Investigating young children's number line estimations using multimodal video analysis

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This research paper reports findings from an exploratory multiple case-study investigating children's cognitive representations of number, in particular during interactions with number line estimation tasks. The paper considers the cases of two Year 1 children (ages 5 and 6) and the findings from their participation in the pilot stage of this larger study. Children participated in individual task-based interviews, which were video-recorded and analysed using multimodal analysis. This methodology identified the cognitive representation of many structural aspects of natural number, and enabled a fine-grained differentiation of children's strategies during number line estimation tasks. Both cases considered invite continued investigation of the connection between strategies and estimation results. The implications for our understanding of children's cognitive representations of number, and the interpretation of number line estimation tasks, are discussed.

Keywords: representation, early number, estimation, multimodal, cognitive representation, primary education

Background

Representations of number are of foundational importance to mathematics, and research links immature number representations with not only lower mathematics performance but also with hindered learning of new mathematics (e.g. Booth and Siegler, 2008). The present study aims to achieve a deeper understanding of children's representations of number during their first year of formal schooling. In particular, the study offers new analysis of a central research task, the number line estimation task.

Key research has been carried out into children's imagistic representations of number, as well as their representations more generally, and found them to progress through clear stages of structural development (Mulligan, Mitchelmore and Prescott, 2005). Separate from this imagistic research is a substantial body of work within cognitive science, which has focused upon automaticised representations of numerical magnitude and their spatial aspects. The key research task used in this area has been the number line estimation task, in which participants position target numbers on an empty number line. Studies using this task have repeatedly documented what appears to be a 'shift' with age from a logarithmically to linearly calibrated mental number line. However, despite a large number of studies in the field, there remain disagreements over key ideas – for example the meaning of 'mental number line' – and the interpretation of existing data (e.g. Thompson and Opfer, 2010).

The present study is designed to deepen understanding in this area in three specific ways. First, it addresses a gap in the literature by explicitly investigating children's *interactions* with number line estimation tasks. Whilst the results of estimation processes are easy to measure, the processes themselves are difficult to reach, so despite these tasks being primary research tasks of the field, very little research has examined children's interactions with them. Petitto (1990) identified two strategies, 'counting on' and those involving midpoints, but was unable to provide further detail based solely on real-time observations. More recent work has made progress using eye-tracking (e.g. Schneider et al., 2008) and fine-grained statistical analysis (e.g. White and Szucs, 2012), with both methods suggesting variation in children's strategies based upon structural features such as orientation or 'anchor' points (e.g. decades). Both eye-tracking and purely statistical analyses have, however, proven less successful with younger children, and the above authors have explicitly noted the need for trial-by-trial analysis and the support of qualitative data.

Hypotheses concerning children's use of 'anchor points' in estimations relate directly to the second aim of the present study, namely to investigate relationships between the structure of children's cognitive representations of number and their number line estimations. This aim is motivated not only by the above hypotheses but also by further empirical evidence pointing to connections between different representations of number. These include the grounding of the mature concept of number in infants' numerosity representation systems (Carey, 2004); spatial similarities between participants' automaticised and imagistic representations (Fias and Fischer, 2005); and the susceptibility of children's magnitude representations to alteration through carefully designed educational activity (Thompson and Opfer, 2010).

A final contribution offered by the full study will be to complement current reliance on cross-sectional studies by following individuals through one school year.

Theoretical framework

The research adopts the theoretical approach to cognitive representations described by Duval (1999), supported by Presmeg's work on imagistic representation (e.g. 2006; Thomas, Mulligan and Goldin, 2002). Duval's framing acknowledges both intentional (semiotic) and automaticised (including perceptual) representations, and that relations exist between them. This inclusive framework is necessary, given the empirically observed connections noted previously. Duval argues that the customary distinction between mental and external representations is a "misleading division" (1999: 5), since it addresses only the "mode of production" rather than "nature" or "form" of representations. In terms of *images*, Duval's classification of representations emphasises that there exist two kinds of mental image: firstly "internalised semiotic visualisations", and secondly "'quasi-percepts' which are an extension of perception" (1999: 6).

