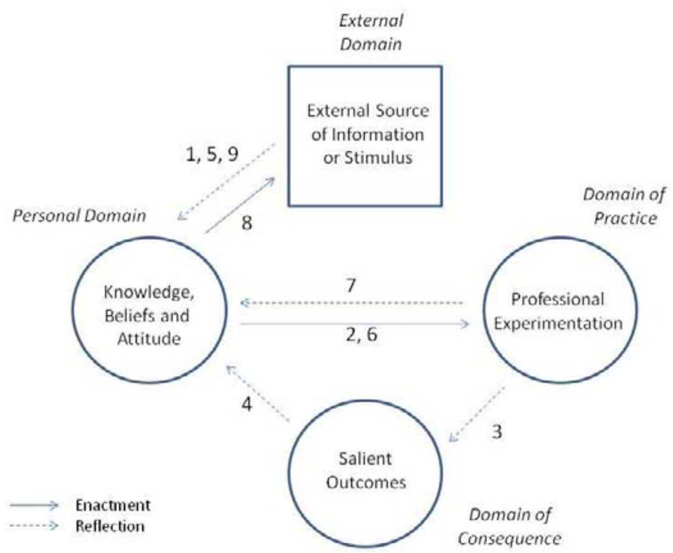

Figure 3: Kimberley's professional growth

Kimberley had been introduced to Geogebra during her PGCE course but she had never used it in her lessons (Arrow 1). The training session on Geogebra, and discussions with Kirsten on topics that it could be used for, led her to decide to use it with her year 11 pupils (Arrow 2). She did not feel the year 11 pupils were positive about Geogebra and they would have been more positive if they had been able to use Geogebra as well (Arrow 3). This led her to consider that Geogebra was better used as a whole class pupil activity rather than a demonstration tool (Arrow 4). However, she was unable to book an ICT room and, realising she would not be able to use Geogebra in an ICT room in future, she changed her goal, focusing on using it as a demonstration tool because she would have more opportunity to use it as a demonstration tool in the future (Arrow 5). She used the internet to search for more information on Geogebra and how she might use it in her lessons (Arrows 8 and 9). Kimberley then taught a lesson using Geogebra as a demonstration tool (Arrow 6) and after the lesson she reflected on her use of Geogebra. One example of her self-evaluation was that she considered Geogebra helpful in making certain tasks more efficient, e.g. drawing straight lines. She also considered how she could manipulate objects on the interactive whiteboard screen (Arrow 7).

Discussion

The first two models of professional development were able to model two, but not all four, teachers' journeys: the first model was suitable for Kirsten's journey, but not Winona's journey; and the second model was suitable for Winona's journey, but not Kimberley's journey. The third model was not suitable for any of the four teachers' journeys. Given that these three models were not able to model all four teachers' professional development, it would be reasonable to assume that the teachers' processes of development were very different. However, the IMPG was utilised to model all four teachers' journeys. Furthermore, there are clear similarities and common features between the four teachers' journeys when viewed through the IMPG.

The first common feature is that the external domain was the stimulus for development. Kimberley, Kirsten and Wendy reflected on the new technology in some way before putting it into practice (reflection from the external to personal domains). The fourth teacher, Winona, due to lack of time, used the provided lesson plans (enactment from the external to practice domains) but she viewed the lessons negatively. When she taught the lessons again, she made changes to the provided lesson plans, which resulted in her positive view of the lessons and her continued use of the software.

A second common feature for all teachers was the value they placed on pupil feedback (reflection from the domain of practice to salient outcomes). Their salient outcomes were pupil-orientated and impacted on the teachers' views of the new technology (reflection from the salient outcomes to personal domain).

A third commonality was the two-way connection between the teachers' lessons and their knowledge, beliefs and attitude (reflection and enactment between the personal and practice domains). Only Kirsten did not have a direct reflection from the domain of practice to personal domain, but she reflected on the salient outcomes of her experimentation.

Conclusion

Although there are only four teachers in this research their experiences were diverse enough to show that some models of professional development are not suitable to model the integration of technology into a teacher's practice. The IMPG was the only model suitable for all four teachers and, moreover, when using this model there were commonalities in the sequence of events for the four teachers. The research suggests that there are specific components of change and a specific order in which these occur when a teacher integrates technology into her/his teaching of mathematics. Further examination and analysis of the relationships between these elements are needed in order to advance our understanding of technology integration.

Acknowledgements

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Researching children's 'self' constructs and their success at solving word problems: a pilot study

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Few studies have investigated self-constructs on primary school age children's achievement in mathematics. However, studies on secondary and tertiary levels suggest that academic achievement is influenced by a person's self-efficacy/self-confidence, (a belief in their own ability), social comparison, (comparing own performance with others), self-concept, (perceived ability in a particular area) and attitude towards mathematics. In preparation for a research study in the following academic year, twenty-one year 5 pupils took part in a pilot study in which they completed a psychometric inventory which measured social comparison, self-concept and attitude. Additionally pupils rated their confidence in solving a range of mathematical word problems prior to solving them. Analysis suggests that these self-constructs influence primary age pupils' academic performance.

Key Words: self-efficacy, social comparison, self-concept, attitude, word problems.

Background

During the first author's career as a primary school teacher, she was aware of pupils' difficulty in solving mathematical word problems. During this time, the UK Government and mathematics education interest groups were also aware of pupils' difficulties. This was demonstrated in the renewed UK Primary Framework for Mathematics (DfES, 2006) in which one of the aims was to provide children with the skills to accurately and efficiently solve word problems.

Many researchers have indicated that there are a variety of reasons for pupils' difficulties (see for example, Muir and Beswick, 2005; Mamona-Downs and Downs, 2005; Gooding, 2009), with the most common reasons being pupils' uncertainty of the method to use and their lack of confidence in their own ability. These views were supported by the UK Government Review reports, such as the Cockcroft Report (1982), the Williams Report (2008) and the Rose Report (2008). The Cockcroft Report suggested that children needed confidence in the use of mathematics and that mathematically able children are not always good problem solvers, due to lack of confidence. The Williams Report (2008) further emphasised that fostering good attitudes towards mathematics was needed and the Rose Report (2008) concluded that numeracy proficiency involved confidence and competence.

In this paper the idea of self-confidence affecting the way that pupils problem solve is based on Bandura's social cognitive theory (Bandura, 1977; 1986). Social cognitive theory indicates that a pupil's performance is affected by both their environment and their own personal affective/cognitive constructs, such as self-efficacy beliefs (Bandura, 1986). Self confidence is usually the operationalised measurement of self-efficacy (Schunk, 1991). According to Bandura, self-efficacy beliefs, are a pupil's own judgement of their capacity to execute actions to attain a particular performance level. For example, in terms of word problems, self-efficacy will be pupils' judgements of their ability to solve word problems and their performance will be solving the problem correctly. Bandura suggested that pupils who had high self-efficacy were more likely to persevere at solving a problem and trying different methods whilst a pupil with low self-efficacy was more likely to give up. The reasons for giving up may be because pupils lack the skills to solve the problem or they are unsure how to use their skills (Bandura, 1993). Based on Bandura's social cognitive theory, research studies found that there were positive relationships between pupils' self-efficacy and mathematics performance (see for example Linnenbrink and Pintrich, 2003; Pietsch, Walker and Chapman, 2003; Marat, 2005; Williams and Williams, 2010). There is also an indication that pupils feel less confident in solving problems on areas of perceived conceptual difficulty like fractions (Nunes and Bryant, 2009) and time (Monroe, Orme and Erikson 2002). Rittle-Johnson and Schneider, (in press) suggest that children's conceptual knowledge is often fragmented which could explain the lack of self-efficacy.

Besides self-efficacy, there are other affective/cognitive constructs that can influence pupils' problem solving skills such as self-concept, although there are arguments whether the two are essentially different (see Pietsch et al., 2003). According to Marsh, Relich and Smith (1983) self-concept is one's perceived ability in a particular area based on environmental reinforcements. Self-concept was found to be significantly correlated to measures of performance and hence suggests that self-concept and self-efficacy may be related. In fact, Zimmerman (2000) found that there was a correlation between self-concept and self-efficacy. Self-concept is sometimes operationalised as a measurement both from the affective (i.e. attitudes) and cognitive constructs (Pietsch et al., 2003). The affective domain relates to pupils' attitudes. In fact, Singh, Granville and Dika (2002) found that there was a positive relationship between attitude towards mathematics and performance that would be expected if attitudes formed part of pupils' self-concepts. Marsh (1990; 1993) suggests that social comparison may also affect performance and influence pupils' self-concept. Marsh believes social comparison is how pupils evaluate how good they are in comparison to others and may be considered to be both an environmental and personal factor within social cognitive theory framework. These studies were with secondary and university students. Little research, of this type, has been carried out with pupils of primary school age and therefore this research will explore the extent to which social cognitive theory applies to primary school age pupils.

The aim of this study was thus to investigate how these cognitive and affective constructs (social comparison, mathematics attitudes and self-efficacy) relate to each other and influence the mathematical word problem solving performance of primary school age children, in England and to inform and focus the approach for the main programme of research, which was to take place the following year. A Year 5 class was chosen as most research in this area has focused at the secondary and tertiary level with little indication whether these constructs and attitudes influence performance at the primary school level.

Method

After receiving approval from the headteacher, twenty one Year 5 pupils at a primary school were given a written questionnaire. The questionnaire was used to measure the pupils' mathematics social comparison, mathematics self-concept, attitude towards mathematics, mathematics self-efficacy and mathematics performance. The social-comparison and the mathematics self-concept inventory were from Pietsch et al. (2003). The mathematics self-concept inventory consisted of six items and was a mixture of both competence and affective items that were measured on a scale from 1: 'I disagree' to 3: 'I agree'. This scale was adapted to represent limited complexity to a ten year old. The attitudes questionnaire was based on eight items from Pampaka, Wo, Kalambouka and Swanson (2012) and used a similar scale as the self-concept inventory. Only one social comparison item was used and was measured on a four-point scale from 1: 'Not Very Good' to 4: 'Excellent'.

As the pupils were unfamiliar with this type of questionnaire, the researcher took time to read the questions aloud and discuss them with the class, prior to its completion, so that their meanings were correctly understood. There is a problem with questionnaires (Menter, Elliot, Hulme, Lewin and Lowden, 2011) in that the answers to the questions depend on how they are interpreted. This is particularly important where there are 'reverse (score) coded questions' e.g. 'I am more worried about mathematics than any other subject'.

The mathematics self-efficacy and performance were measured following a similar method and survey design to Pampaka, Kleanthous, Hutcheson and Wake. (2011). Pupils were first asked to assess their self-efficacy in solving fifteen word problems and were then asked to solve these problems. Each correct problem was awarded two marks. These word problems were modified from Pampaka et al. (2011) as their questionnaire was based on a Year 6 class. Self-efficacy was measured on a four point scale from 1: 'Not confident at all' to 4: 'Very confident'. This method was used in order to compare perceived performance with actual performance (Pajares and Miller, 1995). The problem solving tasks were related to mathematical problems in the Year 5 curriculum and included tasks related to fractions, time and height (see Figures 1-3 for examples of problems asked). In the problem solving test, questions were read to the class, again where necessary, but no further explanation was given. The researcher ensured that all of the questions had been attempted and that the data set was complete.

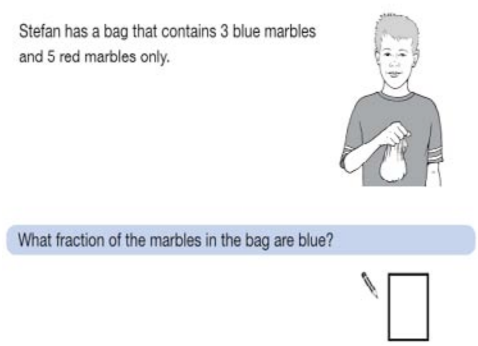

Figure 5: Example of a fraction problem

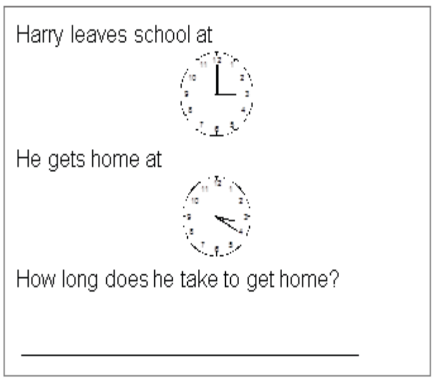

Figure 2: Example of a time problem

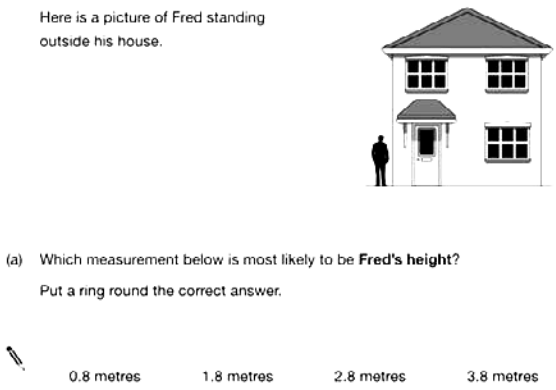

Figure 3: Example of a height problem

Data on pupils' gender and their mathematical level grouping were also collected. Pupils' mathematical level grouping was determined by the teacher, based on the pupil's past performance and on-going assessment. There were two groupings: High and Low level.

Results

Table 1 below shows the make up of the class by attainment and gender.

Table 1: Y5 class make up by level grouping and gender

	Boys	Girls	
High level	8	2	10
Low level	4	7	11
	12	9	21

Pupils were fairly confident in solving each problem, with mean scores on a range of 2.62 to 3.71 (out of 4). The mathematical problems were marked out of 2, and the mean scores for the problems ranged from 0.19 to 2.0. Pupils generally performed lower in problems where their confidence was low. For example, pupils had the lowest confidence in solving the fraction problem in figure 1 (2.62) and also performed poorly (0.57). In the time problem in figure 2, pupils also had low confidence (2.95) and also performed poorly (0.86) and in the height problem in figure 3, whilst pupils were fairly confident of finding the correct solution (3.14) they had the lowest performance of all (0.19).

Using the coding for each of the questionnaire items, the totals for each of the affective/cognitive constructs for the pupils were calculated (see Table 2). Pupils' self-concept and attitudes were quite positive for this class. However, in terms of social comparison, that is, how well the pupils thought they were doing in comparison with their classmates, this was relatively low.

Table 2: The mean totals for the affective//cognitive constructs for the 21 pupils in Year 5

Construct	Mean	Mean as Percentage of Maximum
Performance (out of 30)	15.2	50
Self-Efficacy (out of 60)	46.9	78
Social Comparison (out of 4)	2.5	62
Self-Concept (out of 18)	15.4	85
Attitude (out of 24)	21.9	91

The Cronbach alpha for both the self-efficacy and word problems' performance were within the range required for reliability (0.85 and 0.75 respectively). Although, the pupils had a relatively high self-efficacy (78%) on the word problems their performance was relatively low, that is, pupils only acheived 50% on average.

For each of the variables, the data was found to follow a normal distribution and this allowed the data to be analysed for relationships between the variables, social comparison, self-concept, attitude and self-efficacy in relation to the performance on word problems. A multivariate analysis of variance (MANOVA) was used to determine if there was any difference in the cognitive/affective constructs between the groups by gender and grouping level. Only self-efficacy was found to be significant and this was for gender by the grouping level interaction ($F(1,17) = 5.19$, $p = 0.04$, $\eta_p^2 = 0.23$). The female pupils had similar self-efficacy in both the Low and High level groups (47.9 and 45.0 respectively). However, the boys in the Low level group had a lower self-efficacy score than boys in the High level group (41.8 versus 53.1).

To determine the relationship between the affective/cognitive constructs and the performance in the Year 5 class, a correlation analysis of all the constructs was completed (see Table 3). The results indicate that for the class there is a significant relationship between their self-efficacy, social comparison and their performance. In addition, pupil's self-concept was found to be related to mathematical attitudes. However, this may be because the self-concept inventory included attitudinal items.

Table 3: Pearson's correlation between self-constructs and problem solving

	Social comparison	Self-concept	Attitude	Self-efficacy	Problem solving score
Social comparison	1				
Self-concept	0.42	1			
Attitude	-0.05	0.52*	1		
Self-efficacy	0.56**	0.29	0.21	1	
Problem solving score	0.52*	0.26	0.02	0.48*	1

** 1% level of significance, * 5% level of significance

As there was an interaction with the mathematics level grouping and gender, an exploratory correlational analysis was undertaken. It is exploratory as the sample size is small and hence findings must be taken with caution. The data was first split by gender and a correlational analysis with the affective/cognitive constructs was conducted separately for boys and girls. This was then repeated by splitting the data into the two grouping levels. Interestingly, significant correlations were found between self-efficacy, social comparison and performance only for the boys but not for the girls. Further, when looking at the level groupings, marginal correlations were found between performance and social comparison ($p = 0.06$) and self-efficacy ($p = 0.07$) for the High level group. There were no correlations found for the Low level group for performance. As most of the boys were in the High level group, this may be a contributing factor for the marginal correlations in the level grouping. It is uncertain whether gender or level grouping only is influencing performance or a combination of both.

Discussion

This study found a weak but modest relationship of self-efficacy on problem solving performance which is consistent with analysis of the 2003 PISA survey (Ferla, Valcke and Cai, 2009); other studies such as Zimmerman (2000) have found a stronger relationship between self-efficacy and performance. Whilst Skaalvic and Skaalvic's (2011) longitudinal study found that self-concept was a mediator in performance, no relationship was found in this study. Skaalvic and Skaalvic conducted their study in Norway with pupils' aged 14-16 years. Pupils' age may have an influence on how they view mathematics as Midgley, Feldlaufer and Eccles (1989) noted that pupils' perception of mathematics changes from primary to secondary school. This they found, was due to pupils' perception that they had less support from teachers and so experienced a sharp decline in their perception of the usefulness and importance of mathematics. It may mean that these personal constructs such as self-efficacy with relation to mathematics performance begin to have a stronger impact the older the student is. Perhaps in further studies, which we are conducting with younger pupils (8 to 9 year olds), this relationship between self-efficacy and performance, may be even weaker. Possibly, within the mathematical context, there may be questions about whether social cognitive theory can be applied to younger age groups.

There is also an indication that the primary school pupils' self-efficacy may be dependent on their gender and their level grouping. As pupils' self-evaluation of themselves can be formed by social comparisons (Festinger, 1954), it may be pupils who are in the High level group may perceive themselves as being better than those in the Low level group and hence have a higher self-efficacy. This appeared to be moderated by gender in this study as the girls' self-efficacy did not differ by level grouping, whereas the boys' did. This might be because the girls compared themselves with the other girls and as there were a small number of girls in the High level group, they were not able to socially compare themselves to the same extent as the boys.

Conclusion

The main aim of this study was to investigate how personal affective/cognitive constructs influence the mathematical word problem solving performance of a Year 5 class (9 to 10 year olds), in England. From the analysis it is clear that there is a weak correlation between self-efficacy and success in solving word problems and pupils' social comparison and problem solving success. Self-efficacy and social comparison were also correlated. Self-efficacy in this class was found to be dependent on gender and the level grouping of the pupils.

This was a small scale pilot study and findings must be taken within that context. However, whilst there may be limits in scale, similar results to studies with secondary and tertiary students will confirm that social cognitive theory can be applied to primary school pupils. Although little research of this type has been done with a primary school age group, the effects of self-constructs on problem solving performance will be investigated further as part of a larger study where both qualitative and quantitative data will be used to confirm the findings.

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Development and evaluation of a partially-automated approach to the assessment of undergraduate mathematics

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This research explored assessment and e-assessment in undergraduate mathematics and proposed a novel, partially-automated approach, in which assessment is set via computer but completed and marked offline. This potentially offers: reduced efficiency of marking but increased validity compared with examination, via deeper and more open-ended questions; increased reliability compared with coursework, by reduction of plagiarism through individualised questions; increased efficiency for setting questions compared with e-assessment, as there is no need to second-guess the limitations of user input and automated marking. Implementation was in a final year module intended to develop students' graduate skills, including group work and real-world problem-solving. Individual work alongside a group project aimed to assess individual contribution to learning outcomes. The deeper, open-ended nature of the task did not suit timed examination conditions or automated marking, but the similarity of the individual and group tasks meant the risk of plagiarism was high. Evaluation took three forms: a second-marker experiment, to test reliability and assess validity; student feedback, to examine student views particularly about plagiarism and individualised assessment; and, comparison of marks, to investigate plagiarism. This paper will discuss the development and evaluation of this assessment approach in an undergraduate mathematics context.

Keywords: assessment, higher education, e-assessment.

Introduction

Coursework carries greater potential validity than other assessment methods, in part because of the ability to access greater depth through more open-ended questions (Cox, 2011; Thomlinson, Robinson and Challis, 2010). However, this is accompanied by concerns about plagiarism (Cox, 2011; Iannone and Simpson, 2012; Thomlinson, Robinson and Challis, 2010). Plagiarism might be simple copying (Beevers, 2006), collaborative working taken too far (Cooper, 2002) or impersonation (Beevers, Wild, McGuire, Fiddles and Youngson, 1999). E-assessment allows randomisation of question parameters, meaning individualised work can be set (Gwynllyw and Henderson, 2009), which may be useful for the avoidance of plagiarism (Hatt, 2007). However, writing "reliable, valid questions" for e-assessment is "a difficult task, requiring expertise" (Sangwin, 2012: 7), and input interfaces may add learning requirements (Lawson, 2002) and cognitive load (Mavrikis and Maciocia, 2003) unrelated to the assessment objectives. E-assessment may not be suitable for testing conceptual understanding (Robinson, Hernandez-Martinez and Broughton, 2012), extended work (Sangwin, 2012) or problem-solving (Beevers and Paterson, 2002). Therefore, a partially-automated approach is proposed, in which assessment is set via an automated assessment generator but printed for completion and marking offline as a traditional piece of coursework. The potential exists to maintain the validity of coursework, while increasing the reliability via a reduction in plagiarism, at the cost of decreased marking efficiency. The main question of this research becomes whether there is a context in which this approach could be more useful than existing methods.

Teaching and learning context

Implementation was during a group project in a final year module (not at the author's current institution). Individual work was used alongside group work, partly to increase the amount of the module mark that reflected individual ability, following concern over groups carrying students as what MacBean, Graham and Sangwin call "passengers" (2001: 7).

The main project saw students spend three weeks answering a brief from a (fictional) client. Specifically, students were to investigate 'Art Gallery Problems', which are concerned with determining the minimum number of point 'guards' necessary for all points in a polygon (the 'art gallery') to be connected by a straight line (line of sight) to at least one guard (O'Rourke, 1987). The brief gave three art gallery floor plans and asked groups to propose the size of a staff which must be hired to guard each of these in a short report. The individual coursework gave a single art gallery floor plan and asked the same question.

The similarity of the individual and group tasks meant the risk of in-team plagiarism or collusion was high, suggesting a need for exam conditions or individualised work. The individual work required students to solve a problem and discuss its solution in the context of the real-world scenario. Challis, Houston and Stirling (2004) say that written examinations are "not useful for assessing extended investigations" (45) and this is beyond the limits of automated marking. In addition, a solution would involve drawing a diagram, which via computer input would introduce additional, irrelevant learning outcomes, such as use of a drawing package. The need to produce individualised work via randomisation, lack of suitability of automated marking and the need for students to be able to hand-write their answers suggests that the partially-automated approach suggested above may be appropriate.

Individualised worksheets were generated using the system Numbas, principally a mathematically-aware e-assessment system (Foster, Perfect and Youd, 2012) that can also provide printable question sheets and corresponding answer sheets (where answers can be generated). Producing this was much like writing questions for an e-assessment system, without the requirement to comply with the limits of automated marking. (Systems other than Numbas could presumably be adapted for a similar approach.) When marking, answers could not be learned and student submissions needed to be matched to an appropriate answer sheet using an ID number, which added to the time taken for marking, though not substantially.

Evaluation method

The partially-automated approach was proposed as having potential to maintain the validity of a piece of coursework while increasing reliability via reduction in plagiarism. In general, an assessment method must not be unduly sensitive to who is doing the marking (Cox, 2011). It is important, therefore, to check that reliability, with respect to who is doing the marking, and validity, specifically what is being assessed, are not adversely affected by the use of this method, and to examine its contribution to reducing plagiarism.

Discussing practicalities of evaluation, Moore (2011) suggests that it is important that evaluation is proportional to the activity being evaluated. This has an effect on the workload incurred by participants, in this case students giving feedback and volunteer second-markers. Data already compiled was used for a comparison of marks to reduce the overall resource need.

Second-marker experiment

Asking a second person to mark an assessment is a straightforward way to test the objectivity and accuracy of multiple markers. We should not expect complete agreement between multiple markers for this deeper, more open-ended form of assessment (Cox, 2011). Also, Bloxham (2009) criticises the inherent assumptions that higher education work can be awarded an accurate and reliable mark and that academics share common views regarding academic standards. Therefore, conclusions about whether the level of agreement found between multiple markers is reasonable or not require context. In order to calibrate expectations and provide reference information, the level of agreement for multiple markers of two more established assessment methods was examined. This used: a class test under examination conditions, a method of assessment recognised as being highly reliable (Cox, 2011); and, an open-ended piece of coursework, a method reported as having problems with consistency of marking (Iannone and Simpson, 2012). Comment on the differences in the marks and the intraclass correlation coefficient (ICC) are presented.

A simple test of validity was to ask the second-markers what they thought the coursework was assessing. They were given enough information to mark student work, but were not told the intended learning outcomes.

Written examination reference experiment

The work arose from an open-book test under examination conditions, during a basic mathematical methods module for first year mathematics students. The test comprised five well-focused, short problem questions for which 50 marks were available. A 10% sample of all scripts was checked by a moderator, with reference to the original marks, as part of the usual departmental process. The moderator agreed with the marks awarded in all cases.

I marked a sample of ten scripts without reference to the marks assigned by the original marker but using the same mark scheme (blind second-marking). The mark scheme was a set of worked solutions with individual marks indicated for components of answers and for working. The original marker was working at the same university as me so was used to marking work from similar students.