The theoretical relation of cognitive representations to mathematics is that mathematical processes consist of transformations of representations, either *processing* (within registers) or *translation* (between registers) (Duval, 1999). Number line estimation tasks are of significant theoretical interest because they require transformation between registers: the *translation* between symbolic and verbal representations of number and spatial representations.

Theories of number concept development (see Nunes and Bryant, 2009) hold that children's understanding of structural aspects of number increases significantly during the early years of schooling, and this has been hypothesised as a cause of changes in children's interactions with estimation tasks (e.g. White and Szucs, 2012). The features potentially expected in children's cognitive representations of number at this age are sequence structure, relative magnitudes of numbers, half/double relationships, and some aspects of the base ten system (Thomas, 2004).

Research design/methodology

The research design of the wider study is an exploratory multiple case study, in which children participate in video-recorded individual task-based interviews. Since the exclusion of representations based on mode of production is taken to be theoretically unjustified, the study necessarily adopts a multimodal approach. The present paper presents analysis of two task-based interviews with Imogen (age 6) and Patrick (age 5), from a south of England primary school. Both children were assessed by their teacher to be of mid- to high- attainment in mathematics.

Task design

In each interview, four tasks were completed, designed to stimulate and require translation of cognitive representations of number. The first task (T1) required children to close their eyes and imagine the numbers 1 to 100, then to draw and describe the picture in their mind (adapted from Thomas, et al., 2002). Following this, children completed a number line estimation task (T2) in which they were asked to position number rocket stickers onto blank number lines (adapted from Thompson and Opfer, 2010). A third task (T3) asked children to estimate the quantity of sweets in clear plastic boxes. Finally, children were asked to estimate the number represented by already-positioned rockets on blank number lines (T4, adapted from Petitto, 1990).

In both number line estimation tasks, children were presented with randomised target numbers across different ranges, to be placed on blank number lines with only the endpoints labelled. The ranges tested, and hence the endpoint labels, were 0-10, 0-20, 5-15, and 0-100. Imogen completed a reduced version of T4 including only ranges 0-20 and 0-100. Before each number line estimation task, children completed a practice trial with the researcher, in which the target number consisted of an endpoint. Children received encouragement but no corrective feedback during trials.

Data analysis

Video data was transcribed separately for speech, gaze, and gesture (both co-speech and co-thought). These data were then analysed alongside children's paper-based representations. Imagistic representations were coded according to their component sign (pictorial, iconic, or notational) and all cognitive representations were examined for structural features of number. Estimation strategies were deduced from the cognitive representations identified during each estimation trial.

In line with previous studies, children's number line estimates were quantitatively analysed for their degree of linearity. In the number-to-position task (T2), analysis first converted estimate positions (measured from left endpoint, in millimetres) into the number line values that would have been 'hit' assuming a linear scale. In order to compare the linear accuracy of estimates in both T2 and T4, the absolute percentage error of each estimate was also calculated, by taking the absolute difference between target number and the number 'hit' by a child's estimate, as a percentage of the number line range for that estimation.

Linear and logarithmic models were fitted to children's target number estimates for each range individually. For each model, the coefficient of determination R^2 was calculated in order to compare model fits. A new analysis offered by the present study was to compare these quantitative analyses to children's other cognitive representations during estimations (e.g. gestures), in order to ask how strategies and cognitive representations might be linked to estimation result.

Findings

The full case study of Imogen (Williamson, 2013) discusses all four interview tasks and the relations drawn between them. The present paper narrows the focus to the number line estimation tasks T2 and T4, in order to present a more detailed analysis and discussion of these.

Number line estimation tasks (T2 and T4)

Representation of structural elements of number occurred in speech, gaze, and gesture during these tasks. For both children, clearly distinguishable estimation strategies were identified. In line with the study's explicit focus, the strategies were named and classified according to the structural features represented within them.