Coursework reference experiment

The work for this reference experiment arose from a task to write an 800-1000 word review of a popular book or textbook on mathematics or the history of mathematics. The marking criteria specified those pieces of information that each review should contain, as well as some general subjective criteria around the quality of the writing and level of critical understanding. Marks were a simple percentage. A sample of work had previously been approved via a departmental moderation procedure, conducted with reference to the original marks.

I marked a sample of 14 scripts via blind second-marking. The original marker was working at a different university with a similar entry requirement to my own.

Second marking of the individualised coursework

Three second-marker volunteers each had experience of marking work at university; one as a senior academic, one as a junior academic and one as a PhD student. One was from a university with a similar entry requirement to where the work was submitted, one had a lower entry requirement and one a higher entry requirement.

A 10% sample of student work was anonymised (5 pieces from 44 submitted). This was provided along with grade descriptions, a mark scheme and a sample piece of marked work (written to be correct on the non-subjective parts of the mark scheme) as a reference piece since the second-markers were not familiar with the topic.

Student feedback

Students were asked via a questionnaire to express their views, anonymously, on the role of individualised work and how this affected interaction with other students, as well as questions about plagiarism in this assignment and other work. This was completed by the cohort taking the assessment task described above (group A) and by a group at a different university (group B) to provide input from an independent cohort of students with which I had not interacted. The lecturer for group B had also used the technique developed for this project via Numbas for an individualised formative in-class question sheet in a final year digital signal processing module. For both groups, questionnaires were administered via Google Docs. For group A, this was six weeks after the group project had been submitted. For group B, this was at the end of the session in which the individualised assessment was used.

Comparison of marks

The risk was around intra-group plagiarism, since group members were working together on similar problems, so individual marks from within groups were examined. Wide variety of individual marks might indicate that intra-group plagiarism is not a large problem. A lack of variety, however, could indicate plagiarism or perhaps just that students had been learning the topic together and so have similar understanding. If group members colluded on the individual work, we might expect to see similarity between individual and group marks, since they certainly colluded on the latter.

The correlation of raw group project marks and rankings (prior to scaling due to peer assessment of contribution) with the individualised coursework is presented via Pearson's ρ and Kendall's τ. The dispersion of marks for the coursework is examined via the range and standard deviation of the marks within each group.

Results

Second-marker experiment

Written examination reference experiment: There were five discrepancies of one or two marks (2% or 4% of the total) in ten scripts. The ICC for the two sets of marks is 0.992. This value is regarded by Landis and Koch (1977) as an "almost perfect" level of agreement (165).

Coursework reference experiment: There were differences in all fourteen pieces of work. Six were differences of around 5% or less, a further six were around 10% and two were greater differences. The ICC for the two sets of marks is 0.586. This value is regarded by Landis and Koch as a "moderate" level of agreement (165).

Second marking of the individualised coursework: The marks are given in table 1. The ICC for the four sets of marks is 0.635. This value is regarded by Landis and Koch as a "substantial" level of agreement (165).

Student	PR	Second-marker A	Second-marker B	Second-marker C
1	56	31	38	49
2	74	64	59	72
3	67	72	74	77
4	67	46	51	51
5	74	59	54	69

Table 1: Original and second marks for five pieces of work submitted for the individual coursework.

Comments on learning outcomes: Second-markers A and B suggested wording very similar to the three intended outcomes (problem-solving, working in depth and communicating results). Second-marker C, with less experience, suggested two of the three but did not identify communication skills. No marker proposed additional outcomes.

Student feedback

Students were asked to indicate their level of agreement with each of four statements, listed with numbers of responses in table 2. Also in table 2 are the p-values obtained for each Likert-type question when comparing the two groups via Fisher's Exact Test. In each case, there is no evidence at the 5% level to reject the null hypothesis that the distribution of answers is independent of the group. Responses to two questions about copying, which were accompanied by a reminder that the questionnaire was anonymous, are given in table 3. Again, p-values from Fisher's Exact Test are listed in table 3 and do not give evidence at the 5% level to reject the same null hypothesis.

Group	'Strongly disagree' 1	2	3	4	5 'Strongly agree'	p-value
"I disliked having different questions because I wanted to work together with another student on our answers."						
A	12	16	13	1	0	0.08851
B	3	8	2	3	0	
"I liked having different questions because it meant I could freely discuss the work with others with no risk of plagiarism."						
A	0	1	10	22	9	0.6193
B	0	0	4	6	6	
"I liked having different questions because it meant that no one could copy from me."						
A	0	4	14	17	7	0.1366
B	2	2	5	3	4	
"If we had been set identical questions, (members of our group [group A]/some students [group B]) would have copied answers from other students."						

A	2	5	11	15	9	0.3132
B	2	1	6	2	5	

Table 2: Number of students indicating level of agreement with four statements about individualised work.

Group	Yes	No	p-value
"While at university, I have copied work from other students"			
A	22	19	0.1513
B	5	11	
"While at university, other students have copied work from me"			
A	35	7	0.2811
B	11	5	

Table 3: Number of students answering yes and no to two questions about copying.

Comparison of marks

The raw group project marks and rankings do not correlate well with the marks and rankings for the individualised coursework (ρ=0.230; τ=0.229). The range and standard deviation of the individual marks within each group are presented in table 4. Individual marks for each group represent a range of at least 23 marks and up to 31 marks, and have a standard deviation of at least 8.216 and up to 11.411.

Group	Individualised coursework marks range for group members	Individualised coursework standard deviation for group members (3 d.p.)
A	31	11.411
B	30	10.706
C	23	8.216
D	28	9.584
E	30	9.513

Table 4: Marks range and standard deviation for the individualised coursework within each group.

Conclusions and discussion

This research proposed a novel partially-automated approach, in which the tools of e-assessment are used to set an individualised assessment that is taken and marked offline. The evaluation focused on whether a reliable and valid assessment had been set and attempted to examine to what extent this addressed the issue of plagiarism via comparison of marks between students working in the same group.

Second-marker reference experiments showed a high level of agreement for an open-book written examination and a moderate level of agreement for an open-ended piece of coursework. For the individualised coursework, a group of four markers showed a level of agreement that was between the two reference experiments, and close to the open-ended piece of work. This suggests a conclusion that the coursework was, despite its unusual status as individualised work, not unduly sensitive to who was doing the marking.

The three second-markers identified the learning outcomes with a fair degree of accuracy and did not recognise unintended learning outcomes being assessed. The conclusion, based on this, is that the assignment was assessing what it was intended to assess, and no more.

Some sources question whether concern about plagiarism is overblown (e.g., Cox, 2011). Among my 42 students, 22 confessed copying work from another student at university and 35 said another student had copied from them at university. Students generally appreciated being able to discuss individualised work with no risk of plagiarism and reported concerns about copying, including that if identical work had been set then some students would have copied from others. The responses from an independent reference group of students at another university are apparently similar.

Individual marks were not well correlated with group marks and dispersion of individual marks in each group was high. We may conclude, therefore, that plagiarism was not a big problem.

Since student feedback indicated a high risk of plagiarism and none was detected, we may conclude that the individualised nature of the coursework did contribute to a reduction in plagiarism. One of the interviewees of Thomlinson, Robinson and Challis (2010) said that it is “not clear what the real benefit is” of coursework, given that copying is a particular problem among weaker students, and Iannone and Simpson (2012) report some departments moving away from coursework towards in-class tests. The partially-automated approach proposed here appears to be capable of adapting a coursework assignment to make it less sensitive to plagiarism while maintaining its reliability and validity, though it lead to a reduced efficiency for the marker. By contrast, converting the coursework to a written examination or e-assessment in order to reduce the risk of plagiarism can result in reduced validity.

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Teachers of Mathematics: those who Mediate and those who are Mediated.

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The notion of an autonomous teacher has long been accepted as an important characteristic of a good mathematics teacher. In this paper, the beliefs and practices of six primary teachers, all of whom are construed against various criteria as autonomous, were examined. Data, which derived from multiple interviews with and observations of each teacher, were analysed using constant comparison and yielded three themes related to the interaction of belief and practice that separated the six teachers into two equal groups. The results show that autonomy depends on one's perception, therefore challenging the sufficiency of the construct as a characteristic of a good mathematics teacher.

Key Words: mediators; mediated teachers; teacher autonomy

Introduction

Possibly as a reaction to the introduction of the National Curriculum in the late 1980s, the term *teacher autonomy* now occupies a position of importance in the professional literature of English teacher education (Alexander, 2009). In the particular context of mathematics teaching the introduction of the National Numeracy Strategy (1999), whereby the non-statutory guidance became, de facto, a script against which teachers were evaluated by the inspectorate, further reduced teachers' sense of professionalism. Indeed, across the board, different government initiatives were seen to erode teachers' professionalism, policy and practice (Jones et al., 2008), reflecting research that where teachers feel controlled they become teachers who control their students (Pelletier et al., 2002). From such circumstances grew a movement to equip newly qualified teachers in becoming confident and reflective teachers (Pollard, 2008). Such teachers have been encouraged to reflect upon what they do and how they engage with the constraints and affordances different contexts demand of them. Moreover, where teachers are supported at 'various levels within the social context' they are working, they acquire not only an advanced knowledge of their professional field, but also a particular confidence and sense of autonomy (Dierking and Fox, 2012: 141). Such autonomy, where teachers feel able to make decisions that are not driven by external forces, leads to teachers loosening their control of student learning (Warfield et al., 2005). However, the nature of autonomy, despite its regularity of use, remains elusive. In this paper, I present findings from a multiple case study of primary mathematics teachers' beliefs and practice that offered considerable insights into how notions of autonomy play out in their respective classrooms.

The nature of teacher autonomy

In the rhetoric of mathematics teacher education practice and policy the *autonomous teacher* is not an unfamiliar topic, and yet there has been relatively little research on the nature and manifestation of this much promoted quality (e.g. Yackel & Cobb, 1996; Warfield et al., 2005: Watson and De GEest, 2010). In coming to a definition, I have examined literature from a range of research fields. For example, Littlewood (1996), writing in the context of language education, defines an autonomous person as one "who has an independent capacity to make and carry out choices which govern his or her actions", a capacity dependent on two main components, "ability and willingness" (p. 428). From the perspective of mathematics education, Ernest (1989) argued that an autonomous mathematics teacher would have an awareness of having adopted specific views on and assumptions about the nature of mathematics and its teaching and learning; the ability to justify these views and assumptions; an awareness of the existence of viable alternatives; and a context-sensitivity concerned with reconciling and integrating classroom practices with beliefs and to reconcile conflicting beliefs about themselves. Such views, concerning the level of a teacher's awareness of his or her beliefs in relation to practice, resonate with the notion of the reflective professional highlighted above. In related vein, Ernest's context-related observation that "teachers in the same school are often observed to adopt similar classroom practices" (1989: 3), has found affirmation in recent research highlighting the influence of the social context of learners on teachers' classroom decision-making (Skott, 2009). The limited literature seems to suggest that an autonomous teacher is one who has the knowledge and dispositions to make independent professional decisions in ways that acknowledge but can mediate contextual factors.

Significantly, the extent to and ways in which professionals negotiate constraints seem to be a key element of autonomy (Hargreaves, 1996). For example, different educational systems impose different curricular expectations with respect to what is permitted deviation from expected norms (Goodson, 2003). Indeed different notions of professionalism have been framed according to their proponents', whether state, municipality or school, ambitions (Hargreaves, 1996). Thus, autonomy appears very much the concept its definers wish it to be. In a related vein, teachers who are able to navigate their professional contexts and act autonomously experience less stress than teachers who are not, particularly with regard to control over what is taught (Pearson and Moomaw, 2005).

Interestingly, throughout the literature teacher autonomy is assumed to be a precursor to learner autonomy (Gavrilyuk et al., 2013). Indeed, teachers who do not view students as autonomous learners do not see the need to focus on students' thinking in their instruction (Pelletier et al., 2002). Moreover, affective factors such as attitude, feelings, rationality, responsibility for actions and values have all been shown to be significant (Pennycook, 1997). Thus, autonomy appears to be a multi-faceted construct linked to teachers' awareness of and ability to negotiate the constraints within which they work. Moreover, the more competent they are in this regard the more effective and less stressed they are likely to be. That is, professional fulfilment seems dependent on the extent to which one is autonomous.

Methodology

As indicated above, this paper draws on a multiple case study examination of the beliefs and practice of six primary teachers. All teachers, drawn from the same locality, were purposively selected as leaders of the subject in their school and considered locally to be ambassadors of mathematics learning and teaching. They were similarly qualified; each teacher had studied 'A' level mathematics before specialising in mathematics in his or her training and each had achieved a 2.1 degree or above. Some had become, or were applying to become, leading mathematics or advanced skills teachers.

Data, focussed on teachers' management of whole class interaction (WCI), they were subjected to a constant comparison analysis (Strauss and Corbin, 1998), collected by means of interviews to tease out individual teacher's backgrounds, their beliefs and attitudes towards the subject and the teaching of the subject; video-recordings of random mathematics lessons, (between three and six lessons per teacher) and stimulated recall interviews (SRIs) following each lesson to elicit the teacher's rationale for various elements of observed practice. These elements are reported below in the three themes that emerged from the data.

Results & Analysis

As the data were analysed it became apparent that a number of dichotomies were emerging. For example, during their initial interviews all six teachers discussed how they had enjoyed learning mathematics at both primary and secondary school, and described several good teachers and what made them memorable. However, the role played by their families in the construction of their memories of early mathematics differed. One group of three teachers, which I label group 1, comprising Caz, Ellie and Louise, spoke of being members of families in which an enjoyment of doing mathematical problems for their own sake was valued. A second group of three teachers, which I label group 2, comprising Sarah, Gary and Fiona, mentioned no such memories but talked only of satisfaction gleaned from pages of correct answers in their workbooks at school. This group valued the rightness and wrongness of mathematics, in contrast to the other. Essentially the initial interviews revealed, within some shared experiences, two distinct groups of strongly-held beliefs about and attitudes towards mathematics.

The lesson observations yielded three distinct themes, which were also later found to dichotomise the six teachers. These three themes concerned teachers' *mathematical intentions,* or their learning aims or goals; their *pedagogical approaches* or the teaching strategies they adopted during the whole class phases of their lessons; and the *classroom norms*, or the repeated patterns of classroom behaviour, they encouraged. In the following, I discuss these three themes, and my interpretation of them, supported by insights from the stimulated recall interviews (SRIs).

Teachers' mathematical intentions

Analyses yielded five characteristic mathematical intentions (MI) as evidenced by either teachers' observed lessons or their SRIs. All five, which are well known in both research and curriculum literature (Askew et al., 1997), were considered by all teachers to be important in ensuring their children's learning of mathematics. Significantly, the manifestations of these five intentions dichotomised the six teachers in exactly the same way as the initial interviews. Details of the two groups' perspectives on mathematical intent can be seen in table 1, highlighting the extent to which all six teachers used the same professional language but frequently meant very different things in practice.

Group 1: MI	Group 2: MI
Prior knowledge is constantly activated through reference to known knowledge throughout lesson	**Prior knowledge** is activated at the start of every lesson to recall previous activity
Connections are made explicitly through opportunities to discuss and explore ideas	**Connections** are made implicitly with little development of ideas
Vocabulary is emphasised with children and high expectations of its use thereafter	**Vocabulary** is emphasised through Q & A episodes, low expectations of use elsewhere

Mathematical Reasoning derives from opportunities to think and play with ideas individually and collectively.	**Mathematical Reasoning** draws on learned facts memorised from games and tricks.
Mathematical tasks provide opportunities for learning how to be a mathematician; playing with and discussing ideas	**Mathematical tasks** are demonstrated by the teacher so that children will know how to do same.

Table 1: A summary of the two groups' mathematical intent

As table 1 indicates, one group of teachers provides opportunities for their children to think and explore ideas with others and to make connections. This group also provides opportunities for children to use their vocabulary in appropriate ways, and emphasises enquiry, argumentation and justification. Despite a similar professional vocabulary, the second group of teachers present only closed and very limited opportunities for their students to engage with mathematics. Finally, and quite unexpectedly, through observations and later discussion, the dichotomisation highlights an interesting perspective on, perhaps unwittingly, who works the hardest during the WCI phases of lessons, the children or the teacher.

Teachers' pedagogical approaches

The observations revealed that teachers' practices did not always match the beliefs espoused during their initial interviews. They all espoused to provide opportunities to discussion, for example, but what individual teachers understood discussion to mean could be quite different. The analyses revealed seven key behaviours characteristic of teachers' pedagogical approaches (PA), which also split the six teachers into the same two groups as before. Table 2 summarises these differing characteristics.

Group 1: PA	Group 2: PA
Discussion: Provided opportunities for all children to engage in **authentic discussion** (Reynolds & Mujis, 1999).	**Discussion**: A **non-authentic discussion** with alternative views neither sought nor reconciled.
Questions: Teachers provided children time to think about their answers before being expected to answer.	**Question:** Little time given for thinking. Often questions were asked to which the children already knew the answers.
Explaining: Teachers spoke of and enacted modelling as a collective Q & A strategy, while demonstration meant 'telling'.	**Explaining:** Teachers repeatedly explained through demonstration/telling. They saw modelling also as showing/telling best practice; children watch and learn the best way.
Resources: All resources were espoused and enacted as scaffolds for relational learning.	**Resources:** Although espoused as scaffold, resources were used in rote ways to support instrumental learning.
Practising: Seen as part of an approach to develop speed and agility.	**Practising:** Perceived as the most important aspect of mathematical learning.
Encouragement of enjoyment: Playing with numbers and shapes was seen as an integral element of mathematical exploration	**Encouragement of enjoyment:** Teachers focused on extrinsic social pleasures rather than intrinsic mathematical pleasures. Keeping children entertained seemed the dominant theme
Expectation and **differentiation.** Teachers emphasised high expectations irrespective of school context. Differentiation was discreet and flexible.	**Expectation** and **differentiation.** Teachers had low expectations fro some groups. Differentiation was explicit, typically in inflexible groupings.

Table 2: Pedagogical Approaches espoused by project teachers

My interpretation of the details of table 2, allied to the mathematical intentions of the two groups of teachers, is that the dichotomies allude to important insights with respect to professional autonomy, particularly as the vocabulary used by both groups is the same. In particular, the teachers in group 1 emphasise a *relational* perspective on mathematical learning in comparison to teachers in group 2 who emphasise an *instrumental* perspective. These perspectives, which were observed in their teaching were not as apparent in their espousals. Although the same vocabulary is used to describe approaches to mathematics teaching each group assigns different meanings.

These elements are not new and reinforce previous work concerning the relationships between: beliefs and practice (Thompson, 1984; Beswick, 2005); classroom discourse (Myhill et al., 2005); and the making explicit of connections (Askew et al., 1997). Pedagogically, the data appear to dichotomise according to Skemp's (1976) relational versus instrumental approaches to learning and teaching. Where group one discursively extends children's knowledge and understanding, group two focuses on tightly controlled routine skills.

Classroom norms

Finally the classroom norms, or the repeated patterns of classroom behaviour, fell into three categories summarised in table 3. The same two groupings emerged from the data, with the norms typifying group 1 teachers falling into what Yackel and Cobb (1996) call socio-mathematical norms. Those of group 2 teachers appeared more social than mathematical.

Classroom Norms		
	Group 1: *Sociomathematical norms*	Group 2: *Social norms*
Structural classroom norms	Quick pace to improve mental agility Flexible lesson structure	Quick pace to improve lesson & amount of material covered. Nonflexible *(do as instructed)* lesson structure
Cognitive class norms	Awareness of learning theories Affordance & constraints of thinking time	Awareness of rhetorical learning experience Superficial time offered to children to think
Attitudinal classroom norms	Enjoyment of mathematics: *through enjoyable, challenging/interesting tasks, games, the recall of knowledge facts and relationships between numbers and shapes.* Achievement (*understanding).* High expectations of all children.	Enjoyment of mathematics lessons: *children enjoy mathematics through laughing together* Achievement (*accomplishment*) of content covered. Low expectations of children from low-socio-economic families

Table 3: Classroom norms recorded

The details of table 3 provide further case study evidence of distinct groups of teachers of mathematics as defined by their espoused and enacted beliefs. This is in accordance with previous case studies, e.g. Twisleton (2002) and Palmér (2012). that have also shown the extent to which a child's opportunity to learn mathematics is determined by what teachers believe and do. Askew et al. (1997) also found different groups of teachers: discovery-, transmission- and connectionist-oriented teachers, with the latter group being designated as *effective*. Characteristics of these three elements can be seen within the two groups reported here. However, while it is not the place here to suggest that group 1 teachers are better than group 2, it is clear that they offer qualitatively different learning experiences for their children, confirming that teachers incorporate assumptions about the nature of mathematics, based on their early experiences as a learner, into a personal philosophy of mathematics with clear classroom consequences (Jennings and Greenberg, 2009).

Discussion

When viewed against typical perspectives on teacher effectiveness, both groups initially seemed indistinguishable. For example, all had good subject knowledge qualifications, which evidence suggests is a characteristic of teacher quality (Rowland and Ruthven, 2011). But the data above seem to suggest that the quality of subject knowledge alone is no predictor of teacher belief or practice. At the espoused level, all teachers exploited the same mathematical pedagogical vocabulary. For example they explicitly mentioned the use of mathematical vocabulary, children's prior knowledge, connections made and rich tasks used, which evidence suggests are crucial to effective mathematical teaching (Petrou and Goulding, 2011). But the data above suggest that the interview articulation of professional vocabulary is not evidence of its use in practice. When general pedagogical approaches to teaching and learning were discussed, all teachers were again in agreement, as in, for example, the use of discussion, questioning, differentiation and choice of resources are thought to be facilitators of learning (Alexander, 2009; Moyer, 2001). But the data above suggest different interpretations. Finally, all teachers espoused high expectations in relation to pupils' achievement and engagement in mathematics, another characteristic of good teaching (McKown and Weinstein, 2008). But the data revealed very different manifestations of such expectations in practice.

All teachers talked in ways that would find approval in the literature as being autonomous teachers, just as others construed them locally. However, Lawson (2004) cautioned us that 'the appeal of teacher autonomy… must be tempered by the recognition that it has the ability to both liberate and deceive' (p. 3). The dichotomisation of these six teachers presents a problematic picture. On the one hand, we see the teachers of group one whose espoused beliefs resonate closely to their practice and expectations of effective teaching found in the literature. On the other hand, we see the teachers of group 2 whose beliefs are not only incommensurate with their practice, but also unlikely to provide children with meaningful learning opportunities. In other words, it could be argued that teachers of group 1 are autonomous, whilst those of group 2 are not. In the following I revisit notions of autonomy in relation to these two groups by introducing the concept of mediation.

Theorisation and Conclusions

I construe the teachers of group 1 as mediators in that they seem able to analyse the contexts in which they work and then act in ways that remain consistent with their core beliefs. For example, they warrant their practice against critical reflections on their own personal and professional learning, frequently drawn from research readings, professional development and classroom-based experiments. They have secure subject and pedagogical content knowledge that allows them to mediate imposed initiatives with confidence and authority. They do not implement permanent changes without first trialling and evaluating them. Flexibility was seen to be a consistent attribute, where individuals' daily progress was analysed and dealt with accordingly. For example, none of these teachers saw low socio-economic factors as barriers to learning, only high expectations were observed.

I construe the teachers of group 2 as mediated in that they seem unable, and unaware, to analyse the contexts in which they work and act in ways that remain consistent with their core beliefs. For example, their approach to mathematics teaching and learning draws more on their core beliefs derived from their own early experiences of learning rather than their training or professional-development. They espoused discussion but actioned only teacher question and answer episodes. Classroom norms exhibited a focus on having fun, but this was through social enjoyment rather than in the mathematics. Mediated teachers relied upon their senior manager to inform them of new initiatives, implying a dependent behaviour and a low-level professional identity. They discussed the demands of increasing paperwork and accountability, but offered few strategies that would help them manage these demands. Mediated teachers had low expectations of pupils from low socio-economic backgrounds, consistently presenting low level learning opportunities.

In sum, although increasing paperwork and accountability were viewed as continuing workplace frustrations for all teachers, clear strategies for handling these were articulated by the mediators only, indicating a high level of professional resilience. They operated at a high level of self-awareness, a component of a strong professional identity, having an ability to recognise and reconcile conflicting beliefs about themselves, their role and their environment. Whereas mediated teachers operated at minimal levels of self-awareness, demonstrating weak professional identities. Where constraints were seen as starting points for structuring their approaches, whereas mediators viewed these as barriers to be negotiated in their teaching of mathematics. Therefore, the notion of autonomy is insufficient to characterise a good teacher of mathematics.

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Reading strategies in mathematics: a Swedish example

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Recent research shows that the dominant practice in mathematics education in Sweden involves students working individually from a textbook. However, to read mathematical texts involves comprehending the global meaning from the page and this requires specific reading skills. In this study, the reading strategies of six 10-year old students, with different levels of mathematical achievement, are identified. The analysis is based on Palinscar and Brown's reciprocal activities prediction, clarification and summarisation and Halliday's Systemic Functional Linguistics (SFL). In this small study, high achieving students more often described that they used appropriate reading strategies.

Keywords: mathematics textbooks, reading strategies, primary school

Introduction

In Sweden, mathematics students spend a large portion of their time working individually with textbooks, with little involvement from teachers except to answer questions about an exercise (Johansson, 2006). Consequently, the textbook influences how students learn and apply mathematical concepts (Bryant et al., 2008). However, mathematics textbooks are generally written in a short dense style with few contextual clues to help decode, for example, the meaning of specialised words (Adams, 2003). As multimodal texts, textbooks require students to synthesise complex meaning-making systems, through identifying the separate meanings and purposes of, for example, charts and written texts, and recognising how these separate purposes combine (Carter and Dean, 2006). Consequently, special reading skills are needed (Carter and Dean, 2006). However, not all students possess these skills and some struggle to gain anything from their textbooks except the location of the exercises (Weinberg and Wiesner, 2011).

To comprehend a mathematics textbook requires students to activate their prior-knowledge, comprehend vocabulary, symbols, and visual images and use strategies to monitor their reading (Carter and Dean, 2006). Identifying the main idea in a particular paragraph or section helps students to make connections between their prior-knowledge and the new information being learnt (Carter and Dean, 2006). As well, understanding the specialist mathematics vocabulary and the concepts behind them support making links to new knowledge (Lee, 2006). However, some mathematical words having different meanings in conversational language (Adams, 2003; Lee, 2006), such as odd and volume, and this may hinder rather than support appropriate connections being made.