The following short example demonstrates three strategies, identified in Imogen's T2 (number-to-position) estimation of target 13 on the range 5-15. The strategies here identified are "Reference to right endpoint (RE)", "Count-on from midpoint (MP) in 1's", and "Count-on from target":

> **Interviewer**: It's thirteen. Where do you think thirteen belongs? [Imogen's gaze goes quickly to RE (15) then to proffered rocket sticker]
>
> **Imogen**: [takes rocket with R-hand, transfers to L-hand, pauses] 'Cause ten is here [R-hand points onto MP and holds] … [R-hand 'hops' to right; both hands stick rocket to right of last 'hop'] Fourteen fifteen [R-hand thumps line between rocket and RE; thumps RE itself]

Multiple aspects of number structure are represented in this example. The number sequence is represented in two count-on strategies, with a confidence that allows Imogen to start counting midway through the sequence. The units represented by gesture during counting on represent a further aspect of structure: the spatial extent of each unit is approximately equally sized, and scaled so that Imogen's sequence from ten to fifteen covers the spatial extent from indicated midpoint to endpoint. Number structure is also apparent in Imogen's use of the right endpoint (fifteen) as an appropriate 'anchor point' for the target number thirteen. The structural role of ten as the midpoint is represented in both speech and gesture.

Representation of number structure also occurred outside of particular estimations, particularly when a new number range was introduced. Imogen, for example, represented evenly spaced multiples of five in an exclamation on seeing the first page of 5-15 trials: "Shouldn't it be five TEN …?" [right hand points to midpoint of line and holds]. Patrick also made verbally explicit some of the scaling evident in Imogen's physical gestures, for example explaining that "we should squeeze them in a bit 'cos it's got ten more" when the number range changed from 0-10 to 0-20.

Unsurprisingly, the most commonly identified strategies during children's estimations were references to both the left and right endpoints of the presented number line. Next in frequency was counting on from the left endpoint (in single units). This form of counting of course features frequently in the classroom, and both children demonstrated imagistic representations of the counting sequence, similarly based upon a left-to-right increasing sequence, elsewhere in the interview.

The strategies identified from Imogen and Patrick's interviews are listed below in Table 1, together with the target numbers of the trials in which they were observed. Patrick and Imogen employed a similar array of strategies, but with application to different target numbers and with differing frequency. Overall, strategies were identified more often in Patrick's interactions with estimation tasks than in Imogen's. For Patrick, references to left and right endpoints were apparent in every single number range in both tasks. The shaded rows represent Patrick's results.

Strategy	T2: number-to-position				T4: position-to-n			
	0-10	5-15	0-20	0-100	0-10	5-15	0-20	0-100
Count on (LE, 1's)	4 6		2				2	3
	2 3 4 6 7 8	6 7 9 10 11 14	2 4 6 9 11 15 16	3 6		7 13		25
Count on (LE, 5's)		9						
Count on (MP, 1's)		11 13	6					
				92			9	
Count on (target)		13						25
	9		7 19		6		9	67
Count back (MP, 1's)			6					
Count back (RE, 1's)								
		13				11 13		
Count back (other)		11						
Ref. to other trial		7 14	7	48			15	18
	3	14	9 11	92	2	9	7	
Ref. to LE	2 3 7 8	6	4 13 18	2 4 6 25 48 67			4 7 16	6 71 86
	1 2 3 4 5 6 7 8	6 10 11 13 14	2 4 6 7 11 16	2 3 6 18 49 71	2 4 5	7 9 10 11 13	7 9 11	2 3 25 50
Ref. to RE	3 7 8 9	13	16 18	2 4 48			7	48 71 86
	2 8 9	6 10 11 13 14	11 15 18 19	71 92	4 6 7	7 9 10 11 13	2 9 11 15 16	2 3 4 6 18 67 71 92
Ref. to MP		11 13	6	3 18 25 67 71				
	5	11		49 50 92			6 9 15	
Ref. to quarter								
				25				
Ambiguous		8	15	86			6 13 18	3 5 67
					8 9		4	18

Table 1: Imogen and Patrick's strategies in T2 and T4 estimation tasks. Patrick's results are shaded. Abbreviations: LE=left endpoint, RE=right endpoint and MP=midpoint.