Symbols, such as those for mathematical operations, can efficiently tell the learner what to do (Adams, 2003). However, Österholm's (2006) study with Year 12 and university students showed that students concentrated on the operative meaning and not on the semantic role of the symbols. By focussing on the numbers without connecting them to the textual meaning, many students were led to using the wrong mathematical operations.

Visual images are thought to make concepts easier to learn and more accessible. If students cannot make a connection between written text and visuals, they might fail to construct the intended meaning from the illustrations (Noonan, 1990). In textbooks, visual material or graphics can be categorised as decorative, related but non-essential, or essential. Decorative material makes the page more attractive, but serves no instructional purpose. Related, but non-essential, material repeats ideas already provided in words and can provide another entry point into understanding the textbook. Essential graphic material can be graphs and tables that is referred to, but not repeated in the text. This graphic material must be 'read' along with the rest of the text and cannot be ignored (Noonan, 1990).

Previous research on students' reading strategies in mathematics have mostly concentrated on investigating the result after implementing different reading strategies, such as Borasi et al.'s (1998) study. However, given that so many Swedish students are expected to work individually in their textbook, it is surprising to find that apart from Österholm's (2006) study there is no information about the reading strategies that they use in classrooms. As a result, the aim of this pilot study is to identify the reading strategies that Year 3 students use. The two research questions were: "What kind of reading strategies do Year 3 students use when they approach a page in a Year 4 mathematics textbook?" and "What kind of comprehension problems do students indicate that they have when reading this page?"

Systemic functional linguistics (SFL) is used as a methodological tool in combination with the reciprocal activities of prediction, clarification and summarisation (Palinscar and Brown, 1984) to view the students' different reading strategies. In an earlier study (Ebbelind and Segerby, 2014 forthcoming), SFL was used to identify the potential difficulties that students face when interpreting a textbook page. SFL provides information about how textbook context, classroom context and social context can have an impact on students' reading strategies. However, because SFL is used to analyse texts, it could not be used by itself to describe specific reading strategies. Consequently, Palinscar and Brown's (1984) reciprocal strategies are used to describe the strategies, but these are linked to SFL. In this way, a more complete understanding of the connection between the textbook page and how it is likely to be interpreted can be made.

Systemic Functional Linguistics (SFL)

Halliday described the relationship between people's linguistic interactions and the kinds of meaning that can be realised. Halliday argued that texts are developed through the context of situation which is surrounded by the context of culture (Halliday and Hasan, 1985). A situation is an instance of culture and culture lies behind the different types of situation that can occur (Halliday and Hasan, 1985). In this study, the context of culture is Swedish mathematics classrooms in which learning is expected to occur as a result of individual work in textbooks. The context of situation is the interaction between individual students and the textbook pages.

According to Halliday (Halliday and Hasan, 1985), a text is any instance of a living language that plays some part in the context of a specific situation. In this research the text in focus is the textbook pages but in relation to how students interpret the textbook. The text, therefore, is not just what is on the page but the interaction that occurs between it and the students as they try to make sense of it. Every text is about something and is constituted by the field, involves participants, constituted by the tenor, and is based on the text structure, constituted by the mode.

The field of the discourse describes the events that are taking place in the context of situation and are realised through naming aspects of the event. In relation to interpreting the textbook, it is about the content that is introduced on the page. Thus it was important to understand how the content of the page affect students' interactions with it.

Who is taking part, the status and roles of the participants, the authors of the page and the students, is constituted by the tenor (Halliday and Hasan, 1985). In this study, the authors of the textbook have a different role to that of the students and that of their teacher, who for a large part is invisible in the interaction, even if physically present in the classroom in which the interaction takes part. These roles were likely to affect the interaction.

The mode refers to the role that language plays and what the participants are expecting the language to do for them in that situation (Halliday and Hasan, 1985). It is about how coherence is achieved through synthesising the different modes such as text, symbols and illustration. Given that the textbook page combines illustrations with words and symbols, it was important to find out how the students integrated them together to gain meaning from the page.

In an earlier study (Ebbelind and Segerby, 2014 forthcoming), an analysis of a textbook page using SFL found that there were a number of potential difficulties for students in interpreting textbooks.

In this study SFL is combined with the reciprocal activities of prediction, clarification and summarisation (Palinscar and Brown, 1984) to identify students' reading strategies.

Reciprocal activities

Palinscar and Brown (1984) described four activities, which were considered to activate relevant background information, to help readers comprehend a written text.

Prediction concerns students predicting future content and drawing and testing inferences. It considers how students make sense of different components, words, picture, symbols, and so is about SFL's mode. However as it also refers to what is happening in the content of situation of the textbook page, there is a relationship with SFL's field.

Clarification requires students to engage in critical evaluation. It is connected with the coherence of words/phrases, symbols and visual representations and, like prediction, draws upon understandings both of the field and mode.

Questioning involves students composing questions on the content to see if they have understood it. However, in this study this reciprocal activity is not included as the study is not about students' learning but about the reading strategies that they use before any intervention is made.

Summarisation is related to identifying the major content and determining what the page is about. Thus, it is connected to SFL's field. It also connected to the mode as students will need to be able to extract meaning from synthesising the pictures with the words, symbols and pictures.

Method

Interviews, based on an interview guide, were used to ascertain what students considered as they approached the textbook page. The interviews were video- and audio-tape recorded. Table 1 shows the connections between the interview questions, the reciprocal activities and Systemic Functional Linguistics (SFL). The questions focused on the field as they mostly referred to what was happening on the page.

Table 1. The questions related to reciprocal activities and SFL (System Functional Linguistics)

Questions:	**Reciprocal activities**	**SFL**
1. What is the first thing you look at on the page?	Prediction	Field
2. How can you find out what the page is about?	Prediction	Field
3. What is a digit and what is a number?	Clarification	Field
4. What is the function of the information box and what does it tell you?	Summarisation	Field and Mode
5. If you constructed a page in mathematics textbook, how would it look and why?	Summarisation	Field and Mode

Six Year 3 students (10 years old) were interviewed individually about how they approached page 12, in the widely-used, Swedish, Year 4 mathematics textbook *Matte Direkt Borgen* (Falck, Picetti and Sundin, 2011), which was about digits, numbers and place value. The page is similar to many other pages in Swedish textbooks in its layout. At the top of the page is a heading "Siffror och tal" (digits and numbers) and then there is an information box that describes the content, followed by five exercises. The text in the information box states: "When we write numbers, we use one or more digits. Our number system contains the digits: 0 1 2 3 4 5 6 7 8 9". This is followed by "The digits in a number have different values" and an illustration showing the different values of the digits of number 2345 (2 in 2345 is worth 2000, digit 3 is worth 300, etc.)". The information box also contains an illustration of a boy holding a board with the number 2345. A speech bubble says "How much is digit four worth here?", the boy's hand points at digit five which is the units digit. The girl answers "Forty!" and her left hand is raised, showing five fingers.

Each exercise on the page has three questions, except the fifth exercise. The first exercise is "How many digits are there in this number?" The two following exercises have the question "How much is the digit worth in this number?" The fourth exercise is "Write a number with four different digits" and the fifth and last exercise on the page is "Moa has one note worth a thousand kroner and another note worth fifty kroner. Can she buy the coat?" Connected to the exercise is an illustration of a coat with a price tag of 1498 kroner.

The teacher was asked to choose two students, considered as high achievers in mathematics (A1 and A2), two as middle achievers (B1 and B2) and two as low achievers (C1 and C2). It seemed valuable to see if the reading strategies of students with different achievement levels in mathematics could be compared. Teachers' use of subjective assessments to identify students' achievement levels can be problematic. However, for a small pilot study, such an approach seemed appropriate.

Results and Discussion

Prediction

The first two interview questions were about the first thing that the student looked at on the page and how they found out what the page was about. They were thus about predicting and connected to SFL's field, as they focused on what was happening on the page.

Four (A2, B1, B2, C2) of the six students first looked at the picture of the coat in the bottom right corner of the page, connected to exercise 5. A1 looked at the heading "Siffror och tal" (Digits and numbers) and C1 focused on the "strange numbers" (2345) on the board, held by the boy in the information box. As the pictures caught almost all the students' eyes, they seemed to have an important role. However, the pictures repeated the ideas expressed in the written text so they are related, but non-essential material (Noonan, 1990). However, it is not clear that the students recognised that the pictures were duplicating written material, as their initial viewing of the page did not suggest that they saw the pictures as being connected to the written text, as can be seen in the answers to the following question.

The students gave different answers as to how they could find out what the page was about:

> A1: Look at the headline in order to know that to do and then in the information box.
>
> A2: Look around in the text (pointed at headline, information box, text in the tasks)

Both of the high-achieving students (A1 and A2) looked at the main heading and at the information box. On this page, the heading is important because it identified the main idea that there was a relationship between digits and numbers (siffor och tal). The high achieving students stated that they had come to understand the value of the heading, by themselves, as they could not remember being explicitly taught about it during their mathematics lessons.

Three students (B1, B2 and C2) focused on the numbers (symbols), a similar strategy to those of the older students in Österholm's study (2006).

> B1: Maybe it is about how much money you will get back (pointed at the picture in the information box). I do not know how to find out what the page is about.
>
> B2: I see what it costs (points at the coat) and that they put numbers under each other (pointed at the information box) digits under each other. I think that it is about training to set up numbers (operations).
>
> C2: If there is a lot of numbers like on this page I start to look at the numbers in the first task.

C1 did not seem to know how to figure out what the page was about except that it was about mathematics.

> C1: I think that it is about maths and if there is some maths.

The predicting strategies that the students used were closely connected to the pictures but it did not seem that they understood that the pictures were replicating information also provided in the text. Only the high achieving students used the heading and information box to find out what the page was about. Yet this information can help students activate their prior knowledge in order to learn new knowledge.

Clarification

In relationship to this textbook page, clarification was linked to students' definitions of the important mathematical words, digit and number. The students were asked to read the page and identify if there were any words they did not understand. None of the students said they had found any words that they did not understand. I then asked them to describe what a digit and what a number (Siffor och tal) were, as these two words made up the heading.

Three students described a digit and number appropriately (A2, B1 and C1). The rest thought that digit and number were the same thing and related to two-digit numbers, such as 22. Four students defined numbers as mathematics tasks (A1, A2, B1 and C2). For example, B1 said "*such as 21 – 5 =*", indicating a task based on an operation. In Sweden, the word "tal", number, is often used colloquially in mathematics classrooms to mean a task, such as "This week you are going to solve twenty numbers (tasks) in the textbook" and "Can you solve the number 5 x 3=". It is perhaps not surprising that some students, independent of level of achievement, could not define digit correctly and did not realise that they did not understand. The information on the page did not seem to help them to learn the correct meaning. Yet, for students to comprehend a page, understanding the meaning of the mathematical words is essential (Adams, 2003; Carter and Dean, 2006; Lee, 2006).

Summarisation

The interview questions about summarisation focused on students' thoughts about the information box and how a textbook page should be constructed. This strategy is related to SFL's field, about the naming of objects, and the mode, about the text's coherence and the connection between the pictures and the text.

All of the students thought that the information box was important. Two of them could not explain why (B1 and C2), while the others stated that it explained what to do. It is interesting to note that they did not see it as providing information about what they were to learn about the topic but rather what they had to do. When the students were asked to identify what the information box told them, the two high achieving students (A1 and A2) read both the text and looked at the picture in the information box and said that it was about numbers, digits and place value. They found it hard to connect the picture of the boy with the sign to the textual information, possibly because the words and the hand signals of both the boy and the girl do not match. The two high achieving students tried unsuccessfully to explain what the picture was meant to convey and became frustrated. This suggests that for these students the different modes, the pictures and the words, were not producing a coherent text (Halliday and Hasan, 1985) from which the students could construct meaningful mathematics. The lack of coherence seemed to affect other students in other ways.

B1 only read the speech bubbles and said that the digit 4 meant 40. B2 read the three first lines in the box and said, "You only use 1 to 9 to write every number. I have never thought about that." C1 looked at the picture where the number 2345 has been split into its composite values and said, "5 is not as much as 40, 40 is not as much as 300." C2 responded similarly to B2 and only read the first three sentences in the information box. She stated:

> 1 has less value than 2 and 6 has more value than 3, 9 has a very giant value compared with 0. Because 0 is nothing but 9 is really high.

C2 only referred to digits and their value and did not seem to understand that the information box was about place value (talvärde), where the digits gain their value from the position that they occupy in the number.

When the students were asked to describe how they would construct a textbook page, all of the students suggested that the page would contain multiplications tasks, which was the area they had currently worked on. Three of the students (A1, A2 and C2) also suggested addition tasks and B1 suggested some subtraction tasks.

C1 and B2 thought that at the beginning the tasks should be easy and at the end become harder. B2 suggested that there should be reading tasks at the bottom of the page. However, only one of the students (B1) suggested that a description of what to do should be included. Nothing was said about what they should learn.

Four of the students wanted pictures on the page but for different purposes. One purpose was to be a decoration (Noonan, 1990) such as an animal at the bottom of the page (B1) and a car in the beginning of the page student (C2).

Another purpose was as a repetition of the idea being expressed in the text (A2 and C1), which would make it related but non-essential information (Noonan, 1990). C1 said that pictures can help you to solve the tasks if you cannot read. On the other hand A2 stated that he thought that some pictures were confusing. He pointed at a picture on the next page in the textbook and said, “Like this picture, with the dragon. I do not understand it and I do not like it”. In the picture, the number 2908 was written in large digits above a dragon’s head and the number 6890 was written in small digits underneath the dragon. The purpose of that page was to order numbers. However, it was clear that rather than supporting the student’s understanding, like the picture of the boy with the board, the picture caused confusion and frustration.

In summary, it would seem that the purpose of an information box and how to read it is not clear to all the students. Two students could not explain the purpose of the information box. Only the high achieving students used both the pictures and the text to understand the textbook page. The other students focused on different parts of the information box and did not seem to be able to connect the parts into a coherent whole. Valuable information was missed such as the description of numbers and digits and the relationship between them. Children’s confusion over the role of pictures was evident in their suggestions for how a page should be constructed, with some just wanting decorative pictures. Only two of them wanted to have pictures that were related but non-essential to the text.

Conclusion

The importance of the textbook in Swedish mathematics teaching means that understanding how children read a textbook page is valuable information. From this small study, it would seem that some students have been able to work out how to obtain more meaning from the page than their peers. A larger study is needed, to determine whether the suggestion of a correlation between mathematics achievement and an ability to decipher a textbook page found in this study can be seen in the wider population of Year 4 students.

Field: It was clear that not all students knew how to find out about the main ideas. Only some of the students could define “digit” and “number” appropriately. The context of culture of Swedish mathematics classrooms interfered with the students recognising that ‘tal’ did not mean exercise in this context of situation. Without being able to identify the main idea the students’ relevant background information may not be activated, thus impeding their comprehension. It may also be that the students who were confused did not have relevant background information.

Tenor: From the students’ perspective, the tasks on the page position them as doers that perform actions and not thinkers that reflect, describe or explain their thinking. This also had an impact on the field as their expectation as doers was that they would not read about what they should learn but what they would do in the exercises.

Mode: The pictures seemed to play an essential role because all the students looked at them first when they approached the page. However, the pictures did not always support the students’ understanding but instead caused confusion. It was also clear that many of the students were not able to integrate the pictures with the written text to create a coherent whole and thus could not make sense of the information that they were expected to learn.

The high achieving students had more successful reading strategies than the others students in the study. However, the study also showed that the high achieving students needed guidance in understanding the mathematical words connected to the main ideas. It is likely that not being able to locate and understand the main ideas expressed in the textbook will affect students' learning. However, a larger study is needed to investigate that further.

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Counting difficulties for students with dyslexia

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Many studies have examined counting skills in young children with language-related disabilities but few studies have examined counting skills in older children with these disabilities. In this study I examined the counting skills of fifteen 9 to10-year-old students with dyslexia. During an individual clinical interview students worked on an object counting task, counting by tens tasks, and word problems. Video analysis of these tasks revealed that twelve students made errors on the counting tasks and that these counting difficulties impacted the students' abilities to complete the word problems accurately. Even in upper primary school students with dyslexia have difficulties with counting and these difficulties with counting impact their abilities to accurately solve more complex mathematical problems.

Keywords: dyslexia, counting, assessment

Introduction

Numerous studies have shown that learning the counting sequence is particularly difficult for young students with dyslexia and other language impairments (Donlan, Cowan, Newton and Lloyd, 2007; Fazio, 1996). An impaired counting sequence may make it difficult for these students to do more complex arithmetic because they have to focus too much attention on counting and do not have enough mental resources to devote to problem-solving or finding efficient strategies (Dowker, 2005). In this study, I examine whether counting difficulties for students with dyslexia continue into upper primary school and how this impacts their ability to solve mathematical word problems.

In this study I use the British Dyslexia Association's 2007 definition of dyslexia,

> Dyslexia is a specific learning difficulty that mainly affects the developments of literacy and language related skills. It is likely to be present at birth and to be life-long in its effect. It is characterised by difficulties with phonological processing, rapid naming, working memory, processing speed, and the automatic development of skills that may not match up to an individual's other cognitive abilities (http://www.bdadyslexia.org.uk/about-dyslexia/further-information/dyslexia-research-information-.html).

This definition includes students who have difficulties reading, writing, and/or processing oral language.

Following a brief summary of relevant research literature and the research methods, I investigate the counting skills of students with dyslexia and factors that adversely affect their counting skills. I then investigate how their counting skills affect their ability to complete word problems accurately.

Literature review

Learning and Assessment

Before we teach mathematics to students we need to know what they already understand about the subject (Allsopp, Kyger and Lovin, 2007), and this requires us to assess their knowledge. Most standardised assessments assess what the students already know independently, or their zone of actual development. However, Vygotsky (1978) posited that in order to assess for instructional purposes we should not assess what ideas have already developed but assess which ideas are in the process of development. He proposed the idea of the zone of proximal development as the "distance between actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance" (p. 86). Determining the zone of proximal development allows us to find out which ideas are in the process of maturing and are therefore in the ideal space for learning, whereas tests that measure the zone of actual development measure what has already been taught.

Ginsburg (1997) proposed clinical interviews as a powerful tool for assessing students and determining a student's zone of proximal development because, unlike in standardised or other written tests, it is possible to scaffold the children's understanding during a clinical interview. The main purpose of a clinical interview is to understand the thinking underlying the student's responses to a standard task. Therefore the interviewer can check that the student understands the question, ask questions that prompt the student to rethink their answer, and adjust the difficulty of the questions to match the children's understanding. As an experimental method the interview method has a good history of replication, in that any competent interviewer can obtain similar results.

The clinical interview is particularly appropriate for students with disabilities because these students may have different understandings of the problems and numbers than most students, and the clinical interview allows the interviewer to probe for these different understandings (Allsopp, Kyger and Lovin, 2007). In this study I probe the counting skills of students with dyslexia and their ability to use these skills in word problems by using a clinical interview.

Counting

Counting is fundamental to many other areas of mathematics, including: understanding the size of numbers, number sequences, patterns and place value (Franke, 2003). The National Curriculum for England (2013) acknowledges that counting is a fundamental skill in mathematics by devoting many of the statutory requirements in number and place value to counting in both years of key stage 1. Students are expected to use the correct count sequence both forwards and backwards by ones to and across 100, object count, read and write numbers to 100, and count in steps of two, three, five and ten. In this study I focus on the students' knowledge and use of the count sequence by ones and tens, and their use of one-to-one correspondence when object counting.

Counting for students with dyslexia

Most of the research on the counting skills of students with language-related disorders has focused on students in the early years of primary school. Fazio (1996) found that many 6 to 7-year-old students with language impairments have difficulties with declarative mathematical knowledge such as counting by ones or tens. Donlan, Cowan, Newton and Lloyd (2007) found that 8-year-old students with specific language impairments often had profound deficits in the production of the counting sequence, while Murphy, Mazzocco, Hanich and Early (2007) found that 8-year-old students with mathematics learning disabilities performing at the 10th percentile level or lower still had difficulties identifying counting errors.

An exception to the focus on students in lower primary is Houssart's (2001) study of the counting difficulties of Julie, a girl in year 5 (9 to 10-years-old). Julie had difficulties with extended counting, which meant that she had difficulties counting across tens and hundreds boundaries, counting forwards and backwards in different steps and from different starting numbers. These difficulties made it hard for her to be successful at various mathematics problems such as counting money, making change, measuring larger distances and subtraction of 3-digit numbers.

As students with dyslexia and language-related disorders tend to have counting difficulties in the primary grades, more research needs to be done to examine how long they continue to exhibit these difficulties and what effect counting difficulties have on their understanding of more complex mathematical concepts.

Current Study

In the current study I look at the accuracy of the counting knowledge of students with dyslexia and the factors that adversely affected their counting knowledge. I also examine how the students used their counting knowledge in solving problems in all four operations. My research questions were: (1) How accurate are students with dyslexia at object counting? (2) What are some factors that affect their accuracy in counting? (3) How do counting skills affect their accurate solution of word problems?

Methods

Participants

Fifteen 9 to 10-year-old students attending an independent school for students with dyslexia participated in this study during the 2012-2013 school year. All students at the school have been given or are in the process of receiving a diagnosis of dyslexia by a diagnostician who specialises in evaluating students with learning disabilities. This school was in the Pacific Northwest of the United States of America.

All the participants spoke English as their first language. Six of the participants were female and nine were male.

Procedures

In order to assess the students' knowledge of counting I conducted an individual clinical interview with each of the participants. The entire interview consisted of 17 questions but in this paper I only analysed the students' responses to eight of the questions. These questions were broken into two sets: one set were counting tasks and the other set consisted of word problems.

The first set of questions examined students' ability to count. First I asked the students to count 120 tiles and then represent their count (Schwedtfeger and Chan, 2007). Then students counted by tens as I placed completed tens frames on the table. There were thirteen completed tens frames so the students counted by tens to 130. Then students again counted by tens as I placed tens frames on the table but this time they started from 14 with one completed tens frame and one tens frame with only four dots. Again there were thirteen completed tens frames so they counted by tens from 14 to 134 (Wright, Martland and Stafford, 2006). I included these questions because Desoete and Grégoire (2006) found that many third grade students with mathematical learning disabilities still have difficulties with the counting sequence, and I wanted to know whether these difficulties with counting by ones and by tens continued into the fourth grade.

The next set of questions I analysed comprised five word problems. There was a Join Result Unknown (JRU), a Separate Result Unknown (SRU), a Join Change Unknown (JCU), a Grouping and a Division by Ten problem (Carpenter et al., 1999). I included these problems to see whether students' difficulties with counting impacted their ability to complete these problems accurately.

For all of the word problems, the students had a selection of manipulatives available to use if they wanted. These manipulatives included connecting cubes stacked in towers of ten, a hundreds number chart, base-ten blocks, tens frames, and coloured tiles. I read the students the word problems, asked them to retell the problem, and then asked them how to solve it.

In these interviews the initial tasks were standard across all of the participants but my follow-up questions were not, as they were designed to elicit more detailed descriptions and explanations of the students' strategies. For follow-up I asked questions such as, "How did you figure that out?", "Can you do it out loud?", "Can you show me how you did it?", "Is there another way to solve this problem?", or "How do you know?" (Ginsburg, 1997).

Data Collection & Analysis

I analysed the students' responses to the questions for accuracy and errors. This analysis helped me understand how much the students knew about counting and how they used this knowledge to solve word problems.

I video recorded the interviews. From the interview videos I transcribed 18 counts where the students made errors and 13 word problems where the students made counting errors.

When I had transcribed the counts and the word problems, I coded for student errors in the number sequence, one-to-one correspondence, units and due to attention. Having coded for these errors I further coded for whether the sequencing errors occurred between decades or above one hundred. I also coded for whether the one-to-one correspondence errors seemed related to the student's uncertainty with the number system or seemed to be more related to attentional issues.

Results

In this section I first examine how accurate these students with dyslexia were at object counting by ones and by tens. Then I examine what factors affected their skills at these tasks. Finally I examine whether their counting skills affect their ability to solve word problems accurately.

Counting accuracy

Eleven of the fifteen students made number sequencing mistakes either when counting by tens or by ones. Nine students counted the 120 tiles inaccurately, eight students made number sequencing errors, and seven made one-to-one correspondence errors. In the tens frames tasks, three students made errors when counting in tens and six students made errors when counting on in tens from fourteen. Most of these 9 to 10-year-old students with dyslexia had difficulties counting accurately.

Counting errors

When students made counting errors they made four types of errors: number sequences between decades, number sequence beyond one hundred, switching units, and attentional lapses (see Table 1). The one-to-one correspondence errors were either due to uncertainty about the number sequence or due to attentional lapses.

Two students made errors between decades. Paul[10] made decade errors when he got into the decades above fifty. In English these higher decade numbers are fairly regular, with sixty, seventy, eighty, and ninety all sounding related to the digits they contain. Paul's difficulties with these higher decade numbers suggest that he did not use the patterns in the numbers to help him remember the counting sequence. Abigail's decades errors came early in the count sequence where the decades are irregular and harder to distinguish. For example, when counting by tens from fourteen she skipped the forties decade, going straight from 34 to 54. "Thirty" and "forty" often sound similar when said by students with language delays and so often get confused for one another.

Table 1 Students' counting errors

Error	Student	How many episodes per student	Examples
Decades	Paul	3	"79, 30[11]"
	Abigail	2	"14, 24, 34, 54"
Above 100	Paul	4	"102, 104"
	Rosie	4	"109, How do you write ten hundred again?"
	Abigail	3	"one hundred [Took one block] and eleven [Took another block]"
	Luke	2	"hundred and four then plus ten more would be um: a hundred forty-four"
	Kevin	1	"one hundred [Takes one block] twelve [Takes another block]"
	Kyle	1	"114, 116"
	Lily	1	"14, 24…104, 204"
	Robbie	1	"90, 100, 120, 120"
	Robin	1	"74, 84…104, 204"
Switching Units	Francesca	2	"14, 24, 34, 35"
	Robin	1	"14, 24, 25, 26, 27"
Attention	Sam	3	"87 [Another teacher entered room], 86, 88, 87, 89" "70, seventy [Dropped tile on floor, picked it up and continued counting] seventy, 71"
	Kyle	3	

10 All names are pseudonyms.

11 "," denotes tagging or that a new tens frame was placed down.
"…" denotes that the counting sequence is accurate between the numbers that are written.