A clear difference can be seen between Imogen's strategies in T4 (position-to-number) compared to T2 (number-to-position). For example, whilst references to the midpoint are made across three ranges and various targets in T2, no midpoint references at all are identified during T4. Within T2, Imogen demonstrated the greatest range and frequency of estimation strategies in estimations on the range 5-15.

Quantitative analysis

In line with expectations from previous studies, the linear accuracy of Imogen's estimates decreased on larger ranges. In T2, the mean absolute percentage error was low for both 0-10 and 5-15 (7.6% and 4.9%), and rose to 25.5% on the range 0-100. In terms of model fit, Imogen's T2 estimates for range 0-10 were best described by a linear model (R^2_{Lin}=.94; R^2_{Log}=.84). On the range 5-15, the comparison was inconclusive for both children (Imogen: R^2=.97 both models, Patrick: R^2_{Lin}=.96; R^2_{Log}=.93). On ranges 0-20 and 0-100, Imogen's estimates were more consistent with logarithmic models (R^2_{Log}=.71 and R^2_{Log}=.57, compared to R^2_{Lin}=.64 and R^2_{Lin}=.39).

In contrast to Imogen, Patrick's T2 estimates demonstrated high linear accuracy across all ranges (PAE 5.9% to 9.3%). Patrick's estimates were better fit by linear models for ranges 0-10, 0-20 and 0-100 (R^2_{Lin}=.89, R^2_{Lin}=.63 and R^2_{Lin}=.99 respectively; R^2_{Log}=.75, R^2_{Log}=.86 and R^2_{Log}=.85).

In terms of differences between number-to-position (T2) and position-to-number (T4) accuracy, a paired samples t-test found a significant difference between Imogen's estimation error in T4 (PAE mean=11.94, SD=10.00) compared to T2 (PAE mean=21.59, SD=8.78); t(16)=2.89, p=.011. Patrick demonstrated no significant difference in linear accuracy between tasks (T4 PAE mean=8.42, SD=7.42; T2 PAE mean=7.55, SD=6.36; t(36)=-.568, p=.573).

Imogen's position-to-number (T4) estimates were better fit by linear models for both ranges 0-20 (R^2_{Lin}=.97) and 0-100 (R^2_{Lin}=.91, compared to R^2_{Log}=.87 and R^2_{Log}=.77). As in T2, Patrick's estimates were better fit by linear models for each range (R^2_{Lin}=.92, R^2_{Lin}=.67, R^2_{Lin}=.97, R^2_{Lin}=.95 for ranges 0-10, 5-15, 0-20 and 0-100 respectively, compared to R^2_{Log}=.86, R^2_{Log}=.61, R^2_{Log}=.95, R^2_{Log}=.74).

Differences between Imogen's T2 and T4 estimates correlate with differences in estimation strategy. In T2, Imogen's estimation strategies included frequent reference to the midpoint, and her estimations fall within a band in the centre of the range (see Figure 1, below). In T4 in contrast, her strategies omit the midpoint, referring only to the endpoints, and the estimations are clustered around the endpoints.