The majority of the errors occurred at or above one hundred, with nine students making eighteen errors at or above one hundred. Students made numerous errors when counting above one hundred: two students skipped one hundred, one student had difficulties counting from 102 to 107, three students had difficulties counting 110, seven students had difficulties counting in the teens above one hundred, and one student had difficulties writing 120. The counting sequence above one hundred was not secure for this population.

Both Kevin and Abigail made one-to-one correspondence errors above one hundred that they had not made earlier in the count. For example, Abigail said, “one hundred [Took one block] and eleven [Took another block]”. In this example she took two blocks but only said one number. She said the number slowly, so slowly that her hands had time to take two blocks in the time that she completed saying the number. Her attention was occupied with remembering the count sequence and she did not notice that she took two blocks.

When counting tens frames from 14, two students counted several tens frames by tens and then switched to counting the tens frames as if each tens frame counted as one. Francesca made several switches of units, she counted the first three tens frames by tens, then she counted the next tens frame as one, counted the next tens frame as a ten, and then counted the remaining tens frames as if they were ones. Robin counted, “14, 24, 25, 26, 27, wait” at which point she went back to the beginning and counted again by tens. Robin made one switch of units from counting by tens to counting by ones, but she realised her mistake and rectified it. These two students were having difficulties distinguishing between counting by tens and counting sets of ten.

Two students made sequencing errors that seemed to be more related to attentional issues than a problem with the counting sequence. Sam made sequencing errors when another teacher entered the room and when he was searching for a particular tile. Kyle made errors when he changed counting strategies or when he dropped a tile on the floor. Although these two boys made counting errors that seemed more related to attention than to knowledge of the number sequence, they both made counting errors elsewhere in the interview that suggest that their knowledge of the counting sequence was insecure.

Miscounts in word problems

Ten students made 13 counting errors when solving the word problems. All of these counting errors occurred as students direct modelled by ones or by tens, counted by ones or skip counted. Students’ difficulties with object counting by ones and tens impacted their ability to solve the word problems correctly, with 17% of the error in the word problems being due to counting errors.

Eight of these counting errors were one-to-one correspondence errors. Four of the seven students who made one-to-one correspondence errors in the word problems had not made this type of errors in the object counting task. It seems that the added burden of making sense of the word problems made it more difficult to keep track of whether they were adhering to one-to-one correspondence.

Five of the counting errors were due to number sequence errors. Paul’s miscount on the JRU was hard to interpret. Both Francesca and Sam made between decades errors when solving the word problems. Francesca counted, “27, 28, 29, that's forty” when solving the SRU problem with direct modelling by ones. In this problem Francesca skipped the thirties decade, which was not an error she had made when counting the tiles. As Sam solved the division problem by counting out 158 tiles in groups of ten he counted, “1…69, seven—eighty, 81…158”, skipping the seventies decade. Unlike during the counting task, Sam did not seem to be distracted by anything other than the task of keeping track of how many tiles there were in each group at the same time as keeping track of how far he had counted.

Lily made a different type of counting error. Lily counted, “5, 10…60, 75, 80” when counting by fives to solve the grouping problem. This error is probably due to the familiarity of the count in fives sequence. When practising multiplication facts, students normally only count up to $5 \times 12 = 60$, so the count in fives beyond 60 is probably less familiar than the counting sequence up to 60. Lily had also made a sequencing error as she counted the tens frames beyond one hundred, which is often as high as teachers ask students to count in tens. For Lily the counting sequence was secure if it was within the range that she had practised extensively.

Robin made a miscount when counting by tens above one hundred to solve the division problem. She correctly direct modelled by tens up to a hundred, putting ten sticks of ten in a box and saying, “that’s ninety, a hundred”, but then she counted out five single cubes to make 150. Robin’s counting error in this problem seems similar to the errors that she made when counting the tens frames. When counting the tens frames she switched from counting the tens frames in tens to counting each frame in ones. In this problem she switched from counting the tens sticks in tens to counting the ones in tens. Robin’s miscount was due to her switching units mid-way through the count.

Discussion

Although the expectations are that children should have mastered object counting by the end of key stage 1, this study shows that many students with dyslexia still have significant difficulties with object counting in upper primary school. In this study twelve of the fifteen students had difficulties with counting on at least one of the three types of tasks. This study confirms and expands previous research that had found that students with language-related disorders have difficulties with counting in lower primary school (Donlan, Cowan, Newton and Lloyd, 2007; Fazio, 1996; Murphy, Mazzocco, Hanich and Early, 2007).

In the counting tasks the majority of the errors these students made were with number sequences, particularly with number sequences above one hundred. Even when students made one-to-one correspondence errors most of these errors were not due to a lack of knowledge of the principles of one-to-one correspondence but because their attention was focused on the counting sequence. Two students had difficulties crossing decades, two students switched from counting by tens to counting by ones, and two students made attentional errors that seemed separate from their knowledge of the number sequence. These results confirm Houssart’s (2001) findings that even in upper primary school students with learning difficulties may have difficulties with extended counting.

In completing the word problems, counting errors accounted for 17% of the problems that the students solved incorrectly. The additional burden of making sense of a word problem made the students with dyslexia even more likely to make one-to-one correspondence errors or to make errors in the counting sequence below one hundred than when they were solving simple counting tasks. Students’ difficulties with the foundational task of counting made it difficult for them to accurately solve more complex tasks like word problems.

The implications of this study are that even in upper primary school teachers should assess students' counting skills, particularly focusing on their ability to count above one hundred and to use their counting skills in context.

The limitations of this study include a small sample size. Future studies should examine how widespread this issue is within the population of students with dyslexia and what effect counting difficulties have on problem solving and computation.

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Designing a clinical interview to assess algebraic reasoning skills

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The Irish Primary Mathematics Curriculum consists of five content strands, namely Number, Algebra, Data, Measure and Shape and Space (Government of Ireland, 1999). Children engage with material from all strands throughout their primary education. Thus the Irish education system fulfills the widespread recommendations in research of commencing algebra early (e.g. Kaput, 2008; Carpenter and Levi, 2000; Cai and Knuth, 2011). However, national and international studies of student attainment suggest that many children in Irish schools may not be developing robust skills in algebraic reasoning (Eivers et al., 2010; OECD, 2009; Eivers and Clerkin, 2011). In my research I plan to investigate to what extent children in Irish primary schools are developing skills in algebraic reasoning. In order to do so, an assessment instrument must be developed which will facilitate an exploration of children's thinking as their skills develop. The clinical interview is an instrument which allows access to children's emergent thinking (Ginsburg, 1997; Piaget, 1929) and in this paper I discuss the design of a clinical interview with specific relevance to children's skills in algebraic reasoning.

Keywords: algebraic reasoning, clinical interview, assessment, primary school.

Introduction

Kaput, Blanton and Moreno (2008) suggest that algebraic reasoning comprises the skills of "generalising; expressing generalisations, and using specialised systems of symbols to reason with the generalisations" (p. 21). Algebraic reasoning relates to children's ability to think logically about quantities (known or unknown) and the relationships between them. Carpenter and Levi (2000) define algebraic reasoning by identifying two central themes, namely "making generalisations and […] using symbols to represent mathematical ideas and to represent and solve problems" (p. 2). Algebraic reasoning skills are a precursor to children succeeding in the study of formal algebra. For formal algebra to be accessible to all children, rather than to a high-attaining minority, it is necessary for instruction in algebraic reasoning to commence early in primary school (Kaput, 1998). Additionally, while in primary school, the engagement of children in algebraic reasoning supports their understanding of the structure of number and provides tools for successful engagement in problem solving (Kaput, 1998; Schifter et al., 2008; Carpenter and Levi, 2000).

In Ireland, the recommendation to commence study in algebra early is satisfied as the 1999 Primary School Curriculum Mathematics (PSMC) advocates that the study of algebra begin in the first year of primary school (Government of Ireland 1999). In the 2003 review of the implementation of the revised Primary School Curriculum the Inspectorate of the Department of Education and Skills (Government of Ireland, 2005) reported that "[i]n the majority of classrooms there was good practice in relation to the implementation of the algebra strand" (p. 28). On the whole the inspectorate seemed satisfied with how the algebra component of the new curriculum was being taught. Considering that algebra commences early in the primary curriculum and that the inspectorate is satisfied with how it is taught, the findings of research into children's attainment in algebra are surprising. The National Assessments of Mathematics and English Reading 2009 ((NAMER 2009) (Eivers et al., 2010) assessed the mathematical attainment of children in 2nd class (aged 8 or 9 years) and 6th class (aged 12 or 13 years). The research found that 90% of pupils in the second class cohort did not "show understanding of the associative property of addition; the connection between two-step word problems and their corresponding numerical expressions; and the correct use of the symbols =, <, >" (p. 39). An examination of the sixth class cohort of NAMER 2009 shows that only 10% of pupils in this study achieved a mastery of algebra that enabled them to evaluate "linear expressions and one-step equations" (ibid. p. 42). NAMER 2009 utilised a set protocol of questions to assess children and is thus limited in identifying whether children used algebraic reasoning in arriving at solutions to test items. The findings are thus taken as an approximation of the level of skill development in algebraic reasoning of the population of sixth class pupils in Irish primary schools.

The purpose of my research is to examine to what extent children are developing skills of algebraic reasoning as they progress through primary school in Ireland. To this end, I am aiming to develop a clinical interview based upon a framework of growth points. The framework of growth points is structured around five broadly defined developmental steps, incorporating interim learning trajectories, to monitor children's skills as they develop proficiency (Twohill, 2013). This discussion paper outlines the theoretical perspective for the clinical interview design and also the relevance of a clinical interview as an assessment instrument in the area of algebraic reasoning.

Theoretical perspective

In order to assess and design assessments, it is imperative to have a clear idea of what constitutes mathematical proficiency. Milgram (2007) discusses the difficulty encountered in attempting to clinically define 'mathematics' and suggests instead a description of the most important characteristics of mathematics, namely "precision" and "stating well-posed problems and solving them" (p. 33). In the same volume, Schoenfeld (2007) suggests that the key element of proficiency in a subject is the ability "to use it in the appropriate circumstances" (p. 59). While Schoenfeld and Milgram adopt differing perspectives on defining mathematical proficiency, their assertions complement each other. Thus, underlying this paper is a theory of assessment as a measure of a participant's ability to apply mathematical knowledge and skills appropriately in solving new problems while always retaining a focus on precision.

Kaput (1998) discusses the development of algebraic reasoning and considers the need to nurture and encourage the 'roots of algebraic reasoning' over the primary school years. Mason (2008) identifies skills of algebraic reasoning which are evident in very young children as imagining and expressing, focusing and de-focusing, specialising and generalising, conjecturing and convincing, classifying and characterising. The immature skills of young children as highlighted by Mason (2008) develop over a length of time into strong broadly applicable skills of algebraic reasoning. To assess algebraic reasoning during this time requires an understanding that the skills under assessment are in development. In discussing the tools and artefacts of assessment, Smagorinsky (1995) warns against presuming that evidence of children's thinking represents a "cystallized, fully formed state of development independent of the artefact's cultural significance and the means through which the learner has appropriated an understanding of how to produce it" (p. 199). In designing an assessment, it may be preferable to engage participants in non-routine tasks that are not derivatives of a rote-learned approach.

Children, whose skills are emergent, may present with a variety of skills while complete solution of a problem remains beyond the range of their ability (Radford, 2012). When operating within their Zone of Proximal Development (ZPD), children are capable of demonstrating skills which they are currently developing (Vygotsky, 1978). In assessing algebraic reasoning skills therefore, an assessment instrument is required which will facilitate children in engaging with tasks within the highest cognitive ranges of their ZPD and also facilitate the researcher in observing, as far as is possible, the mental processes underlying the approach each participant adopts. In this paper, I aim to discuss the characteristics of the clinical interview and how it may be suited to the assessment of young children's skills in algebraic reasoning.

Assessing algebraic reasoning skills

The development of algebraic reasoning skills involves more than the rote-learning of routine algorithms or processes. To reason algebraically, it is necessary for children to develop appropriate strategies and habits of mind (Cai and Knuth, 2011). Kieran (2007) discusses the shift in thinking which is required for students to progress from a purely arithmetical approach to developing algebraic thinking. Among other skills, she suggests that it is necessary to focus on relationships and not simply on the calculation of an answer. In considering methods for assessing children's algebraic reasoning, it is necessary to develop a strategy for assessing such constituent elements as foci, skills and habits of mind. The clinical interview creates an opportunity to assess the reasoning that underlies decisions a participant makes in the solution of tasks (Ginsburg, 1997). Van de Walle (2004) suggests that algebraic reasoning is more pervasive than the specific skills of pattern spotting, equation solving or variable use and as such requires assessment across many areas of mathematics. Students with strong skills in algebraic reasoning may seek to identify generalisations in many areas of mathematics while children with less robust skills might experience difficulty in identifying and expressing relationships (ibid.).

There is an inherent challenge in skills-based assessments of algebraic thinking, particularly among young children. It is not sufficient to score participants on the basis of correct or incorrect answers, but rather assessments are required to incorporate questions that afford insight into how the participant is thinking. Participants may arrive at a correct answer in an item that purports to assess algebra without applying algebraic reasoning. For example, Osta and Labban (2007) found that there was a need to alter the test question in their study of seventh graders' pre-algebraic problem solving strategies. In the solution of a numerical sentence with an unknown, children reverted, when possible, to a trial-and-error application of arithmetic, rather than adopting an algebraic approach. In interpreting the findings of research into Irish primary school teachers' mathematical knowledge for teaching Delaney (2010) expressed concern regarding findings in the area of algebra. There existed a possibility that in solving a question designed to assess algebraic skills, some teachers may have adopted an arithmetical approach that was sufficient to solve the problem correctly.

Equally, as suggested by Van de Walle (2004), in problem-solving items that are not designed to assess algebraic reasoning skills specifically, some children may utilise an algebraic approach but their doing so may not be identified by the assessment. An alternative to a standardised assessment is that of the clinical interview. Ginsburg et al. (1983) assert that "to establish different aspects of competence, it is useful to use flexible, nonstandardised procedures" (p. 14). The clinical interview method affords a researcher insight into the level of cognitive competence at which the child is operating (ibid.). In the following section I will discuss the clinical interview and why I consider it to be a more appropriate form of assessment of algebraic reasoning skills.

The clinical interview

Piaget (1929) presents a view that assessments with a set protocol of questions are destined to skew the possible information about the subject. Piaget suggests that within assessment there are occasions when a fixed questionnaire is not an appropriate instrument as it may yield insufficient information regarding the internal reasoning of the participant. In solving a task that is purporting to assess algebraic reasoning, a student may use trial and error or purely computational strategies (Osta and Labban, 2007; Delaney, 2010). Piaget's clinical method "claims to unite what is most expedient in the methods of test and of direct observation, while avoiding their respective disadvantages" (p. 19). Utilising a clinical method of assessment, the researcher allows the child to guide the direction of the assessment, while constantly maintaining the focus on the area of research. Rather than a single linear sequence of tasks, the clinical interview consists of a number of pathways, along which the researcher will guide the participant depending on his/her reaction to each item. In designing a clinical interview to assess algebraic reasoning skills, I intend to include, among others, patterns presented in Radford (2011) and Cooper and Warren (2011). Tasks will include extending the pattern to the following term, describing the pattern and also explaining how the pattern will continue to a far term. If the participant fails to complete a task, the researcher will choose a subsequent task based upon the nature of the participant's error, creating a scenario similar to a "choose your own adventure" story (Clarke, 2013). The methods involved in the administration of a clinical interview are sensitive to the child's interests and concerns, are interpretive in nature and involve an ethic of caring (Ginsburg, 1997).

Zone of proximal development

Vygotsky (1978) discusses the Zone of Proximal Development (ZPD) within the context of assessment and advises examiners to remain cognisant of the exact focus of assessment and of what assessments inform us about children. Without support from a teacher or more able peer children may only complete tasks for which they have already acquired the requisite knowledge and skills. The ZPD deals with the knowledge and skills that the children are in the process of acquiring, which may be as yet not fully formed but are in gestation (ibid.). It is very probable among children in primary school that algebraic reasoning skills will be partially developed and children's failure to complete a task independently may not offer a true reflection of their algebraic reasoning ability. In solving a task therefore each participant will be supported in progressing to the most challenging task that he/she is capable of solving with mediation. Whilst it is necessary to maintain a differentiation between the tasks that a child completed independently and those completed with assistance, the supported successes should also be recorded. Vygotsky (1978) asserts that there is a distinction between tasks of which the child is capable with support and those that are beyond his/her ZPD. It may be necessary for task design, researcher-subject interaction and analysis to reflect the continuum of independent success to supported success to unfulfilled task. In analysing the performance of children on the clinical interview, there is a necessity to examine the level of mediation required in order to complete a task.

If the aim of an adult-child interaction is to work within the child's ZPD it is necessary to pay attention to the balance of autonomy within the researcher-child dynamic (Jordan, 2004). Jordan contends that in order for teachers to gain insight into how young children are thinking, it is necessary that there is a co-construction of understanding between the two parties. Co-construction occurs when adult and child are "interpreting and understanding activities and observations as they interact with each other" (p. 33). It is necessary therefore that the activity be meaningful and that the child and adult are engaged in interpreting information in the process of acquiring information. Not only must the adult remain vigilant to the pre-existing understanding of the child but also he/she must maintain a position of shared autonomy in the activity at hand. Jordan explains that co-construction depends on the extent to which a shared understanding is developed, and that this in turn depends upon the metaphorical distance between researcher and participant and on how power is shared between them. A questioning paradigm which best supports co-construction involves questions to which the adult does not have ready access to the answer, where silence is allowed and the child's lead is followed. In designing a clinical interview where the aim is to uncover a child's thinking at the highest cognitive level within his/her ZPD, questioning and progress of the interview will be mandated by the necessity of co-constructing understanding.

Mediation

Smagorinsky (1995) discusses Vygotsky's assertion that assessment when positioned after learning produces very limited and often misleading information regarding a participant's ability. In order to ascertain current skill-level within a field, it is preferable to engage the participant in a learning activity positioned within the upper cognitive range of his ZPD. While the clinical interview underpinning my research will be an assessment rather than a teaching activity, the tasks presented to the students should be sufficiently challenging so as to facilitate the child's learning. Indeed, Schoenfeld (2007) proposes that participation in an assessment should always involve learning. Smagorinsky (1995) concurs and adds that rather than viewing a researcher as potentially contaminating an assessment by his/her presence, it is more beneficial to view a researcher as a mediator in the participant's operating at the upper end of his/her ZPD.

Replicability of findings from a clinical interview

In discussing the use of the clinical interview in assessing children, it is pertinent to mention concerns regarding the level of finesse required in administering this assessment instrument. Piaget (1929) suggests that only after a year of daily use of the method may practitioners move beyond "the inevitable fumbling stage of the beginner" (p. 20), which for teachers may involve excessive talk and suggesting answers. Piaget cautions that there is a challenge involved in striking the balance between preconceived ideas that may skew the direction of the interview and approaching the interview with too little knowledge about the subject matter to form a reasonable hypothesis for research. Equally, it is imperative to adopt a balanced approach to the utterances of the child, which may express factors extraneous to algebraic reasoning, such as a willingness to impress, an inclusion of spontaneous imagination, or fatigue. Piaget (1929) cautions that the greatest risk to the clinical interview is the researcher who considers every utterance of the child as either 'gold' or 'dross' and thereby the findings are rendered relatively meaningless. There is sophistication required in the approach taken to consider and analyse the findings, incorporating information regarding the child's temperament and mind-set at the time of interview.

In order to assess fully children's proficiency in their emergent skills of algebraic reasoning, it is preferable that children be facilitated in engaging with tasks within their ZPD and to that end, mediation is required. There are implications however for the reliability and validity of research when the role of the researcher is so embedded in the research design, in design of tasks, mediation during assessment and analysis of results. The social background of the researcher and his/her understanding of how children think will play a powerful role in the results obtained. Smagorinsky discusses the implications of mediation and cautions that "higher mental processes are culturally shaped rather than universal in nature" (p. 203). He suggests therefore that assessments of children's ability will in effect measure the extent to which the participants' high mental processes mirror that of the researcher. Some highly developed cognitive strengths may be undetected such as those identified by Moll and Greenburg (1990) in their study of Latino students who displayed great prowess in many skills but were deemed to be of low cognitive ability in school settings.

In order to maximise the replicability of results, Goldin (2000) suggests ten methodological principles for designing quality interviews. Among his suggestions are the development of explicitly described interviews and established criteria for major contingencies; deciding in advance what will be recorded and recording as much of it as possible; a robust pilot-test of the interview; and leaving a margin for compromise when appropriate. There is a need for researcher mediation to be scripted precisely preparing for all predictable contingencies and for the mediation to be recorded as much as possible in order to identify when mediation may have been inappropriate or excessive. There may be a need for the researcher to compromise and she/he should therefore be prepared to record such interactions and discuss their implications in the analysis (ibid.).

Conclusion

Among primary school children skills of algebraic reasoning are emergent and not fully developed. Skills-based assessments are optimised when the researcher has some access to the thought processes underlying the participant's solutions to tasks. Administering a clinical interview requires the researcher to apply her/his personal judgement and interpretation during the interview and rigour is required in both design and administration to maximise validity. Analysis of the data gathered can be both difficult and time-consuming. The clinical interview, however, affords participants an opportunity to engage in tasks at the highest cognitive range of their ZPD and also allows the researcher insight into the thinking underlying a participant's solutions to a task. As such, the clinical interview offers a model of assessment that will allow me to investigate to what extent children in Irish primary schools are developing algebraic reasoning skills.

Adopting the theoretical perspective outlined in this paper, I intend to pilot a clinical interview designed to assess children's algebraic reasoning skills. The sample of participants for the pilot study will be taken from a single school situated in a small rural town. The socioeconomic background of the children attending the school is mixed and sixteen participants will be selected at random from four different class levels.

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Teacher knowledge for modelling and problem solving

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This article reports on a study that has researched teacher professional learning in lesson study communities that enquired into how we might better support students develop skills in problem solving and mathematical modelling. A rationale for professional development of this type, both in in terms of its structure and focus, is presented followed by an illustrative description from the study of a typical research lesson and issues raised in the post-lesson discussion. This is used to provide insight into some of the key issues to consider in developing teacher knowledge for modelling and problem solving.

Keywords: teacher knowledge; professional learning; lesson study; modelling; problem solving

Introduction and background

It is noticeable in education that there is a convergence of national curricula around the world with common structures emerging that are particularly affected by international studies such as TIMSS and PISA. The PISA framework (OECD, 2003) that gives structure to mathematics as a domain of study is perhaps particularly influential in this regard. PISA attempts to measure student ability to

> identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage with mathematics in ways that meets the needs of that individual's life as a constructive, concerned and reflective citizen. (ibid., p. 24)

However, curricula tend to embrace epistemologies that emphasise the PISA framework's mathematical *content* areas: quantity, space and shape, change and relationships and uncertainty. The mathematics signalled by such terms is well understood by mathematics teachers around the world, who tend to focus their teaching on ensuring students' technical facility with mathematical procedures associated with these areas. Such teaching is potentially well-informed by the work of researchers who have also, until relatively recently, in the main, prioritised research into children's understanding of key mathematical concepts (for example see Watson, Jones and Pratt, 2013). Issues surrounding the teaching and learning of problem solving skills and modelling are much less well understood. Although in the international mathematics education community there is a strong and active group that is concerned with the development of mathematical modelling and its applications (ICTMA: the international community of teachers of mathematical modelling and applications), this contributes a relatively small proportion of the output of the active mathematics education research community. (For an overview of current areas of concern and focus, see the summary of recent research activity of the international group for the Psychology of Mathematics Education (Matos, 2013).) What is known about how students learn to solve problems and how teachers might support their development in this is at a much earlier stage of development than research that, for example, seeks to explore students' understanding of mathematical concepts, the affective domain of learning, or teachers' pedagogies.

As Cai and Howson (2013) argue, there is discernible evidence of some convergence of mathematics curricula due to the international comparative studies, and there has been a noticeable increase in interest in mathematical modelling and problem solving around the world, instigated in no small part by the influence of PISA. In this paper, therefore, we focus on this important emergent area in school mathematics and report research into the development of teacher knowledge and teaching practice in relation to mathematical modelling and problem solving within a professional learning community of teachers. The research involved a sub-group of teachers from four of nine schools that worked for a year on a project enquiring into classroom practice using a lesson study model. The impetus for the project arose from earlier work[12] that had produced innovative teaching materials aimed at motivating learners with utility and purpose (Ainley, Pratt and Hansen, 2006) through working on substantial problems set in a variety of 'case study' contexts. These 'case studies' and additional assessment tasks stimulate problem-solving and modelling activities in lessons. Such activity was the focus of the research lessons within the lesson study cycle that provided the focus of the professional learning of the teachers, and indeed the researchers that worked on the project reported here.

Lesson study and theoretical perspectives

Fundamental to our research is a concept of professional learning that is focused on enquiry into teaching, learning and classroom practice. The aim of this project, therefore, was to develop professional learning communities in which teachers worked together and learned from each other. The communities were informed by 'knowledgeable others', whose role was to stimulate the community by drawing on a range of expertise that is research-informed.

The project was fortunate to be able to work with colleagues from IMPULS in Japan[13] who, working in their own culture of well-established lesson-study communities, were at the same time, in reaction to developments in the Japanese curriculum, beginning to tackle some of the same issues in relation to problem solving. Lesson study based on the Japanese model has become increasingly widely known and adapted for use across geographical and cultural boundaries since the publication of Stigler and Hiebert's book *The Teaching Gap* (1999). The model is perhaps particularly attractive as it has the potential to meet the requirements that we know facilitate effective professional learning (Joubert and Sutherland, 2009); namely, that it is:

- sustained over substantial periods of time
- collaborative within mathematics departments/teams
- informed by outside expertise
- evidence-based/research-informed
- attentive to the development of the mathematics itself.

As Doig and Groves (2012) point out, drawing on their experiences in Australia, there is a need to adapt rather than adopt the Japanese model when working in another culture. However, in our work as a community we maintained what we saw

[12] The Bowland Maths project provides teacher support, including motivating materials/tasks for use with students in classrooms that promote problem solving.
[13] IMPULS is a project funded by the Japanese government that aims to establish teacher development systems for long-term improvement in mathematics instruction.

as crucial aspects of the Japanese model. Fundamental to our model is the expertise brought to partnerships by 'knowledgeable others' and the focus on the interaction of learning with materials and the mathematical experiences and learning they generate (Lewis et al., 2006). The importance of these in the Japanese model is perhaps signified by the fact that the words used to describe them (*koshi* and *kyozaikenkyu* – Takahashi and Yoshida, 2004; Doig et al., 2012, respectively) are often left in the original Japanese in the literature, as they embody meaning that is often not well-understood outside Japan.

Our model for professional learning communities is one that draws on Beach's construct of collateral learning (Beach, 1999). This does not only place value on the usual notion of learning having to be in a constantly 'upward', that is hierarchically vertical, direction, with knowledge becoming forever more abstract and divorced from the everyday. It also recognises and values the different professional expertise that participants bring as they come together to expand the object of their activity, so that learning together is facilitated in what might be thought of as a *horizontal* direction.

In general we adopt a Cultural Historical Activity Theoretically (CHAT) informed view of the work of the different communities involved. It is not intended to give a detailed account of this here (for a more detailed overview see Wake, Foster and Swan, 2013). From this perspective, central to the work of the professional learning community is the activity system of the mathematics classroom, with teacher and pupils working together as a community in pursuit of the learning of mathematics. As Brousseau (1997) recognised, such communities are culturally and historically situated and evolved in their ways of working and can be considered to operate with a *contrat didactique* that embodies expectations of all as to what should constitute practices in such situations. Lesson study brings into the shared experience of teachers and other educators a new activity system, the lesson study group, which has as its object professional learning through inquiry into practice. Important in providing a bridge between these two activity systems is the 'lesson plan' for the research lesson that is used to identify specific aspects of teaching and learning in relation to students' problem solving. In activity theory terms we consider this document to be a boundary object (Star and Griesemer, 1989), having different meanings in the two settings yet retaining a common essence focused on student learning and teaching practices. The lesson plan provides a script with which the teacher works in the classroom. Prior to this it has been developed collaboratively by the lesson study community, and consequently provides a central focus for communication between participants, eventually coming to embody their values, understandings, beliefs and intentions. Again, in the post-lesson discussion the lesson plan as a document is of central importance in providing a mediating instrument that facilitates discussion of planned intentions and their enactment as pedagogical practices in the classroom.

In this conceptualisation of lesson study in activity theory terms we consider that professional learning takes place at the *boundaries* between the different activity systems in which community participants operate, and the lesson plan, as a *boundary object*, plays a crucial role in facilitating reflection on action and perspective making and taking (Boland and Tenkasi, 1995) on issues in relation to teaching and learning.

The research

To provide insight into the model of professional learning and the issues that arise, the detail of one research lesson is summarised here. This comes from a cluster of four schools that worked collaboratively over the space of one year on thirteen research lessons (a further fifteen research lessons were carried out by the cluster of the remaining five schools). This draws on data that includes video and audio recording of collaborative planning meetings, pre- and post- lesson discussions and the research lesson itself. Additional data used includes the different iterations of the lesson plan for the research lesson, classroom materials and student productions.

Case study: '110 years on'

The lesson was in a Midlands academy with a year 9 class that had little experience of working on problem-solving/modelling tasks. The students had worked on the task in the lesson prior to the research lesson, providing the teacher with insight into the different ways in which they were understanding and representing the situation presented. The focus of the research lesson was to understand better how mathematical representations may assist structuring and supporting mathematical thinking.

The task

<table>
<tr><td></td><td>110 years on
This photograph was taken about 110 years ago.
The girl on the left was about the same age as you.
As she got older, she had children, grandchildren, great grandchildren and so on.
Now, 110 years later, all this girl's descendants are meeting for a family party.

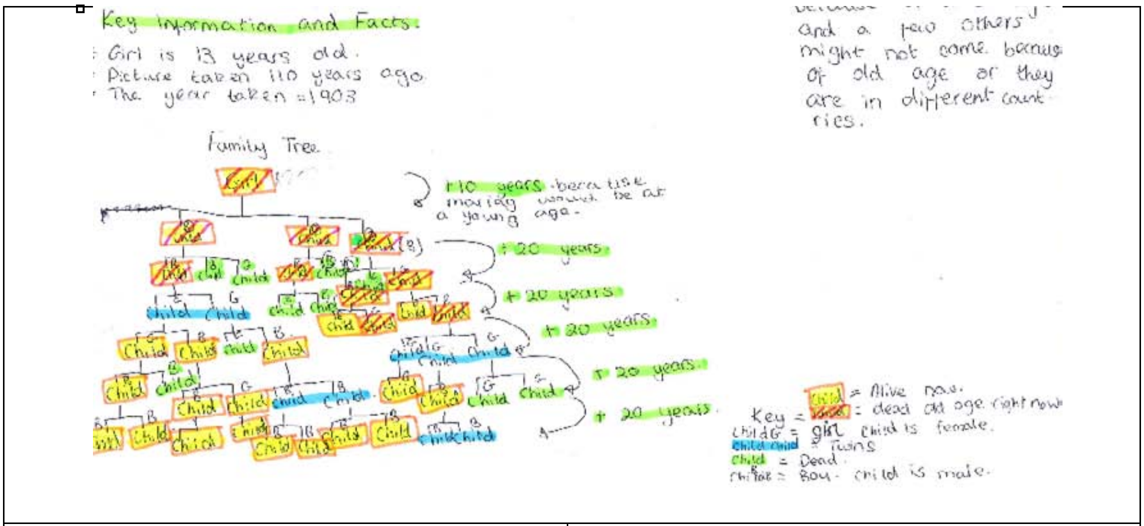
How many descendants would you expect there to be altogether?</td></tr>
<tr><td colspan="2">Twentieth Century facts</td></tr>
<tr><td>At the beginning of the 20th century the average number of children per family was 3.5.
By the end of the century this number had fallen to 1.7</td><td>In 1900, life expectancy of new born children was 45 years for boys and 49 years for girls.
By the end of the century it was 75 years for boys and 80 years for girls.</td></tr>
</table>

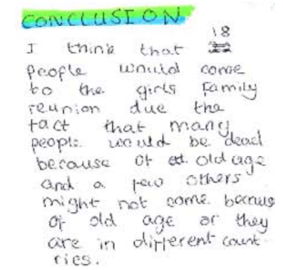
Figure 1. Task: '110 years on'. *(Source: Bowland Maths Assessment tasks)*

The lesson

1. Introduction (1 minute): The teacher reminded students of the task they had been working on in the previous lesson and asked them to respond to questions posed about one student's work (Figure 2) that was distributed to all students in the class.

2. Individual work followed by student discussion in groups of 2 or 3 (8 minutes): Students worked individually, writing answers to the questions (Figure 2) on the sheet provided and then, when asked, discussed their responses in small groups. As an example, one observed group spent time discussing the level of detail that was missing from the diagram (e.g. there may be children who would die young).





Assumptions
1. Can you state the assumptions made?
2. Are there any assumptions that are not useful?
Working out
Is there any mathematical calculation that is not clear to you?
Are the time periods clear?
Conclusion
Can you write the conclusion in your own words?

Figure 2. One student's initial response to the task '110 years on' and the questions asked of students in phase 1 of the lesson.

3. Whole-class discussion (6 minutes): The teacher focused a whole-class discussion on the questions they had just answered and discussed. Some groups contributed their thinking and the student whose work had been used explained how he had attempted to work with the information about the average number of children being 3.5 per family. The remainder of the discussion focused on whether or not the assumptions, calculations and conclusion were clearly communicated. At this point again the student whose work had been scrutinised explained how he had been able to find a solution by counting the yellow-coloured boxes in his diagram.

4. Individual work followed by whole-class sharing (4 minutes): Pupils were asked to write down a checklist that they had developed in the previous lesson to set out a problem-solving strategy. Pupils appeared to have memorised the key steps and were able to write these down.

5. Individual work with informal group discussion (36 minutes): In the main part of the lesson students were asked to improve their work, "That doesn't mean you have to start again. You are just improving your work, so you are correcting the parts that don't look correct to you." As the teacher circulated she initially assisted students to focus on communicating a more complete solution, referring students to their 'checklist'. After about 15 minutes most of the groups of students had started to discuss their work, often trying to make sense of how each other's diagrams related to the assumptions they had made. In this time the teacher circulated, often questioning individual students as she attempted to make sense of how the conclusion they had reached could be found from their mathematical diagram.

6. Individual work followed by whole class discussion [Neriage] (15 minutes): Students were all given an individual copy of one student's work. This student was asked what changes he had made in today's lesson and he indicated that he had redrawn the diagram to arrive at a different conclusion. Following an opportunity for individual students to take a careful look at the work, having been asked to write down on the diagram what they liked and any improvements that he could make, the whole class were able to ask questions of the boy who had produced it. In general this discussion allowed students to gain a better understanding of the student's thinking that he had not communicated in the diagram. Comments often focused on whether or not the assumptions that this particular student had made were reasonable.

The post-lesson discussion

In the post-lesson discussion the lesson study community, together with a member of the IMPULS project, acting as the 'knowledgeable other', raised the following points in relation to the problem-solving focus of the project in general and the research question for this lesson in particular.

Comparing and contrasting approaches: The teacher had changed the lesson at an early stage so that students had an opportunity to critique only one piece of student work (Figure 2). Similarly, the *neriage* phase of the lesson had been curtailed to allow students to spend more time on updating their own work. In this section of the lesson there was therefore time to again consider only one piece of student work, as opposed to the two pieces planned.

Using diagrams for individual mathematical thinking and for communicating with others: Much of the lesson had focused on diagrams as communicative devices, with students thinking about how they could make sense of someone else's diagram. However, the lesson was designed to unpack how mathematical diagrams can assist mathematical thinking. It was noted that even when such thinking is flawed, and this is embodied in the diagram, it can be helpful in assisting a student to see where the problem lies. A particular example of this was illustrated by one of the observers. The diagram that had been considered at the end of the lesson demonstrated evidence of the student moving to more abstract understanding, that is, away from the detailed structure of the situation being directly mapped by the diagram. Similar shifts in student thinking had been noted by others.

For students (and teachers) what is the mathematics? For many students, the focus throughout was on obtaining an answer; not necessarily a reasonable one. Many widely varying answers were obtained. Students prioritised calculations over drawing diagrams, even where it was clear that diagrams would assist their mathematical thinking.

The important role of making assumptions: Many students did not seem to understand the purpose of making assumptions. Most appeared to interpret assumptions as 'filling out details', both relevant and irrelevant, such as wars, death due to disease, occurrences of twins, and so on. Students did not understand making assumptions as being important in simplifying the problem in ways that allow the structure to surface and exploration of key parameters to be facilitated. In this regard, the structure of the lesson plan and the key role that the framing of the task and subsequent written and oral questions played in the initial critiquing phase of the lesson, was raised.

The range of answers obtained in modelling problems, as in this case, was considered as a potential way of focusing students to explore how their assumptions interact with the structure of the problem (and how these are encapsulated in diagrams).

Collaborative working: The structure of the lesson in invoking collaborative, as opposed to individual working, and learning was discussed. The role of the task and the pedagogic moves made by the teacher during the lesson were considered and the question of how to provoke all of the groups to discuss the role of diagrams in supporting mathematical reasoning was raised. The issue of the usual role of diagrams in mathematics where students learn to develop a specific type of diagram in response to mathematical needs was considered as a potential barrier, with students thinking that there is possibly a 'correct diagram' for any given problem.

Discussion

The research lesson and post-lesson discussion described here provide insight into the problems associated with focusing lessons on the complexities of aspects of problem solving. At issue in this particular lesson was how diagrammatic representations might afford, or indeed constrain, individual mathematical thinking. Although there was plentiful evidence in the lesson that a student's diagram affects their individual thinking and understanding, this was not explicitly highlighted or discussed as part of the lesson, even though in some instances students were observed discussing such matters in small groups. Such issues were not formalised and shared in the lesson and students did not, therefore, gain insight into how in future problems they might develop diagrams in ways that prove helpful. In this particular problem, and in general, the assumptions that students make play an important role in enhancing or reducing the complexity of the reality that they end up exploring, and consequently the level of detail that the diagram needs to encompass. As can be seen from the issues raised in the post-lesson discussion, it was felt that, even if discussion of the use of mathematical diagrams to support mathematical thinking had failed to occur at an early stage of the lesson, at a later time attention could have been (re-) focused by considering the wide variation in students' solutions and/or by provoking students to consider varying key parameters in the problem.

The outcomes of the illustrative lesson recounted here in some detail were not atypical of our experiences to date. We are left with the question of why is it difficult to focus teaching on, and how do we improve learning of, problem-solving skills/strategies/competencies? Perhaps key in discussion of this is the question that was raised in the post-lesson discussion: '*For students (and teachers) what is the mathematics?*' Brousseau's construct of the *contrat didactique* suggests that important in this regard is the teacher's epistemological stance, which the students eagerly adopt, even though this remains below the surface. In this particular research lesson, as we observed in others, the solution and its communication were prioritised, as they often would be in 'standard lessons', with both teacher and students having difficulty in shifting their attention to mathematical process. This leaves us with a thorny problem of how we might develop new epistemologies that prioritise and emphasise how mathematics is used to solve problems. Alongside such a restructuring of what it means to learn and use mathematics we need to be sensitive to how tasks as initially posed, and altered 'in the moment' by pedagogic moves by teachers in lessons can, if we are not careful, shift the attention of students in ways that might realign their actions so as to re-attain the usual state of equilibrium of mathematics lessons. Fundamental to these requirements is the need for the teacher to recognise their existing epistemological position, a vision of what this might become and the journey that they will take to achieve this. How as an on-going research project we tackle this issue provides a significant challenge. Our proposed approach is to return to our theoretical view and consider how we might design for such reflection and realignment of classroom goals and objectives. Perhaps a way forward is to consider the mode of professional learning that Engeström facilitated in his 'change laboratory', in which he experimented in problematising and rethinking ways of professional working (Engeström, 2001).

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SKE courses and bursaries: examining government strategies to tackle mathematics teacher quantity and quality issues

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The shortage of secondary mathematics teachers is a concern for UK government. In order to increase the supply of teachers, the government has sponsored subject knowledge enhancement (SKE) courses for non-specialist graduates ('SKE Policy'). Additionally, the government has offered financial incentives to attract high attaining graduates into teaching through differentiated bursaries ('Bursary Policy'). Existing research into the efficacy of these measures is limited but studies on two individual teaching courses suggest that student teachers who have taken SKE courses do no worse in their teaching course and that those who achieve more highly in their first degree do not achieve significantly higher outcomes in their teaching course respectively. This study corroborated results of these smaller studies through testing over 100 secondary mathematics student teachers on a sample of 'mathematical knowledge for teaching' (MKT) items, designed by the University of Michigan. Results suggest that whilst the SKE Policy may be a good strategy in that there was no overall significant difference in MKT scores between SKE students and mathematics graduates, the Bursary Policy may be a flawed since there is no evidence that higher degree classes lead to greater success on PGCE courses.

Keywords: mathematical knowledge for teaching, trainee secondary teachers, SKE courses, PGCE courses, bursaries, policy

Background

There is a shortage of secondary mathematics teachers within the UK (Department for Education, 2013). In order to address this problem the government have implemented two strategies. Firstly, they have introduced 'subject knowledge enhancement' (SKE) courses to graduates without mathematics degrees to enable them to 'top-up' their mathematics knowledge and subsequently take a PGCE (teaching) course (TDA, 2010). Thus, PGCE students can be divided into two groups: traditional entry students (those with mathematics related degrees) and those who have taken an SKE course beforehand. Secondly, attractive bursaries are offered to graduates to train to be teachers. However, in an attempt to avoid compromising teacher quality, larger incentives are offered to graduates with higher degree classes (Department for Education, 2013). These two strategies will be referred to as the SKE Policy and Bursary Policy respectively within this paper.

Limited research exists regarding the efficacy of these strategies. Regarding the SKE Policy, a report evaluating the effectiveness of SKE courses at equipping students to meet teaching standards as well as comparing SKE students with traditional entry trainee teachers was commissioned by the Teaching Agency (see Gibson et al., 2013). This evaluation involved surveys and interviews with SKE candidates from a range of subjects (not just mathematics), PGCE students and course tutors. The report concluded that whilst "SKE courses provide trainees with a high level of subject knowledge and confidence…" (p. 11), the SKE students felt they had a lower level of subject knowledge than their peers who were subject graduates. Nevertheless, SKE students felt their knowledge may be more relevant to a school context than subject graduates (Gibson et al., 2013). Further, a small study with a single cohort of mathematics PGCE students by Stevenson (2008) compared final PGCE scores between SKE students and the whole group. Although mean grades for SKE students were lower than for the whole group the difference was not significant.

With regard to the Bursary Policy, Tennant (2006) investigated the relationship between first degree classification and final PGCE course grades of students at one institution and found no correlation. At another institution, Stevenson (2008) also found no correlation. These studies suggest that prior attainment as measured by formal qualifications does not necessarily predict success on a teaching course. However, sample sizes were small and final PGCE scores are not standardised across institutions but are determined by course tutors and school-based mentors.

The above studies suggest that whilst SKE students' feel their subject knowledge is at a lower level than their traditional entry peers, having a higher degree classification does not predict higher PGCE scores. Perhaps the results found by Tennant and Stevenson are not representative of the national picture. However, the research literature since Shulman's (1986) introduction of 'pedagogical content knowledge' recognises that teachers need more than just knowledge of the subject matter they are to teach in order to be effective teachers. Indeed, researchers at the University of Michigan have defined the construct 'mathematical knowledge for teaching' (MKT) which includes sub-divisions of Shulman's categories of 'subject matter knowledge' and 'pedagogical content knowledge' (Ball, Thames and Phelps, 2008). Further, a pool of multiple-choice questions has been developed to measure teachers' MKT (Learning Mathematics for Teaching, 2007). The MKT questions are kept secure but one of the 'released' questions is included for illustration (see Figure 1). It could be hypothesised that SKE students possess higher levels of MKT - "the *mathematical knowledge that teachers need to carry out their work as teachers of mathematics"* (Ball, Thames and Phelps, 2008: 4) - which would then compensate for their lower level of mathematical content knowledge allowing them to achieve similar PGCE course results. To test this hypothesis, a sample of the MKT questions was administered to PGCE students in England as a pre- and post- (PGCE course) test in the academic year 2012-13. A pre-post test design was used to investigate the effects of both prior experience and the PGCE course itself on MKT. Participants' degree classifications as well as final PGCE grades were also collected to refute or substantiate Tennant and Stevenson's findings in a larger sample of PGCE students.

Mathematical Knowledge for Teaching (MKT) questions

The MKT questions were chosen to compare knowledge between SKE students and traditional entry PGCE students as they were designed "to make statements about how content knowledge differs among groups of teachers, or how a group of teachers performs at one or more time points" (Learning Mathematics for Teaching, n.d.).

The MKT questions have been extensively validated by the authors and other researchers by psychometric analysis (Hill, Schilling and Ball, 2004; Warburton 2013a). Further, scores on the questions have been shown to positively correlate with the 'mathematical quality of instruction' of teachers in the classroom (Hill et al., 2008) as well as positively predicting pupil achievement (Ball, Hill and Bass, 2005).

The MKT questions have primarily been used within the US but have also been used in other countries, namely, Ireland, Norway, Ghana, Indonesia, and Korea (see Ball, Blömeke, Delaney and Kaiser, 2012) where the MKT questions were adapted to the new context. For this study, instead of adapting the questions, those which required alteration were omitted (for example, if a 'state exam' or 'dollars' were mentioned). This approach was taken due to a caution from the question authors:

> Changing items – even by making small alterations in wording – means the psychometric properties no longer hold... Even small wording changes (e.g., changing a preposition) can make the item mathematically ambiguous, and thus unusable in a measurement context (Hill, Ball, Schilling and Bass, n.d.: 2).

This left a sample of questions felt to be appropriate for use in England. Twelve item stems, some consisting of several parts, were used for this study; 18 questions in total.

8. As Mr. Callahan was reviewing his students' work from the day's lesson on multiplication, he noticed that Todd had invented an algorithm that was different from the one taught in class. Todd's work looked like this:

$$\begin{array}{r} 983 \\ \underline{x\ 6} \\ 488 \\ +\underline{5410} \\ 5898 \end{array}$$

What is Todd doing here? (Mark ONE answer.)

a) Todd is regrouping ("carrying") tens and ones, but his work does not record the regrouping.

b) Todd is using the traditional multiplication algorithm but working from left to right.

c) Todd has developed a method for keeping track of place value in the answer that is different from the conventional algorithm.

d) Todd is not doing anything systematic. He just got lucky – what he has done here will not work in most cases.

Figure 6: Example of MKT multiple-choice question.

Method

65 institutions in England were initially allocated[14] by the Training and Development Agency for Schools (TDA)[15] to run full-time PGCE secondary mathematics courses in 2012-13. All institutions were contacted and their students invited to respond to a two-part questionnaire either on paper or online both at the beginning and end of their PGCE course. The first part sought biographical information including date of birth, gender, degree class, and whether an SKE course had been taken. The second part comprised the MKT questions.

[14] Those allocated small numbers of students (<10) may not have run their courses in 2012-13.
[15] Now the National College for Teaching and Leadership (NCTL).

Results

Out of the initial allocation of full-time secondary mathematics PGCE places (1,773 students) 141 PGCE students responded meaningfully[16] to both the pre- and post-questionnaire (8%)[17] from 18 different institutions. Out of these respondents, there were slightly more females (n=72, 51%) than males (n=69, 49%). There were 69 (49%) respondents who took an SKE course prior to their PGCE and 72 (51%) respondents who did not. However, since lengths of SKE courses vary from two weeks to one academic year, the PGCE students were grouped as follows for this study: 43 (30.5%) respondents took an SKE course of five months or longer ('Long SKE'); whilst 98 (69.5%) took either no SKE course or took a short (two or four week) SKE course ('Non SKE'). These groups were chosen as short courses are typically for refreshing specific aspects of knowledge. Indeed, most short course candidates had mathematics degrees. Since mean scores on both tests were very similar for those taking a short SKE course as for no SKE course, this suggests this division was justified. Further, short courses are no longer funded by the government from 2013-14 but "incorporated as part of the normal refresher learning that forms part of an ITT course" (NCTL, 2013:2).

Since a sample of the MKT questions was used in a new context (student teachers in England) the reliability was calculated and found to be reasonable (0.67 – pre-test, 0.71 – post-test) as measured by Cronbach's alpha. A dichotomous Rasch model was used to analyse the psychometric properties of the questions. It confirmed that the questions can be used appropriately in England and that they measure one underlying construct (for further details of the Rasch analysis, see Warburton, 2013a).

Differences between SKE and non-SKE PGCE students (SKE Policy)

PGCE scores

138 final PGCE teaching scores (Ofsted style grades) were collected and, similar to Tennant's study, awarded points as follows: Outstanding = 3 points; Good = 2 points; Working towards good = 1 point; Cause for Concern = 0 points. In addition to scores being unstandardised across institutions, anecdotal evidence from PGCE course tutors suggests that due to inspection by Ofsted students must achieve high grades.

Similar to Stevenson (2008), final PGCE scores were compared between Non SKE and Long SKE groups. Long SKE students did better on average (M = 2.18 points, SE = 0.135) than Non SKE students (M= 1.96 points, SE = 0.115) but this difference was not significant (t = 1.184, p = 0.238, r = 0.102).

MKT scores

A repeated measures ANOVA was used to analyse any changes in MKT scores (dependent variable) over time (between pre-and post-test - within subjects factor) and by group (Non SKE and Long SKE - between-subjects factor). There was a significant increase in overall scores of 1.34 marks[18] (7.4%) between the pre- and post-test ($F(1, 139) = 25.088$, $p < 0.001$ with a medium effect size: partial η^2= 0.153). Both other effects were non-significant, but the difference between groups (Non SKE and Long SKE) in mean scores was sustained over time (Figure 2).

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[16] For example, those who only responded to the first few questions were not counted.

[17] For further details on the pre-questionnaire results (n=239) see Warburton (2013b).

[18] The terms of use of the MKT questions state that raw scores, including mean scores, cannot be reported so are omitted here. Instead, differences between mean scores are provided.

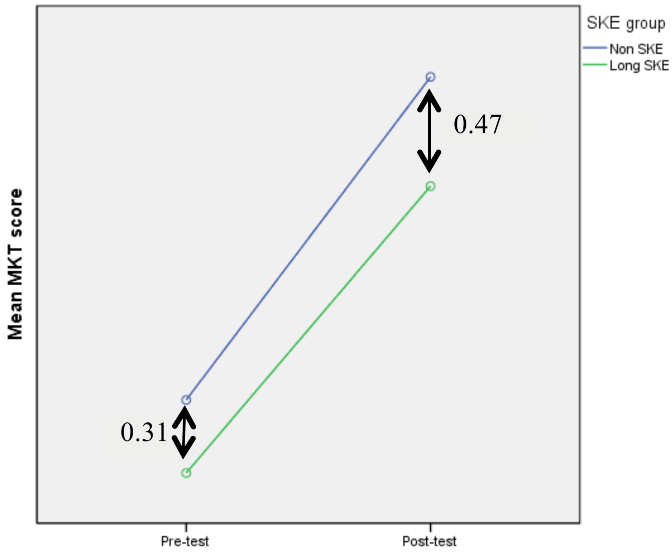

Figure 2: Graph showing differences in mean scores on MKT questions for the pre- and post- test for Non SKE and Long SKE groups.

MKT score sub-analysis

Some of the older MKT questions selected for use in this study had been categorised by the authors as assessing specific components of the MKT construct. This practice was discontinued as it was found to be difficult to place questions into mutually exclusive categories. Thus, more recently developed questions are not classified. I attempted to classify those questions which had not been classified by the authors, as assessing either 'common content knowledge' (CCK) – mathematics knowledge

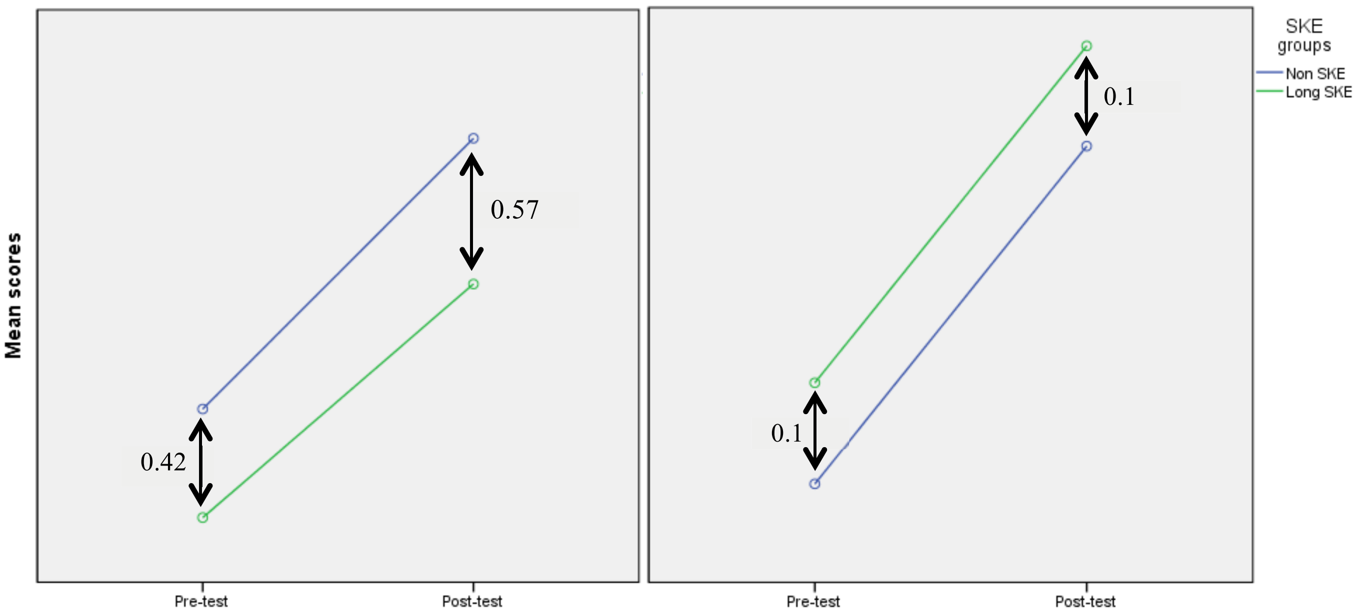

Figure 3: Graphs showing differences in mean scores on the CCK and Other questions on the pre- and post- tests by SKE group.

which any well-educated adult should know - or other aspects of MKT (Other) beyond that expected of an educated adult but specific to the work of teaching (see Ball, Thames and Phelps, 2008 for their other sub-categories of MKT). Whilst the following results provide some insight, they should be treated with caution given the difficulty of classifying questions as recognised by the question authors.

Thirteen questions were classified as requiring CCK in order to answer them correctly, with only five felt to require additional knowledge related to teaching (Other aspects of MKT). The repeated measures ANOVA was reiterated twice with scores on the CCK questions and Other MKT questions respectively as the new dependent variable. For CCK questions there was a significant increase in overall scores of 1.01 marks (□(1, 139) = 24.04, □<0.001, partial η^2= 0.147), yet the other effects were non-significant at the 0.05 level. The same was true for the Other questions, that is, there was a significant increase in overall scores of 0.32 marks (□(1, 139) = 6.24, □ = 0.014, partial η^2= 0.043) and the other effects were non-significant. Although there was no significant effect of group on CCK and Other scores, Figure 3 shows that, on average, the Long SKE group performed worse on the CCK questions, but better on the Other questions than the Non SKE group. These differences are small, but were sustained over time.

Relationship between degree class and PGCE/MKT scores (Bursary Policy)

Using methodology similar to Tennant (2006), the correlation between prior degree classification and final PGCE teaching scores was calculated to be □ = 0.185. This correlation is small and similar to Tennant's (□ = 0.11), corroborating Tennant (2006) and Stevenson's (2008) conclusions that there is no connection between degree results and success on the PGCE. Indeed, whilst a linear regression established that prior degree classification could statistically significantly predict PGCE scores, □(1, 135) = 4.804, □ = 0.03, degree classification only accounted for 3% of the explained variability in PGCE scores. The regression equation was: PGCE score = 1.04 + 0.254 □(degree classification). However, this equation should be treated with caution as the assumption that residuals of the regression line are approximately normally distributed was violated. Further, there are potential issues with PGCE grades as discussed above. Given these issues, the correlations between degree class and scores on the MKT questions were also calculated and also found to be small: □= 0.189 (pre-test), □ = 0.128 (post-test). Alternatively, another reason provided by the Department for Education (2013) for implementing the Bursary policy is that: "Degree class is also a good predictor of whether a trainee will complete their course and achieve QTS". However, there was no significant association between degree classification and whether or not the student completed the course (□ = 0.07, Fisher's exact test, Cramer's V = 0.155, small effect) for 244 PGCE students involved in this research.

Discussion and conclusion

The Efficacy of the SKE Policy

When comparing mean scores on the MKT questions, there was a significant increase in scores between the pre- and post-test. For this sample of PGCE students, this is good evidence that the PGCE courses helped to improve MKT and is reassuring. However, there was no significant effect of group (Non SKE and Long SKE) on MKT score, suggesting that SKE and traditional entry PGCE students have similar levels of MKT both at the beginning and end of a PGCE course. Further, this study corroborated the findings of Stevenson (2008) that there is no significant difference in final PGCE scores between SKE and Non SKE PGCE students.

These results provide evidence that SKE students commence their PGCE course with similar levels of MKT to their traditional entry peers and that PGCE courses help both groups to increase in MKT over the course and to achieve similar final PGCE scores. However, whilst overall MKT scores do not differ significantly between groups, SKE students performed slightly worse on average than traditional entry PGCE students on questions felt to be testing CCK but slightly better on questions related to teaching and pupils (Other aspects of MKT) over time. This fits with findings of Gibson et al (2013) that whilst SKE courses are intended to focus on subject knowledge, many SKE courses also include an aspect of pedagogy (though the amount varies between courses). Thus, PGCE students within the Long SKE group have potentially spent at least five months learning mathematics with the intention of teaching as well as acquiring some knowledge of how to teach it.

Nevertheless, further research is needed to understand the differences in types of knowledge between SKE and traditional entry PGCE students given the difficulties of classifying the MKT questions as discussed.

The efficacy of the Bursary Policy

This study, involving larger numbers of PGCE students across 18 institutions in England, corroborated results found by smaller studies at single institutions that degree classification does not strongly predict success on a PGCE course. Additionally, degree scores do not predict success on the MKT questions. Therefore the higher financial incentives offered by the government to graduates with higher degree classes may not be achieving the purpose of "[raising] the quality of new entrants to the teaching profession" (Department for Education, 2010: 20). Furthermore, this study suggests there is only a small effect of degree classification on whether a student completes their PGCE course.

The Bursary Policy awards £20,000 to graduates with a first class degree, £15,000 to those holding a 2:1 and £12,000 to a 2:2, regardless of degree title. This study suggests that awarding £5,000 more to first class graduates is perhaps not justified when predicted gains in PGCE and MKT scores are small. Additionally, as regression analysis suggests a linear relationship between degree class and PGCE scores, this raises the question of why increments in bursaries are not equal.

In conclusion, this study provides evidence that the introduction of SKE courses is a good strategy to address the shortage of mathematics teachers whilst not compromising on teacher quality. However, offering differentiated bursaries by degree class may be a flawed policy since there is no evidence from this and other studies conducted in England that higher degree classes lead to substantially greater success on PGCE courses.

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The impact of professional development on the teaching of problem solving in mathematics: A Social Learning Theory perspective

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Teachers' professional development (PD) has been described as an 'unsolved problem', particularly where there is an expectation to change teaching practices from teacher-centred orthodoxies to more student-centred approaches. This paper considers PD that has been designed to support the teaching of problem solving in secondary classes. One of the main problems in PD design and research has been identified as the limitations of existing professional learning theory. To respond to this, the research reported in this paper is intended to contribute to the theorisation of PD. I will present a case study of one teacher – taken from a larger multiple case study − as they take part in a programme of school-based PD. Social learning theory (SLT) is used to analyse and explain the learning processes as a result of participation in the PD programme. This reveals that SLT provides a useful theoretical approach. As a result, I suggest the approach could be used more extensively in professional learning and PD, to understand, evaluate and develop programmes more effectively.

Key words: professional development; PD; CPD; problem solving; inquiry-based learning.

Introduction

The new National Curriculum for England, which comes into effect from September 2014, features problem-solving as one of the aims for teaching and learning in mathematics: "The National Curriculum for mathematics aims to ensure that all pupils can solve problems by applying mathematics to a variety of routine and non-routine problems …" (DfE, 2013: 3). Across Europe there has also been considerable interest, at policy level, in developing inquiry-based learning approaches in the teaching of STEM subjects (Wake and Burkhardt, 2013). Problem-solving and inquiry-based approaches involve students collaboratively working on complex and unfamiliar problems where, according to Schoenfeld (1992), the methods to use are not obvious or there may be a choice of different methods.

There are two main arguments for the wider use of these approaches, which I will refer to as *student-centred problem solving* as a pedagogic approach. First, there is an economic argument: developing transferrable and flexible skills is necessary in an increasingly complex and mathematically formatted world (Skovsmose, 2008). Second, there is the issue of student engagement, but this has also been argued to have economic consequences. It is suggested that *student-centred problem solving* approaches are more engaging (Martino and Zan, 2010), this can lead to higher take-up of STEM subjects in more advanced study (Rocard, 2007).

At policy level then, there is an interest in student-centred problem solving orientations. However, there is strong indication that teaching in secondary schools in England is predominantly traditional, teacher-led and focussed on teaching methods (Ofsted, 2008; 2012). It is also been suggested that attempts to reform extant practices through professional development have not been successful (Borko, 2004; Guskey and Huberman, 1995). This has been attributed to under-developed theory in the field of professional learning (Opfer and Pedder, 2011).

This paper addresses these issues by presenting an aspect of the research and evaluation of a PD solution (Bowland, 2008). The PD was designed to support teachers implement *student-centred problem-solving* approaches in their teaching. It was also designed to be used, by departments, without external expertise.

In this research, I use a theoretical approach to understanding PD and professional learning that has not, until now, been used in the context of teachers' professional learning. The theory used is based on social learning theory (SLT) (Bandura, 1977). I illustrate its affordances by considering the case of a teacher who participated in the wider PD project and evaluation. From this I will demonstrate the power of this theoretical approach and its potential in the design, development and evaluation of professional development. In so doing, I attempt to offer a theoretical approach that counters the criticism aimed at extant theories used in professional development and professional learning.

The research question for this study is: How effective is SLT in accounting for teachers' professional learning in the context of a PD programme? For this I will consider the case of a single teacher who was a participant in a wider study.

Social Learning Theory

Social learning theory (SLT) or social cognitive theory represents a theoretical and research pathway that began in the latter part of the nineteenth century and has developed and grown subsequently. The originators of the ideas in SLT have been attributed to William James and then developed this century by Millar and Dollard and, through the latter part of the last century, by Albert Bandura (1977; 1986; 1997). The key features of SLT are:

- *Observational learning* – learning takes place by observing behaviour directly; or by using guides or principles for action and adapting them.
- *Self-efficacy* – the belief an individual has that they will be successful in any given endeavour or course of action.
- *Reciprocal triadic determinism* – this is the inter relatedness between individual thinking and beliefs, the social context and individual behaviour. These three components, in SLT, are reciprocally influencing.

SLT affords a number of advantages and facilities in respect to teachers' professional learning. One of which is the way learning is conceptualised and this, as I will explain, satisfies criteria identified in the literature. Sfard (1998) argues the equal importance of 'two metaphors' for learning; which are, individual, cognitively-oriented knowledge *acquisition* conceptualisations *and* social and participatory conceptualisations of learning. Borko (2004) suggests this is also true for teachers' professional development and professional learning. She refers to the theorisation of professional learning that draws on both metaphors as a *situative* approach. This is consistent with *reciprocal triadic determinism* in SLT.

However, cognitively-oriented and participatory theories have come to represent two distinct strands of research in professional development, mathematics education and educational research. It has been argued that these two approaches are too distinct; they have separate philosophical underpinnings and are incompatible. It is not possible, therefore, to unite the two to develop a genuinely *situative* analysis as suggested by Borko (see, for example, Greeno, 1997). However, SLT arrives from a different tradition and implicitly takes a *situative* perspective on learning. Individual acquisition is captured in the construct of self-efficacy and the participatory aspect reflected in SLT's reciprocal determinism. So let us turn to the key concepts and structures in Bandura's formulation of SLT.

A central tenet of SLT is observational learning. While this may suggest replication or mimicry, the mechanisms of observational learning within SLT involve a process through which novel behaviours are formed. That is, an observer can watch somebody's behaviour and consciously adapt and develop what has been observed and subsequently behave in a way that is related to what was observed but has been transformed and developed by the observer.

The processes by which novel behaviours are formed are regulated through self-efficacy. Self-efficacy represents the belief an individual has in their ability to achieve certain levels of attainment using a particular approach or behaviour in a particular domain (Bandura, 1997). For example, an individual will observe a set of behaviours in a situation or context and then adapt them to a form consistent with what they believe they will be effective. If, as in the context of teaching, existing behaviours (practices) are dominated by norms and expected routines then lower levels of efficacy would result in the implementation of behaviours that are similar to existing approaches. Higher levels of efficacy may prompt the introduction of more novel approaches.

Self-efficacy reflects individual cognitive aspects such as underlying knowledge as well as affective aspects such as confidence, motivation and 'underlying skill' (Bandura, 1997). Self-efficacy combined with observational learning provides a *situated* or *participatory* component through which behaviours are observed and self-regulated. In the context of a school, the self-regulating processes of modelled behaviour serve to 'conservatise' teachers' practices. High levels of efficacy are required by individual teachers to implement and sustain practices that vary from the orthodox routines and organisation of teaching.

Methods

This study uses a case study approach as characterised by Yin (2009). In this paper I will present the analysis of a single case, that of Imran (pseudonym), as he participated in a PD programme run by his department, and then, as he attempted to implement *student-centred problem-solving* in one of his classes.

The PD was designed to support the teaching of *student-centred problem solving.* The programme involved two cycles of PD sessions, each with an hour-long *introductory* session: an *into-the-classroom* phase – where teachers 'try out' the ideas presented in the introductory session – finally there is a *follow-up* session in which the department meets for an hour to reflect on their experiences in teaching the approaches suggested in the *introductory* session. In Imran's school the sessions were led by the head of mathematics, the materials provided detailed instructions and resources to run the PD sessions.

The PD materials, developed by the Shell Centre team at the University of Nottingham, were originally released as part of the Bowland materials (Bowland, 2008). There are seven PD modules in all, two of these were used for the two cycles of PD sessions described above. The modules, as well as having a general emphasis on teaching using a student-centred problem-solving approach, each have a more specific pedagogical focus. The modules used in Imran's school focussed on *Questioning and reasoning* and *Involving pupils in peer and self-assessment.*

Imran was selected for the case study in this paper as he appeared to illustrate the challenges teachers face in adopting new approaches in teaching. That is, he explained that his teaching was traditional, with an emphasis on teaching methods and presenting students with extensive opportunity to practise with textbook- or worksheet-based exercises. He taught in a way consistent with the norms of practice described above. It was assumed that Imran is representative of many teachers who generally teach mathematics using a teacher-centred approach. And so, it is likely that the exploration of Imran's attempts to develop and potentially change his approach serves to investigate the efficacy of SLT in the context of professional learning.

Data collection included questionnaires administered to all teachers before and after the programme; the observation of PD sessions and lessons; and interviews with the PD leaders, heads of departments, after the PD sessions. There were also interviews with teachers after each observed lessons. Imran was observed on five occasions through the project, two of the observations involved the *into-the-classroom* phase of the PD. Teachers were asked to teach a 'problem solving' lesson in each of the observed lessons. This was in order to understand how teachers developed in their teaching of student-centred problem solving through the project.

The analysis of data involved a two-stage process in which data were first reduced and organised. The second stage featured a range of analytic approaches, involving the analysis of video recordings of PD sessions, observed lessons and interviews. Interviews with teachers were analysed and coded in order to identify the challenges teachers faced in teaching using the suggested approaches and the way in which they felt they had developed through the programme. These were compared and triangulated with interview responses from heads of departments and lesson observations. From this a case study report for each teacher was generated with a particular focus on the way in which s/he had developed through the programme. In the next section I will present an analysis of Imran's experience using SLT as the analytic framework.

Results

Imran had been teaching for about nine years at the time of this study. This was his third teaching post. He had been at the school two years. It is an average sized comprehensive school serving students in the 14-18 age range and located in a village outside of a city in central England. GCSE results were below the national average and pupils also made progress below the national average.

Imran described himself as a traditional teacher, although he indicated that he would like to try more problem solving and use more student discussion and collaboration in his lessons. The head of department, Deborah, had strongly encouraged Imran to be a participant in this study in order that he had the chance to develop his approach to using student-centred problem solving. Deborah had decided to do the PD modules on *Questioning and reasoning* and *involving pupils in peer and self-assessment*. Imran attended all sessions along with most other members of the department.

In terms of SLT, Imran's practice was consistent with the norms of school mathematics teaching. In order for his practice to change, then the PD would need to help him develop self-efficacy in the suggested alternative approach. It would also need to provide observable 'models' of the suggested approach.

The head of the department described the department as having a core team of four teachers of which Deborah counted herself a member. This group had a particular interest and commitment to developing teaching and learning. Imran, although a valued member of the department, was peripheral to the 'core' group. Deborah also recognised Imran as a traditional teacher who found the behaviour of lower-attaining students challenging.

From an SLT perspective this suggested that there was an efficacious group of teachers in the department who were willing to experiment, work together and try things out. Imran, who was less efficacious in respect to his teaching, was on the periphery of this group. It is assumed he is less efficacious because he is less confident and motivated to try things out. This assumption is supported by the analysis of interviews and observations.

At the beginning of the project Imran explained how he teaches in a traditional way using exercises and worksheets having explained a method:

> Most of the time we are doing [something like] SOH-CAH-TOA [mnemonic for trigonometry ratios] and they use it and apply it straight away.

The first lesson that was observed involved students working on two problems involving the optimisation of the volumes of two objects. In the interview afterwards Imran explained that the difference between the lesson and the way he usually taught was that he did not provide an explanation of how the problems should be solved at the beginning of the lesson. Imran had interpreted a problem solving lesson as much like a traditional lesson but not involving teacher exposition or an explanation of methods at the beginning of the lesson. Imran did not appear to be confident in handing over decision-making to the students. This is consistent with SLT, if more efficacious in respect to the teaching of problem solving it is likely that Imran would have given greater authority to the class.

In the second lesson, Imran was expected to use the ideas presented in the PD introductory session: the lesson planned in that session or the suggested lesson plan, to try out the student-centred problem solving approaches described in the PD. Like many of the teachers participating in the PD, Imran chose to use the tasks and adapt the model lesson plan included in the PD materials. This can be considered in terms of observational learning. Bandura extends observational learning to include, as well as the direct observation of behaviour, the use of 'guides' or 'principles for action' (Bandura, 1997). Here the lesson plan represents an observable behaviour. In addition, included in the PD materials and shown in PD sessions, are videos of lessons as examples. What is interesting from a SLT perspective is the way in which Imran uses and adapts the models.

The suggested lesson plan includes a brief introduction by the teacher and a few minutes for students to look at the problem individually. It is then suggested that the teacher collects ideas at the board before students work in groups for 20 minutes on the problem. This is followed by a whole-class discussion after which students are given further time to work on the problem. Finally, it is suggested, students present their solutions.

Planning and organising: **Aircraft turn-round time**

Between landing and taking off, the following jobs need to be done on an aircraft.

	Job	Time needed
A	Get passengers out of the cabin and off the plane	10 minutes
B	Clean the cabin	20 minutes
C	Refuel the plane	40 minutes
D	Unload the baggage from the cargo hold beneath the plane	25 minutes
E	Get new passengers on the plane	25 minutes
F	Load the new baggage into the cargo hold	35 minutes
G	Do a final safety check before take-off	5 minutes

What is the shortest time needed to do all these jobs?

Would it make any difference to this time if the people could get off more quickly (from the front and rear of the plane)?

Figure 1: Example task from PD materials

Imran adopted a similar structure when he attempted the lesson with his group. However, he made one substantial adaptation. In the lesson he used two of the problems from the PD session and used the suggested structure twice in the lesson. He halved the suggested times. He explained why he adapted the suggested lesson:

> …when I look at these problems I tend to just adapt to my class, I won't particularly go with what's on the lesson plan. I would sort of adapt to my lesson.

In the first half of the lesson he used the aircraft turn-round problem. He introduced the problem and gave the class three minutes to consider the problem on their own. He opted not to collect ideas at the board, as suggested, but explained to students that they should think about how they write the answer down.

Imran stopped the class and questioned them about the effects of passengers leaving the aircraft from both the front and the rear. He presents this as a closed question but attempts to foster a whole-class discussion. However, it appears likely that he is not used to leading a more open-ended whole-class discussion.

It is clear that Imran has taken the ideas presented in the PD and adapted them in order to develop a lesson that he felt comfortable with. Imran explained in the post-lesson interview that he thought the tasks were too easy and this had prompted his decision to include the two tasks in the lesson. It was interesting that he did not ask the students to extend the tasks. It appeared that Imran preferred them to find a solution and then move on, rather than explore the situation in more depth. It is possible that Imran did not understand the aims of the PD that were to use student-centred problem solving approaches; his interpretation appeared to be that it was about getting answers in the context of more open-ended problems.

Furthermore, it seemed likely that Imran was concerned about transferring authority and decision-making to the students. As such he would have felt a loss of control. As a result of observing Imran for five lessons it seemed highly probable that the control issue was the most likely explanation for the adaptations Imran had made to the suggested approaches.

What appeared to be happening was that Imran wanted students to be 'busy' on the problem, allowing students space to explore may have challenged Imran in terms of managing behaviour or possibly in the mathematical questions that may have arisen. The character of the lesson involves short periods of group work with frequent interjections by the teacher.

In interview, Imran explained that he found it difficult to allow students more opportunity to think about and discuss problems and he attributed this to his own lack of confidence and experience in using student-centred approaches. Although over the course of the PD and through the lessons observed he felt he had become more comfortable with it, he still found it very difficult. He accounted for his reluctance in terms of 'not being confident enough' to give greater authority to the students.

Discussion

In the case presented, SLT provides a useful theoretical explanation for what was observed. From an SLT perspective Imran adapted the models and ideas suggested in the PD and this perhaps reflects the way in which teachers adapt and interpret reform ideas in order that they can reproduce an approach in the classroom that they feel is achievable and are comfortable with. In Imran's case the adaptations were sufficient to suggest that what he was implementing in his teaching was some way distant from the approach suggested in the PD. I would suggest this is quite common and would explain why so much reform-oriented PD does not result in fundamental changes to teaching.

It was pointed out at the beginning of the paper that a key assumption in this research is that teaching is predominantly traditional, teacher-centred and oriented towards teaching and learning methods. An explanation for this observation can be offered in terms of the practical demands of teaching day-to-day (see, for example, Cuban, 2009). A second related reason is related to the idea of teaching following 'cultural script' (Stigler and Hiebert, 1999). In sum, traditional teacher-centred teaching offers an approach that is manageable and economic in the school institutional context. At the same time it offers routines that students, parents and teachers have familiarity with.

Deviation from this norm represents a demand for many teachers. Imran, in this case study, illustrates how teachers respond to an expectation to teach using more student-centred approaches. He takes the modelled approach – in video or as a lesson plan – and modifies the approach to a format that he was comfortable with. Indeed, something that was more teacher-centred.

An analysis using SLT is useful in this respect because it offers a route to understanding the way in which reform attempts interact with existing practices, in schools as well as with the knowledge and beliefs of the teacher. It highlights the importance of the need for appropriate models, suggested-lesson plans, video examples and classroom activities. It also shows that it is important to consider the self-efficacy of teachers in respect to the implementation of the suggested approach. A final point is to consider how the PD supports the development of teachers' self-efficacy.

Concluding comment

In this paper I have illustrated how SLT can be used to explain both the professional learning and the constraints that teachers experience. I argue, based on the limitations of existing theory and the evidence of this case study, SLT has potential in improving the design, research and evaluation of PD and professional learning. This paper serves to illustrate and exemplify the use of the theory and hopefully prompt further research in this area.

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Changing attitudes: undergraduate perceptions of learning mathematics

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The purpose of my research is to explore first year undergraduate perceptions in learning mathematics and their identification of strategies to support them in this area. I work as a senior lecturer in mathematics education on a BA Applied Education Studies course where students consistently identify the mathematics education units as a source of anxiety. Having established a potential link between mathematics anxiety in teachers and trainee teachers, and the potential that this anxiety could be passed on to children in classrooms, my aim is to identify whether there may be strategies that could support adults in learning mathematics. To explore this area, I tracked a sample of first year undergraduate students through their initial mathematics education unit, establishing their perceptions before and after the teaching of this unit. Initial findings demonstrated that the students had negative perceptions about learning mathematics (twice as many negative as positive) and that this was reversed following the completion of the first mathematics education unit. A range of factors were identified as affecting how they felt about learning mathematics, including the role of the teacher, personal perceptions regarding learning mathematics and the role of discussion.

Keywords: mathematics; attitudes; perceptions; confidence; understanding

Introduction

This research stems from my experiences of working with undergraduates on a BA Applied Education Studies course at a university in the UK, where I have been a senior lecturer since 2006. The discussion is based on the initial findings of a doctoral research project exploring the views of 75 first year undergraduates, with the aim of identifying strategies that might support them in learning mathematics. The students attending the course study part-time, and are required to work or volunteer within a school based setting for at least one day a week, although many of the students work full time as teaching assistants and unqualified teachers. Over half of the students who graduate go on to train as primary teachers. The course consists of a range of units focussing on education within primary settings, and includes the development of personal subject knowledge and pedagogical understanding in learning mathematics. On commencement of the course, students complete audits in mathematics, science and English, and from these audits it has been possible to identify a trend where students express anxiety regarding the learning of mathematics, hence leading me to explore this issue in more depth.

Perceptions regarding learning mathematics

In terms of adult perceptions, there are those who suggest that anxiety in learning mathematics is an issue that exists for many individuals who exhibit symptoms such as tension, fear and panic when faced with carrying out mathematical problems (Buxton, 1981; Tobias, 1993; Boaler, 2009). Both Crook and Briggs (1991) and Tobias (1993) suggest that adults bring their past experiences with them when learning mathematics and that when these experiences are negative ones, their disposition towards the subject is affected. Additional concerns are raised regarding the potential effect of mathematics anxiety on those who are working within primary settings, including those who are training to teach as well as those who are fully qualified (Hembree, 1990; Relich, 1996). There is the suggestion that those who are anxious about mathematics themselves may pass this onto their pupils, creating a cycle of anxiety (Ashcraft and Moore, 2009; Haylock, 2010).

A range of influences in regards to how adults feel about learning mathematics have been identified, with the most consistent influence being that of the role of the teacher (Hodgen and Askew, 2006; Ashcraft and Krause, 2007; Bekdemir, 2010). Other factors for consideration include the effect of parents and peers (Evans, 2002; Ashcraft and Moore, 2009) and the lack of perceived connections between mathematics and reality (Tobias, 1993; Boaler, 2009). Others suggest that the cause of mathematics anxiety is related to personal perceptions limiting a person's ability to carry out mathematics (Buxton, 1981; Dweck, 2007; Bekdemir, 2010).

Coben (2006) identifies that the teaching of mathematics to adults is an area that needs further consideration. A number of approaches have been explored that might support adults in learning mathematics, including the need to make such learning relevant to real life (Lindeman, 1926; Knowles, Holton III and Swanson, 2005). Others suggest that the development of a range of pedagogical approaches could support understanding, to include collaboration and group work (Gresham, 2007; Ashun and Reinink, 2009; Johnston-Wilder and Lee, 2010), alongside the need to support adults in making connections between one aspect of mathematics and another (Klinger, 2011). It is also suggested that the role of the teacher in the learning of mathematics remains a consistent influencing factor, with the teacher needing to be knowledgeable, approachable and able to use a range of teaching approaches (Tobias, 1993; Johnston-Wilder and Lee, 2010; Welder and Champion, 2011).

Based on the concerns raised by undergraduates in learning mathematics at the start of their degree course, and the potential link between their perceptions and the effect on the children they may work with, I decided to explore this area further. The purpose of my research is therefore, to explore first year undergraduate perceptions in learning mathematics and their identification of strategies that may support them.

Methodology

The sample of students chosen for the study was 75 first year undergraduate students enrolled on the BA Applied Education Studies course in 2011. The mean age of these students was 30 (SD 9.5) and there were just nine males (13%). My aims were to examine the students' perceptions of learning mathematics before and after their first undergraduate mathematics education unit to see if there were any changes in their perceptions; if there were any such changes I wanted to identify what strategies may have contributed to this. In order to explore these aims I took a pragmatic approach to my research, whereby I aimed to identify the most appropriate tools to explore my issue (Feilzer, 2009; Denscombe, 2010). In this case I wanted to be able to utilise a combination of quantitative methods to establish students' perceptions regarding learning mathematics and qualitative methods to probe into the reasons behind such perceptions.

To establish greater detail regarding the context and background of the students prior to starting the course, my initial method utilised an existing dataset comprising two strands. Strand 1 contained the group statistics collected by the

University administration team and Strand 2 was the data collected from an initial audit of the students' mathematical skills on entry to the degree course. This audit included a number of mathematical questions and a confidence rating scale, where the students were asked to rate their perceived confidence in learning mathematics. Students were also given the opportunity to identify anything they wished to be known about their mathematical capabilities. This data was collected in May/June 2011.

I used survey research to explore my research issue further, comprising of two anonymous questionnaires and a series of focus group discussions. Questionnaire 1 (November 2011) was a pre-teaching questionnaire designed to explore the students' perceptions of learning mathematics prior to embarking on their first mathematics education unit. The questionnaire consisted of a range of closed questions about the students' feelings towards learning mathematics, their perceived levels of understanding and confidence, and their past experiences. Alongside this they were asked to describe an experience which had affected their feelings towards learning mathematics.

Questionnaire 2 (February 2012) was a post teaching questionnaire designed to explore the students' perceptions of learning mathematics once they had completed their first mathematics education unit. The questionnaire followed a similar structure to that of the pre-teaching questionnaire, using a range of closed questions to explore their feeling towards mathematics, perceptions of levels of understanding and confidence, any changes in these levels and factors that may have affected their views. They were also asked to provide a narrative account expanding on their views regarding factors affecting their learning of mathematics over their first unit.

Due to the limitations of a paper of this length, detailed discussion regarding validity and reliability is not appropriate here. However, in terms of triangulating the data this was achieved by using two or more methods to examine the research aims (Cohen et al., 2007). I acknowledge my own role as practitioner-researcher, as I was the teacher of the first undergraduate mathematics education unit, this was disclosed this role to the participants at the start of the research. Through triangulation I planned to minimise my personal bias, but I could not discount the fact that some participants may have responded in order to please me as their teacher (BERA, 2011).

Results

Audit Data

Using Pearson's r correlation coefficient, the audit data demonstrated a moderate to strong correlation between the students' audit percentage score and their perceived level of confidence (positive correlation coefficient 0.620, $p < 0.01$). 55 chose to provide additional comments and these were analysed for the use of positive, negative and aspirational language. The ten most commonly used words (five positive and five negative) formed the first part of the pre-teaching questionnaire to allow for a comparative analysis. When analysing the narrative comments, some students chose to make both positive and negative comments. Over half of these comments were negative (43 negative comments compared to 17 positive comments) and 14 of the students related their negativity to past experiences affected by others. This was reflected in comments such as "I felt a failure in school and this was backed up by home when I asked for help and was made to feel stupid and terrified". As the audit data was originally designed to gain background information for the purposes of teaching, further detail in regards to the research aims was gained from the pre-teaching questionnaire.

Pre-teaching questionnaire

Of the total sample of 75, 68 students completed the pre-teaching questionnaire and were first asked to identify as many of the following words that they associated with mathematics:

Strong, interest, easy, confident, enjoy, weak, fear, unconfident, struggle, difficult

Of the 247 words identified, the ratio of positive to negative vocabulary chosen was 87 to 160. In terms of understanding mathematics, 17 (25%) students perceived they had a good, or very good level of understanding and 24 (35%) students identified themselves as being confident or very confident in learning mathematics. Using Spearman's rho correlation, a strong correlation between perceived understanding and perceived confidence was identified, with a positive correlation coefficient of 0.707, $p < 0.01$. It was the narrative accounts of students' past experiences that provided an insight into what might have shaped these perceptions. 65 of the 68 students provided such an account, and these were analysed using a thematic coding system. A range of themes were identified that are shared in Table 1, with the top two influences being the effect of the teacher and students' personal perceptions, both identified by over half of the sample. A point to note here is that there were almost twice as many negative comments as positive comments, consistent with the earlier proportions identified within the questionnaire.

Table 1: Thematic Analysis of Pre-Teaching Questionnaire

	Positive comments	**Negative comments**	**Total number of comments**	**Percentage of students commenting**
Attendance	0	2	2	3
Behaviour	1	6	7	11
Effect of the teacher	14	28	42	65
Personal influences outside school	0	6	6	9
Tests and exams	2	3	5	8
Public nature of doing mathematics	0	5	5	8
Personal perceptions	22	16	38	58
Setting arrangements	2	17	19	29
Current role	5	0	5	8
Specific aspects of mathematics	3	4	7	11
Support	1	4	5	8
Total comments	50	91	141	

The effect of the teacher was the highest rated influence on learning mathematics (42 students, i.e. 65%), where fourteen students identified the teacher as a positive influence and twenty-eight as a negative influence. Where the teacher was seen as a positive influence, the comments were related to the supportive nature of the teacher who was described as encouraging, enthusiastic and helpful, demonstrated by one student who stated that she felt the change in her attitude at school was, "... due to an understanding teacher who supported the class well." Where the teacher was seen as a negative influence, comments related to the unsupportive nature of the teacher and being made to feel humiliated and scared.

Students' perceptions of their ability to carry out mathematics emerged as the second highest rated influence (38 students, i.e. 58%). Twenty-two students said that their enjoyment and willingness to work hard in learning mathematics had a positive effect on how they viewed the subject. Sixteen students perceived mathematics as a difficult subject and not within their capacity to learn, demonstrated in comments such as "I find maths a struggle because it has never been simple or easy for me to understand it".

Together, the audit data and the pre-teaching questionnaire demonstrated that a greater proportion of students had a negative rather than positive perception of learning mathematics and that two key themes emerged as having influenced these perceptions.

Post-teaching questionnaire

The post teaching questionnaire aimed to explore the students' perceptions having completed their first mathematics education unit as undergraduates. Of the total sample of 75, 64 students completed the post-teaching questionnaire and as for the pre-teaching questionnaire, they were first asked to identify words that they associated with mathematics. Of the 314 words identified the ratio of positive to negative vocabulary was 211 to 103, a reversal of the ratio identified in the pre-teaching questionnaire. 33 (52%) students perceived that they had a good or very good level of understanding of mathematics, in comparison with 17 (25%) students on the pre-teaching questionnaire, and 37 (59%) students identified themselves as being confident or very confident in learning mathematics with a strong positive correlation maintained between these two variables (Spearman's rho positive correlation coefficient 0.643, $p < 0.01$). Students were also asked to rate their comparative levels of understanding and confidence, with 53 (83%) identifying themselves as having a higher or much higher level of confidence and 55 (86%) as having a higher, or much higher, level of understanding. A point to note here is my concern about the students wanting to please me, so I compared the initial audit scores with the scores from the tests the students took as part of their assessment. 64 of the 75 students (86%) students had an increased score, suggesting that these perceptions regarding confidence and understanding may not be invalid.

In terms of what may lie behind this increase in positive perceptions, I turned to the narrative accounts, where the students were asked identify any factors that may have influenced how they felt about the learning of mathematics. 59 of the 64 respondents chose to respond, and as for the post-teaching questionnaire, these comments were analysed using a thematic coding system and the range of themes identified is summarised in Table 2. The ratio of positive to negative comments in this instance is 9 to 1, maintaining a consistency in students' increase in positive perceptions regarding learning mathematics.

Table 2: Thematic Analysis of the Post-Teaching Questionnaire

	Positive comments	Negative comments	Total number of comments	Percentage of students commenting
Online materials	9	0	9	15
Teaching	34	2	36	61
Teacher characteristics	7	0	7	12
Discussion/working with others	12	1	13	22
Practice	10	0	10	17
Tests and exams	0	3	3	5
Personal perceptions	11	3	14	24
Total comments	83	9	92	

The effect of teaching as an influencing factor was identified by over half of the students (36, i.e. 61%). 25 respondents made specific reference to the process of teaching mathematics, referring to the step by step break-down of methods, clear explanations and modelling techniques, reflected in responses such as "The teaching has been broken down and this has made me have a clearer understanding of areas of mathematics that I have previously worried about." Where students commented on the pace of the teaching sessions, this was either in reference to an appropriate pace, or in the case of two students, too slow a pace in areas where they already felt confident. Alongside reference to the teaching, seven students commented on the characteristics of the teacher in supporting them, including a sense of humour, and not being made to feel 'silly'.

Personal perceptions remained the second highest rated influence, identified by 14 (24%) students. As in the pre-teaching questionnaire, the positive perceptions (11 students, 18%) related to those students who found mathematics enjoyable and who were motivated by such enjoyment. Three students felt limited by their perceptions, with one student commenting, "I feel overwhelmed by the amount I have to learn, due to my lack if understanding in the first place."

The third rated theme was the role of discussion and working with others identified as a positive influence by 12 (20%) students.

Summary and next steps

This paper aimed to explore first year undergraduate perceptions of learning mathematics. I wanted to identify if there were any changes in the way the students perceived mathematics during their first year as undergraduates, and if so, what may have contributed to these changes.

The audit and the pre-teaching questionnaire showed that prior to their first mathematics education unit about two thirds of the students had a negative view of learning mathematics. Key themes contributing to this view included the role of the teacher, personal perceptions regarding mathematics and setting arrangements.

In exploring whether or not there had been a change in perceptions of learning mathematics, the post-teaching questionnaire provided data that identified that the ratio of negative to positive perceptions about learning mathematics had been reversed. Alongside this, the perceptions of understanding and confidence had increased in both categories and approximately four in five students believed that they had an improved level of understanding and confidence. This was supported by data demonstrating that almost all the students (64, 86%) showed an increase in attainment between their audit data and post unit tests. These combined factors demonstrate that there had been a change in the students' perceptions in learning mathematics in that they felt more positive, more confident and had a higher level of understanding in the subject.

As there had been a change in the students' perceptions following the completion of their mathematics unit, the final aim was to identify any strategies that might have contributed to these changes. The students' narrative accounts demonstrated that teaching was rated as the highest positive influence in learning mathematics, consistent with those who also identify this as a key influence (Tobias, 1993; Hodgen and Askew, 2006; Johnston-Wilder and Lee, 2010). In terms of the role of personal perceptions as the second rated influence on learning mathematics, this might be compared to the views of those who suggest that learning is affected by self-perception (Tobias, 1993; Dweck, 2007). Finally, the third theme linked to the role discussion and working with others could be considered alongside those who suggest that group work and discussion is an important factor in supporting learning (Askew et al., 1997; Gresham, 2007; Ashun and Reinink, 2009).

It is possible to infer from this research so far, that for some students there has been a change from a negative to positive view of learning mathematics. These students may have been affected by the factors discussed in the initial analysis of data; however, focus group discussions have been designed to probe more deeply into the themes identified and also to give consideration to any additional influences that may have not yet been identified. These discussions form the next steps of this project and once fully analysed will be shared within the full presentation of my thesis.

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Investigating young children's number line estimations using multimodal video analysis

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This research paper reports findings from an exploratory multiple case-study investigating children's cognitive representations of number, in particular during interactions with number line estimation tasks. The paper considers the cases of two Year 1 children (ages 5 and 6) and the findings from their participation in the pilot stage of this larger study. Children participated in individual task-based interviews, which were video-recorded and analysed using multimodal analysis. This methodology identified the cognitive representation of many structural aspects of natural number, and enabled a fine-grained differentiation of children's strategies during number line estimation tasks. Both cases considered invite continued investigation of the connection between strategies and estimation results. The implications for our understanding of children's cognitive representations of number, and the interpretation of number line estimation tasks, are discussed.

Keywords: representation, early number, estimation, multimodal, cognitive representation, primary education

Background

Representations of number are of foundational importance to mathematics, and research links immature number representations with not only lower mathematics performance but also with hindered learning of new mathematics (e.g. Booth and Siegler, 2008). The present study aims to achieve a deeper understanding of children's representations of number during their first year of formal schooling. In particular, the study offers new analysis of a central research task, the number line estimation task.

Key research has been carried out into children's imagistic representations of number, as well as their representations more generally, and found them to progress through clear stages of structural development (Mulligan, Mitchelmore and Prescott, 2005). Separate from this imagistic research is a substantial body of work within cognitive science, which has focused upon automaticised representations of numerical magnitude and their spatial aspects. The key research task used in this area has been the number line estimation task, in which participants position target numbers on an empty number line. Studies using this task have repeatedly documented what appears to be a 'shift' with age from a logarithmically to linearly calibrated mental number line. However, despite a large number of studies in the field, there remain disagreements over key ideas – for example the meaning of 'mental number line' – and the interpretation of existing data (e.g. Thompson and Opfer, 2010).

The present study is designed to deepen understanding in this area in three specific ways. First, it addresses a gap in the literature by explicitly investigating children's *interactions* with number line estimation tasks. Whilst the results of estimation processes are easy to measure, the processes themselves are difficult to reach, so despite these tasks being primary research tasks of the field, very little research has examined children's interactions with them. Petitto (1990) identified two strategies, 'counting on' and those involving midpoints, but was unable to provide further detail based solely on real-time observations. More recent work has made progress using eye-tracking (e.g. Schneider et al., 2008) and fine-grained statistical analysis (e.g. White and Szucs, 2012), with both methods suggesting variation in children's strategies based upon structural features such as orientation or 'anchor' points (e.g. decades). Both eye-tracking and purely statistical analyses have, however, proven less successful with younger children, and the above authors have explicitly noted the need for trial-by-trial analysis and the support of qualitative data.

Hypotheses concerning children's use of 'anchor points' in estimations relate directly to the second aim of the present study, namely to investigate relationships between the structure of children's cognitive representations of number and their number line estimations. This aim is motivated not only by the above hypotheses but also by further empirical evidence pointing to connections between different representations of number. These include the grounding of the mature concept of number in infants' numerosity representation systems (Carey, 2004); spatial similarities between participants' automaticised and imagistic representations (Fias and Fischer, 2005); and the susceptibility of children's magnitude representations to alteration through carefully designed educational activity (Thompson and Opfer, 2010).

A final contribution offered by the full study will be to complement current reliance on cross-sectional studies by following individuals through one school year.

Theoretical framework

The research adopts the theoretical approach to cognitive representations described by Duval (1999), supported by Presmeg's work on imagistic representation (e.g. 2006; Thomas, Mulligan and Goldin, 2002). Duval's framing acknowledges both intentional (semiotic) and automaticised (including perceptual) representations, and that relations exist between them. This inclusive framework is necessary, given the empirically observed connections noted previously. Duval argues that the customary distinction between mental and external representations is a "misleading division" (1999: 5), since it addresses only the "mode of production" rather than "nature" or "form" of representations. In terms of *images*, Duval's classification of representations emphasises that there exist two kinds of mental image: firstly "internalised semiotic visualisations", and secondly "'quasi-percepts' which are an extension of perception" (1999: 6).

The theoretical relation of cognitive representations to mathematics is that mathematical processes consist of transformations of representations, either *processing* (within registers) or *translation* (between registers) (Duval, 1999). Number line estimation tasks are of significant theoretical interest because they require transformation between registers: the *translation* between symbolic and verbal representations of number and spatial representations.

Theories of number concept development (see Nunes and Bryant, 2009) hold that children's understanding of structural aspects of number increases significantly during the early years of schooling, and this has been hypothesised as a cause of changes in children's interactions with estimation tasks (e.g. White and Szucs, 2012). The features potentially expected in children's cognitive representations of number at this age are sequence structure, relative magnitudes of numbers, half/double relationships, and some aspects of the base ten system (Thomas, 2004).

Research design/methodology

The research design of the wider study is an exploratory multiple case study, in which children participate in video-recorded individual task-based interviews. Since the exclusion of representations based on mode of production is taken to be theoretically unjustified, the study necessarily adopts a multimodal approach. The present paper presents analysis of two task-based interviews with Imogen (age 6) and Patrick (age 5), from a south of England primary school. Both children were assessed by their teacher to be of mid- to high- attainment in mathematics.

Task design

In each interview, four tasks were completed, designed to stimulate and require translation of cognitive representations of number. The first task (T1) required children to close their eyes and imagine the numbers 1 to 100, then to draw and describe the picture in their mind (adapted from Thomas, et al., 2002). Following this, children completed a number line estimation task (T2) in which they were asked to position number rocket stickers onto blank number lines (adapted from Thompson and Opfer, 2010). A third task (T3) asked children to estimate the quantity of sweets in clear plastic boxes. Finally, children were asked to estimate the number represented by already-positioned rockets on blank number lines (T4, adapted from Petitto, 1990).

In both number line estimation tasks, children were presented with randomised target numbers across different ranges, to be placed on blank number lines with only the endpoints labelled. The ranges tested, and hence the endpoint labels, were 0-10, 0-20, 5-15, and 0-100. Imogen completed a reduced version of T4 including only ranges 0-20 and 0-100. Before each number line estimation task, children completed a practice trial with the researcher, in which the target number consisted of an endpoint. Children received encouragement but no corrective feedback during trials.

Data analysis

Video data was transcribed separately for speech, gaze, and gesture (both co-speech and co-thought). These data were then analysed alongside children's paper-based representations. Imagistic representations were coded according to their component sign (pictorial, iconic, or notational) and all cognitive representations were examined for structural features of number. Estimation strategies were deduced from the cognitive representations identified during each estimation trial.

In line with previous studies, children's number line estimates were quantitatively analysed for their degree of linearity. In the number-to-position task (T2), analysis first converted estimate positions (measured from left endpoint, in millimetres) into the number line values that would have been 'hit' assuming a linear scale. In order to compare the linear accuracy of estimates in both T2 and T4, the absolute percentage error of each estimate was also calculated, by taking the absolute difference between target number and the number 'hit' by a child's estimate, as a percentage of the number line range for that estimation.

Linear and logarithmic models were fitted to children's target number estimates for each range individually. For each model, the coefficient of determination R^2 was calculated in order to compare model fits. A new analysis offered by the present study was to compare these quantitative analyses to children's other cognitive representations during estimations (e.g. gestures), in order to ask how strategies and cognitive representations might be linked to estimation result.

Findings

The full case study of Imogen (Williamson, 2013) discusses all four interview tasks and the relations drawn between them. The present paper narrows the focus to the number line estimation tasks T2 and T4, in order to present a more detailed analysis and discussion of these.

Number line estimation tasks (T2 and T4)

Representation of structural elements of number occurred in speech, gaze, and gesture during these tasks. For both children, clearly distinguishable estimation strategies were identified. In line with the study's explicit focus, the strategies were named and classified according to the structural features represented within them.

The following short example demonstrates three strategies, identified in Imogen's T2 (number-to-position) estimation of target 13 on the range 5-15. The strategies here identified are "Reference to right endpoint (RE)", "Count-on from midpoint (MP) in 1's", and "Count-on from target":

> **Interviewer**: It's thirteen. Where do you think thirteen belongs? [Imogen's gaze goes quickly to RE (15) then to proffered rocket sticker]
>
> **Imogen**: [takes rocket with R-hand, transfers to L-hand, pauses] 'Cause ten is here [R-hand points onto MP and holds] … [R-hand 'hops' to right; both hands stick rocket to right of last 'hop'] Fourteen fifteen [R-hand thumps line between rocket and RE; thumps RE itself]

Multiple aspects of number structure are represented in this example. The number sequence is represented in two count-on strategies, with a confidence that allows Imogen to start counting midway through the sequence. The units represented by gesture during counting on represent a further aspect of structure: the spatial extent of each unit is approximately equally sized, and scaled so that Imogen's sequence from ten to fifteen covers the spatial extent from indicated midpoint to endpoint. Number structure is also apparent in Imogen's use of the right endpoint (fifteen) as an appropriate 'anchor point' for the target number thirteen. The structural role of ten as the midpoint is represented in both speech and gesture.

Representation of number structure also occurred outside of particular estimations, particularly when a new number range was introduced. Imogen, for example, represented evenly spaced multiples of five in an exclamation on seeing the first page of 5-15 trials: "Shouldn't it be five TEN …?" [right hand points to midpoint of line and holds]. Patrick also made verbally explicit some of the scaling evident in Imogen's physical gestures, for example explaining that "we should squeeze them in a bit 'cos it's got ten more" when the number range changed from 0-10 to 0-20.

Unsurprisingly, the most commonly identified strategies during children's estimations were references to both the left and right endpoints of the presented number line. Next in frequency was counting on from the left endpoint (in single units). This form of counting of course features frequently in the classroom, and both children demonstrated imagistic representations of the counting sequence, similarly based upon a left-to-right increasing sequence, elsewhere in the interview.

The strategies identified from Imogen and Patrick's interviews are listed below in Table 1, together with the target numbers of the trials in which they were observed. Patrick and Imogen employed a similar array of strategies, but with application to different target numbers and with differing frequency. Overall, strategies were identified more often in Patrick's interactions with estimation tasks than in Imogen's. For Patrick, references to left and right endpoints were apparent in every single number range in both tasks. The shaded rows represent Patrick's results.

Strategy	**T2: number-to-position**				**T4: position-to-n**			
	0-10	**5-15**	**0-20**	**0-100**	**0-10**	**5-15**	**0-20**	**0-100**
Count on (LE, 1's)	4 6		2				2	3
	2 3 4 6 7 8	6 7 9 10 11 14	2 4 6 9 11 15 16	3 6		7 13		25
Count on (LE, 5's)		9						
Count on (MP, 1's)		11 13	6					
				92			9	
Count on (target)		13						25
	9		7 19		6		9	67
Count back (MP, 1's)			6					
Count back (RE, 1's)								
		13				11 13		
Count back (other)		11						
Ref. to other trial		7 14	7	48			15	18
	3	14	9 11	92	2	9	7	
Ref. to LE	2 3 7 8	6	4 13 18	2 4 6 25 48 67			4 7 16	6 71 86
	1 2 3 4 5 6 7 8	6 10 11 13 14	2 4 6 7 11 16	2 3 6 18 49 71	2 4 5	7 9 10 11 13	7 9 11	2 3 25 50
Ref. to RE	3 7 8 9	13	16 18	2 4 48			7	48 71 86
	2 8 9	6 10 11 13 14	11 15 18 19	71 92	4 6 7	7 9 10 11 13	2 9 11 15 16	2 3 4 6 18 67 71 92
Ref. to MP		11 13	6	3 18 25 67 71				
	5	11		49 50 92			6 9 15	
Ref. to quarter								
				25				
Ambiguous		8	15	86			6 13 18	3 5 67
					8 9		4	18

Table 1: Imogen and Patrick's strategies in T2 and T4 estimation tasks. Patrick's results are shaded. Abbreviations: LE=left endpoint, RE=right endpoint and MP=midpoint.

A clear difference can be seen between Imogen's strategies in T4 (position-to-number) compared to T2 (number-to-position). For example, whilst references to the midpoint are made across three ranges and various targets in T2, no midpoint references at all are identified during T4. Within T2, Imogen demonstrated the greatest range and frequency of estimation strategies in estimations on the range 5-15.

Quantitative analysis

In line with expectations from previous studies, the linear accuracy of Imogen's estimates decreased on larger ranges. In T2, the mean absolute percentage error was low for both 0-10 and 5-15 (7.6% and 4.9%), and rose to 25.5% on the range 0-100. In terms of model fit, Imogen's T2 estimates for range 0-10 were best described by a linear model (R^2_{Lin}=.94; R^2_{Log}=.84). On the range 5-15, the comparison was inconclusive for both children (Imogen: R^2=.97 both models, Patrick: R^2_{Lin}=.96; R^2_{Log}=.93). On ranges 0-20 and 0-100, Imogen's estimates were more consistent with logarithmic models (R^2_{Log}=.71 and R^2_{Log}=.57, compared to R^2_{Lin}=.64 and R^2_{Lin}=.39).

In contrast to Imogen, Patrick's T2 estimates demonstrated high linear accuracy across all ranges (PAE 5.9% to 9.3%). Patrick's estimates were better fit by linear models for ranges 0-10, 0-20 and 0-100 (R^2_{Lin}=.89, R^2_{Lin}=.63 and R^2_{Lin}=.99 respectively; R^2_{Log}=.75, R^2_{Log}=.86 and R^2_{Log}=.85).

In terms of differences between number-to-position (T2) and position-to-number (T4) accuracy, a paired samples t-test found a significant difference between Imogen's estimation error in T4 (PAE mean=11.94, SD=10.00) compared to T2 (PAE mean=21.59, SD=8.78); t(16)=2.89, p=.011. Patrick demonstrated no significant difference in linear accuracy between tasks (T4 PAE mean=8.42, SD=7.42; T2 PAE mean=7.55, SD=6.36; t(36)=-.568, p=.573).

Imogen's position-to-number (T4) estimates were better fit by linear models for both ranges 0-20 (R^2_{Lin}=.97) and 0-100 (R^2_{Lin}=.91, compared to R^2_{Log}=.87 and R^2_{Log}=.77). As in T2, Patrick's estimates were better fit by linear models for each range (R^2_{Lin}=.92, R^2_{Lin}=.67, R^2_{Lin}=.97, R^2_{Lin}=.95 for ranges 0-10, 5-15, 0-20 and 0-100 respectively, compared to R^2_{Log}=.86, R^2_{Log}=.61, R^2_{Log}=.95, R^2_{Log}=.74).

Differences between Imogen's T2 and T4 estimates correlate with differences in estimation strategy. In T2, Imogen's estimation strategies included frequent reference to the midpoint, and her estimations fall within a band in the centre of the range (see Figure 1, below). In T4 in contrast, her strategies omit the midpoint, referring only to the endpoints, and the estimations are clustered around the endpoints.

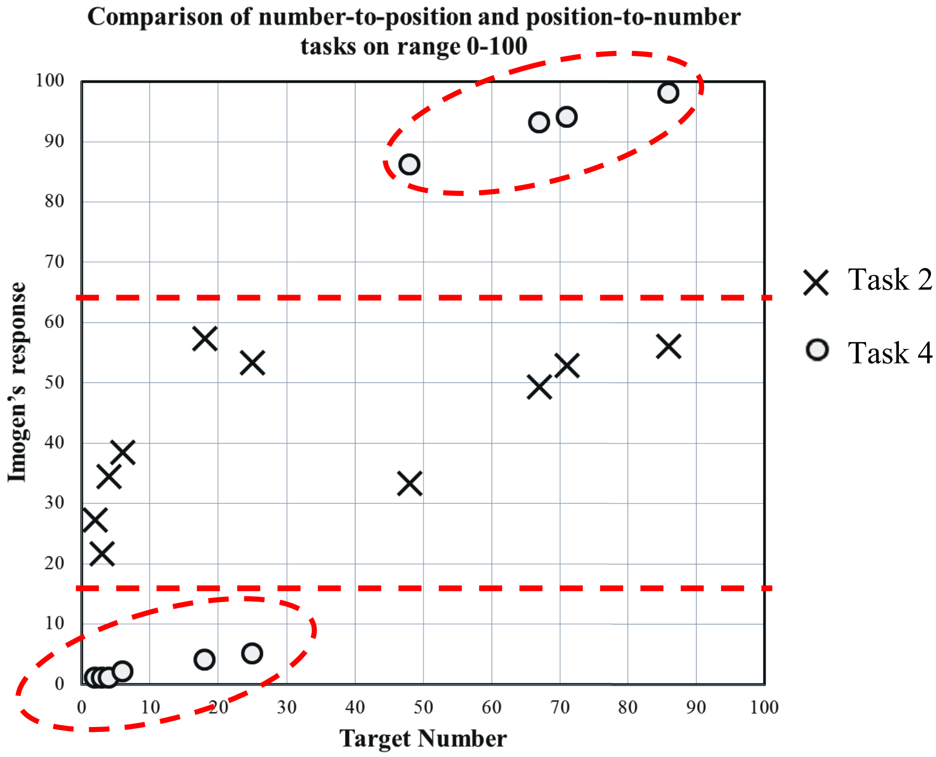

Figure 1: Imogen's T2 and T4 estimates on the range 0-100

Discussion

Both children cognitively represented natural number in ways that encoded significant structural elements, and the chosen methodology enabled the identification of differing strategies used by children in their number line estimations.

The qualitative data from this study supports White and Szucs' hypothesis (2012: 9) that estimation strategies may vary depending on "familiarity with the number range, proximity to either external or mental anchor points, as well as knowledge of arithmetic strategy". The use of particular strategies differed between the two children, and between ranges, tasks and target numbers. As noted, both children made frequent reference to the left endpoint of the line and were most likely to begin any count-on strategies from the left. To this extent, both children appeared to use the left endpoint as a default 'anchor point'. Trials in which target numbers were in proximity to other potential 'anchor points' revealed both children varying their estimation strategies. For example, both referred to the right endpoint *and made no reference to the left* when estimating the position of 9 on the range 0-10; the target 9 appeared to 'cue' them directly to the endpoint 10. Similarly, both children made reference to the midpoint when estimating target 11 on the range 5-15. Overall, Patrick demonstrated 'anchor point' strategies more frequently: for example in referring to midpoints whilst positioning targets 49 and 50 on 0-100; commenting "Quarter of the way" whilst positioning target 25; and explaining of target 9 on the range 0-20 that "it's getting quite close ... could have been half but it's a tiny bit off".

A limitation of the methodology is that it is of course not able to capture all cognitive representations during a given estimation. What this paper has attempted to show is that sufficient aspects can be captured to reveal scientifically interesting variations in children's task interactions. Certainly, the data further challenge the idea that it is meaningful to describe children's estimations in terms of a fixed model or representation for a given number range, whether quantitatively, as queried by White and Szucs (2012), or qualitatively. There are good examples of variation dependent on the particular target. In T4 range 5-15 for example, Patrick deduced from a rocket's midpoint location that it must represent 10. The next trial presented a target further to the right, in fact representing 11. In this second trial, there was no evidence of Patrick representing a midpoint structure, and the strategy he adopted – counting back from the right endpoint – led him to an estimate ("8") which in fact conflicted with the identification he had just made of the midpoint 10. Discussing these two consecutive trials in terms of 'Patrick's representation of the range 5-15' obscures something important.

The findings do not indicate any simple relationship between estimation strategy and result. Both children applied counting strategies, commonly regarded as an unsophisticated approach, with some success, emphasising that the detail of children's interactions must be attended to, such as adjustment of unit size depending on scale. Overall, Imogen's linear accuracy was highest for her T2 trials on the range 5-15, during which she demonstrated each of the strategies seen in her interview overall, as well as spontaneous cognitive representations with structural detail such as the subdivision of the 5-15 line into equally sized fives. This is suggestive, as is the observed correlation between strategy difference and estimation result for Imogen's estimates on the range 0-100 (see Figure 1).

This paper has attempted to show the potential of the current methodology to illuminate children's interactions with number line estimation tasks. The findings discussed are to be followed up in the full multiple case study, to contribute further to understanding the relationships between cognitive representation, estimation strategies and results, and the interaction between task difference and strategy difference.

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Teaching mathematics for social justice: translating theory into classroom practice

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There is growing consensus, amongst teachers, teacher educators and researchers, that a more engaging and relevant school mathematics curriculum is needed, with greater emphasis on problem-solving skills and the development of conceptual understanding. Too much focus on factual recall and procedural understanding has led to unacceptable levels of disengagement and disaffection amongst learners.

This paper reports on initial findings from a project involving a group of teacher researchers who share a commitment to addressing the alienation of learners, raising their awareness of the nature of mathematics and the reason for learning it, developing student agency and an appreciation of how mathematics can be used to better understand the world around them.

I make use of a participatory action research methodology to explore how being part of a research group can help teachers to begin to develop their classroom practice in ways which resonate with a commitment to teaching mathematics for social justice. I identify four themes emerging from an analysis of an initial series of semi-structured empathetic interviews with the teacher researchers that provide a useful theoretical framework for the development of the project and useful insight for others wishing to carry out research in a similar field.

Keywords: mathematics education, social justice, action research.

Rationale behind the research project

Teachers hold a range of views on the aims for teaching mathematics including developing an appreciation of the beauty of the subject and enabling learners to develop important reasoning and communication skills in preparation for solving problems in 'real' life. However, these aims are increasingly in conflict with the educational policies of successive governments, which have placed renewed emphasis on basic skills and the functional aspects of mathematics.

Consensus amongst mathematics teachers, teacher educators, academics, local authority advisors and school inspectors in England, is that a more engaging and relevant mathematics curriculum is needed with greater emphasis on problem-solving and conceptual understanding (ACME, 2011; Ofsted, 2012). However, the high stakes nature of mathematics assessment, and resultant pressure to 'teach to the test', means that most mathematics classrooms remain based on the teaching of a school mathematics curriculum lacking relevance and interest, focusing on factual recall and procedural understanding, and leading to the disengagement and disaffection of learners (Nardi and Steward, 2003).

Gutstein (2006: 10) describes the demands of modern capitalist economies for "an ever-growing army of low-skilled, compliant, docile, pleasant, obedient service sector workers … [to perform] tasks that do not require workers to do mathematics in the same ways as the people they serve", explaining why many students, particularly those from low-income and ethnic minority families, receive a predominantly functional mathematics education.

This research project is intended to challenge this status quo by promoting the teaching of mathematics for social justice (TMSJ), an aim that is best overlooked, and at worst discouraged, by current policies, practices and discourses. With the aim of bringing about desirable change in the teaching of mathematics, the following question is posed: “How can mathematics teachers translate a commitment towards ‘education for social justice’ into pedagogy and classroom practices which resonate with such a commitment?”

Whilst there is an abundance of research literature on mathematics education and social justice, most studies tend to be theoretical or philosophical in nature. What limited published research is available, focusing on translating such ideas into classroom practice and pedagogy, is based mainly on studies from the US, where conditions within schools are significantly different to those in England.

A ‘participatory action research’ methodology

My methodology is based on the premise that mathematics is a human construct, created by a need to make sense of the world. As such it is value-laden rather than neutral, with its content determined by those in positions of power, such as those who fund research into new areas of mathematics. I consider privilege, equity and social justice to be critical to the study of mathematics education, which is fundamentally a social and political practice (Ernest, 2004).

The primary aim of my research is to bring about desirable social change, in this case working with teachers to help them translate their commitment to TMSJ into practice. Such a stance necessitates considering the power relationships between all participants in the research, including teachers, students and school leaders, and recognising the important role that I, as researcher, will play in constructing knowledge and meaning. From the acceptance of my position of partiality in the research process, reflexivity, i.e. being able to stand back and observe your own subjectivity from a distance, is essential in establishing validity (Reason, 1994). Ladkin (2005) argues that concepts such as inequity and social justice cannot be fully understood without experiencing them, necessitating the construction of knowledge through action.

Bearing in mind the focus and aim of my research, I have adopted an action research methodology based on fostering collaborative relationships between myself as researcher and teachers. Torrance (2004: 199) highlights how a collaborative approach between academics and teachers through action research generates research data that is “crucial to developing an understanding of theory-in-practice”.

A ‘critical research’ design

The TMSJ Research Group was established in June 2013, comprising five teacher researchers, who responded to my invitation, and me, as university-based researcher. I had sent out invitations to former student teachers who completed their PGCE mathematics qualification and gained qualified teacher status in 2012, all of whom I have worked with in my role as a university-based teacher educator. An accompanying leaflet included detailed information on the aims of the project and made clear the requirement that research participants should share a commitment towards TMSJ and be prepared to attend seven research group meetings, participate in three interviews, keep a reflective diary and try out at least one classroom activity per action research cycle (three in total).

I have based my research design on Skovsmose and Borba's (2004) 'critical research' model, which aims to carry out research 'with' others rather than 'on' others and to uncover 'how' and 'why' a situation can be different. The model involves participatory action research cycles based on current, imagined and arranged situations. The 'current situation' (CS) is, in this case, the reality of mathematics teaching in schools as described above. Because of the importance of developing a critical understanding of the CS, and acknowledging that this situation should not be taken as given, teacher researchers were encouraged at the initial meeting to reflect upon, and develop a critique of their own classroom practice. The 'imagined situation' (IS) is a vision of what could be, in this case what teacher researchers envisaged as a desirable alternative to the CS after engaging with my conceptualisation of TMSJ. The 'arranged situation' (AS) represents an attempt to put some aspect of the imagined situation into practice, bearing in mind the reality and constraints of the current situation. The aim of the three critical research cycles, to be spread over the 2013-14 academic year, is to coordinate the sharing of ideas and the planning, teaching and evaluation of classroom activities. I anticipate that, through evaluating the success of these interventions (AS_1, AS_2, …) and discussing their significance in relation to the imagined situation, teacher researchers will begin to develop their own conceptualisations of TMSJ (IS_1, IS_2, …), and develop their own classroom practices (CS_1, CS_2, …).

An initial seminar-style meeting was held in July 2013 at which I provided further information about the project. I presented my research methodology and my initial conceptualisation of TMSJ (described in the next section below) for further discussion and invited teacher researchers to relate this to their current classroom practice. Planas and Civil (2009) highlight the critical role that the university-based researcher should play in establishing and facilitating a research group of teachers, in particular, by raising teachers' awareness of issues of social justice, promoting collaboration and sharing of ideas amongst group members, taking responsibility for the collation and analysis of data and reporting research findings.

My initial conceptualisation of 'teaching mathematics for social justice'

During my fifteen years of teaching mathematics, I witnessed first-hand the effects of educational policies which serve to marginalise and alienate certain groups of students in the mathematics classroom. I have seen how the dominant school mathematics discourse has changed, from the promotion of student-centred learning and mixed-ability teaching in the mid 1980s, towards a culture of rigid testing and setting based on prior attainment. This has strengthened my commitment towards mixed-ability teaching, collaborative inquiry-based learning and empowering students through mathematics.

In developing my own conceptualisation of TMSJ, I draw upon Gutstein's (2006) work on 'reading and writing the world with mathematics', in turn based on Freire's ideas on literacy and conscientisation, and upon Skovsmose's (2011) model of 'critical mathematics education'. Gutstein's (2003) 'real world projects' and Boaler's (2008) *Railside Project* aim to develop student agency, through using mathematics to understand and change the world around them, and making decisions and taking responsibility for their own learning in mathematics. Skovsmose (2011) describes these processes as reflecting 'with' and 'through' mathematics. However, he also contends that students need to reflect 'on' mathematics by developing a critical understanding of the nature of mathematics and its role in society.

Ernest (1991) characterises epistemologies of mathematics as being either 'absolutist', i.e. based on a view of mathematical truth as objective and unquestionable, or 'fallibilist', i.e. based on a view of mathematical truth as a human construct and open to constant revision. He argues that a 'public educator' ideology, which is similar to what I describe as a commitment towards TMSJ, is associated with a 'fallibilist' epistemology, and is the only ideology of mathematics education to acknowledge that the assertion that mathematics is 'value-free' in nature is a myth.

I also draw upon Bourdieu's theory of reproduction (Bourdieu and Passeron, 1990), which regards one of the school's primary functions as reproducing the social order and maintaining unequal power relations between different classes and groups in society. Mathematics contributes to this process by functioning as a 'critical filter', with success in school mathematics providing much higher levels of access to further education and employment opportunities (Black et al., 2009). Mathematical attainment remains strongly correlated to social class, more so than to any other difference such as ethnicity or gender (Noyes, 2007). Bernstein's (2000) analysis suggests a danger that well-meaning attempts to encourage student-led learning and collaborative problem-solving pedagogies, might actually contribute to stronger reproduction of inequities as a result of middle class students benefiting from less explicit recognition and realisation rules. However, rather than avoiding the use of such pedagogies, I argue that the solution is to make the 'rules of the game' and rationale behind classroom practices more explicit to students.

In summary, my conceptualisation of TMSJ includes five aspects: using collaborative, problem-posing and problem-solving pedagogies; emphasising the cultural relevance of mathematics and recognising learners' real-life experiences; using mathematical inquiries that enable learners to develop greater understanding of their social, political, economic and cultural situation; developing agency that enables learners to realise opportunities which these situations provide; developing a critical awareness of the nature of mathematics and its position and status within education and society.

Data collection and analysis

Following the initial meeting of the research group, I completed a series of semi-structured empathetic interviews with teacher researchers in September 2013. A thematic analysis approach was used to analyse the data from these interviews, making use of "meaning condensation" and "meaning interpretation" (Kvale and Brinkmann, 2009: 207). This approach draws on methods from 'grounded theory', described by Gibson and Brown (2009: 26) as "the process of developing theory through analysis, rather than using analysis to test preformulated theories". Such methods are consistent with my methodology as there is no pre-existing hypothesis, on how to translate a commitment towards TMSJ into practice, to be tested.

Since my intention is to analyse and report teacher researchers' experiences as a "readable public story" (Kvale and Brinkmann, 2009: 181), interviews were audio-recorded and then transcribed using a literary style by, for example, ignoring pauses, fillers, intonations, colloquialisms during conversations. By reading and re-reading the transcripts, and listening again to the original audio recordings, the meaning within the transcripts was then condensed by breaking down the text into units of meaning and summarising these as simply as possible using descriptive text. Drawing upon the "constant comparative method" (Gibson and Brown, 2009: 28), I then assigned a category to each unit of meaning, allowing me to re-read the data, focusing on selections of text which shared the same category, in order to explore commonalities, differences and relationships between the units of meaning. This process enabled me to draw out four themes from the data, whilst avoiding the danger of losing meaning through the use of codes and categories in too rigid a way.

Jackson and Mazzei (2012: viii) outline the need to avoid the "simplistic treatment of data and data analysis in qualitative research that … reduce complicated and conflicting voices and data to thematic 'chunks' that can be interpreted free of context and circumstance". As an alternative they suggest "plugging in" the data to the texts of theorists whose work underlies the research. This process, characterised by "reading-the-data-while-thinking-the-theory" (p.4) allows new analytical questions to emerge that can give new meaning to the data. I conclude this paper therefore by relating the four emerging themes back to the research question and the theories underlying my initial conceptualisation of TMSJ in order to identify wider themes and create meaning from the data (Kvale and Brinkmann, 2009).

Emerging themes

Four themes emerged from the analysis of the first set of interviews with teacher researchers, whose names have been replaced below with pseudonyms. The first theme was that of changing epistemologies of mathematics. All teacher researchers had a close relationship with mathematics and considered themselves successful learners of the subject. However, involvement with the project had led teacher researchers to begin to question seriously, often for the first time, their previous assumptions about the nature of mathematics. Each of them described a shift from a perception of mathematics as 'value-free', e.g. as a child Anna saw it as just calculations and algebra, towards a view of mathematics as more 'value-laden'. Rebecca highlighted how the apparently value-free nature of mathematics had been part of its attraction for her when she was at school. However, her new-found desire to help students to see the relevance of the subject to their real life situation had made her begin to rethink her relationship with mathematics, potentially placing her in a position of epistemological conflict. The suggestion to make the nature and status of mathematics more explicit to students, as discussed at the first research group meeting, was welcomed with interest by teacher researchers, although this was a new idea for all of them with the exception of Brian, who had previously discussed with students their feelings towards the subject.

The second theme emerging from the data was around beliefs in developing student agency, related to a common desire amongst teacher researchers to change society for the better, described by Anna, Brian and George as the primary reason for becoming teachers. Student agency was seen as having two characteristics. Firstly, there was a strong belief that students should take responsibility for their own learning, for example, by posing their own problems and developing their own problem-solving skills and mathematical reasoning. Sarah and George, in particular, placed a strong emphasis on helping students to see the rationale behind learning specific mathematical procedures, thus making school mathematics more meaningful. Secondly, agency was seen as empowering students to use mathematics as a tool to make sense of the world around them, particularly important, as highlighted by Rebecca, given the amount of information in the modern world including misleading arguments based on the misuse of data. Thus teacher researchers saw it as legitimate to tackle social justice issues within mathematics lessons, including common misperceptions of the level of benefit fraud (Anna), levels of global inequality (Brian), water usage and sustainability (George) and exploring data about lifestyle (Sarah). There was general agreement that the boundaries between mathematics and other subjects are too rigid and such barriers should not stand in the way of tackling issues such as those outlined above.

The third theme to emerge was an appreciation of the opportunity provided by the research project to share ideas and work collaboratively with other teachers in a research group, described by all five teacher researchers as a key motivation for taking part. There was a wide range in levels of previous engagement with social justice issues amongst teacher researchers, with Rebecca beginning to consider their relevance to teaching for the first time, whilst Brian and George had been active in development organisations in the field of education for a number of years. However, all teacher researchers demonstrated a strong commitment to learning from each other and a strong desire to engage with research on participatory action research and TMSJ. They saw the project as coming at an ideal time in their careers, all having just completed their NQT year, when they were beginning to see themselves as established classroom practitioners and to consider the direction they would like the development of their classroom practice to take. Anna, Ben and George saw the project as an opportunity to re-engage with the reasons why they came into teaching in the first place. The collaborative nature of the project was also seen as a way of providing mutual support for overcoming challenges to bringing social justice issues into the mathematics classroom, including the additional time, energy and creativity required to plan such lessons and pressure to get through the scheme of work and prepare students for high-stakes school mathematics examinations. Brian described how high levels of monitoring and scrutiny of lessons by senior staff within the school resulted in 'low-risk' teaching. He saw collaboration with others through the project as giving legitimacy to tackling issues of social justice in his mathematics lessons that he might otherwise be more reluctant to do.

The fourth theme to emerge related to dominant discourses within mathematics education around 'ability', equity and attainment. All five teacher researchers had recently completed an initial teacher education programme that promoted a discourse of addressing educational inequities by focusing on raising attainment of students in schools with relatively disadvantaged intakes. Related to this was a perceived conflict between focusing on raising the mathematical attainment of students and teaching mathematics for social justice, e.g. Anna articulated a reluctance to lose the focus on preparing her students for exams. A focus on raising student attainment at the expense of tackling structural inequities and injustices was described by some teacher researchers as counter-productive, e.g. George highlighted the mismatch between the demands of higher education and employers for more creative independent thinkers, and a curriculum and system of schooling which stifles creativity and promotes compliance. Most teacher researchers suggested that raising issues of social justice would be easier with higher attaining groups of students due to their generally better behaviour and dispositions towards mathematics, and that it would need to be approached differently for lower attaining groups. This is ironic considering that students in lower attaining groups are generally those most disadvantaged by the current school system that TMSJ sets out to challenge. The assumption that setting in mathematics should be the norm in secondary schools, despite little research evidence to suggest that it works, generally went unchallenged by teacher researchers, although this may reflect the fact that it is something that they are currently powerless to change and is beyond the remit of the project. George did acknowledge that teachers' own views of 'ability' and expectations of students should be considered as problematic.

Conclusion

By 'plugging in' the four themes emerging from the interviews to the theories underlying my initial conceptualisation of TMSJ, four wider themes have emerged which I use to develop further the theoretical framework underlying the project.

Insight into changing epistemologies of mathematics amongst teacher researchers might be gained from relating this development to Ernest's (1991) association between a 'fallibilist' view of mathematics and a 'public educator' ideology. The teacher researchers, whose views of mathematics are very much under self-review and liable to change, have already demonstrated a strong commitment to TMSJ. This suggests that Ernest's association is less clear-cut, perhaps reflecting the nature of school mathematics, in which questions about its nature, and the rationale for its study, are rarely considered.

A strong belief in developing student agency resonates with Gutstein's (2006) ideas on 'reading and writing the world' with mathematics, i.e. using mathematics to explore issues of social justice relating to students' real life situations is essential for making mathematics more meaningful and developing mathematical understanding.

The high levels of interest of teacher researchers in the collaborative nature of the project, and the research methods employed, highlights the potential of a participatory action research methodology for engaging teachers in research, as well as generating research findings based on the genuine interaction between theory and practice (Reason, 1994; Torrance, 2004). The active involvement of the teacher researchers 'in' the research process, including a willingness to engage critically with the research methodology, contrasts with the views of those advocating evidence-informed practice who are more likely to emphasise the need for teachers to engage 'with' research findings, i.e. implement what other researchers have shown to work.

Bourdieu would argue that the predominance of setting within mathematics classrooms, whilst being used to support the claim of the existence of a meritocratic system, is in reality disguising the primary function of schools, which is to reproduce inequities and hierarchies within society, hence the need to stifle creativity and promote compliance (Bourdieu and Passeron, 1990). He claims that teachers who wish to challenge this situation are in a contradictory position as, by the very fact of being teachers, they are giving legitimacy to the schooling system.

These wider themes will inform the development of the research project and help to capture the stories of how teacher researchers' conceptualisations of TMSJ, their own relationships with mathematics, and their success in developing student agency, evolve as they develop their own classroom practices and strategies for working within the constraints of the current school system. They also provide insight for other teachers and researchers, sharing a similar interest and commitment towards TMSJ, who wish to explore ways of translating such a commitment into practice.

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