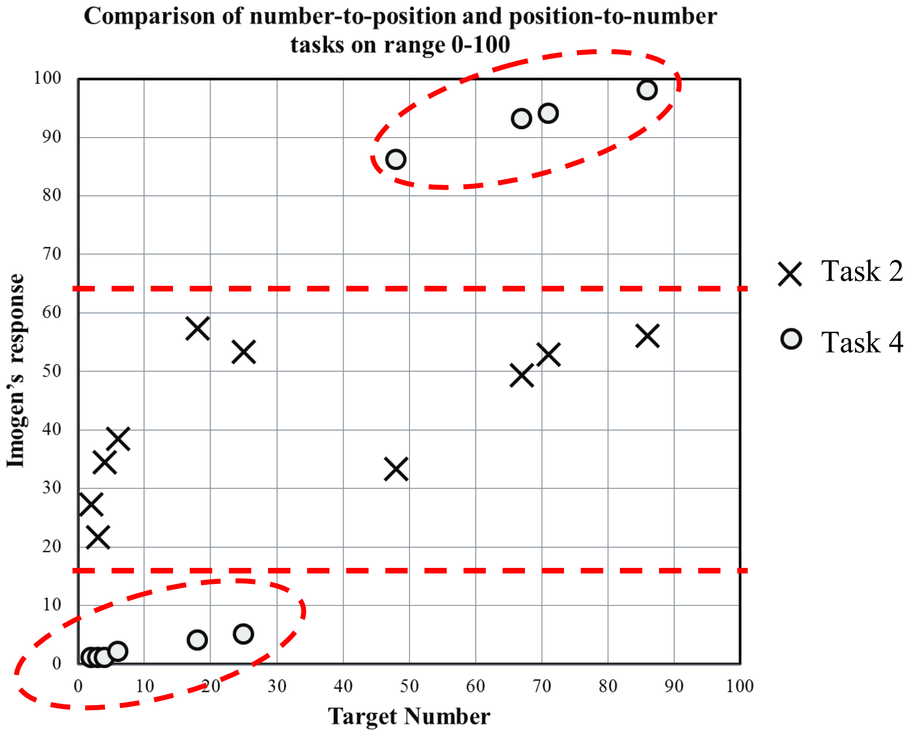

Figure 1: Imogen's T2 and T4 estimates on the range 0-100

Discussion

Both children cognitively represented natural number in ways that encoded significant structural elements, and the chosen methodology enabled the identification of differing strategies used by children in their number line estimations.

The qualitative data from this study supports White and Szucs' hypothesis (2012: 9) that estimation strategies may vary depending on "familiarity with the number range, proximity to either external or mental anchor points, as well as knowledge of arithmetic strategy". The use of particular strategies differed between the two children, and between ranges, tasks and target numbers. As noted, both children made frequent reference to the left endpoint of the line and were most likely to begin any count-on strategies from the left. To this extent, both children appeared to use the left endpoint as a default 'anchor point'. Trials in which target numbers were in proximity to other potential 'anchor points' revealed both children varying their estimation strategies. For example, both referred to the right endpoint *and made no reference to the left* when estimating the position of 9 on the range 0-10; the target 9 appeared to 'cue' them directly to the endpoint 10. Similarly, both children made reference to the midpoint when estimating target 11 on the range 5-15. Overall, Patrick demonstrated 'anchor point' strategies more frequently: for example in referring to midpoints whilst positioning targets 49 and 50 on 0-100; commenting "Quarter of the way" whilst positioning target 25; and explaining of target 9 on the range 0-20 that "it's getting quite close ... could have been half but it's a tiny bit off".

A limitation of the methodology is that it is of course not able to capture all cognitive representations during a given estimation. What this paper has attempted to show is that sufficient aspects can be captured to reveal scientifically interesting variations in children's task interactions. Certainly, the data further challenge the idea that it is meaningful to describe children's estimations in terms of a fixed model or representation for a given number range, whether quantitatively, as queried by White and Szucs (2012), or qualitatively. There are good examples of variation dependent on the particular target. In T4 range 5-15 for example, Patrick deduced from a rocket's midpoint location that it must represent 10. The next trial presented a target further to the right, in fact representing 11. In this second trial, there was no evidence of Patrick representing a midpoint structure, and the strategy he adopted – counting back from the right endpoint – led him to an estimate ("8") which in fact conflicted with the identification he had just made of the midpoint 10. Discussing these two consecutive trials in terms of 'Patrick's representation of the range 5-15' obscures something important.

The findings do not indicate any simple relationship between estimation strategy and result. Both children applied counting strategies, commonly regarded as an unsophisticated approach, with some success, emphasising that the detail of children's interactions must be attended to, such as adjustment of unit size depending on scale. Overall, Imogen's linear accuracy was highest for her T2 trials on the range 5-15, during which she demonstrated each of the strategies seen in her interview overall, as well as spontaneous cognitive representations with structural detail such as the subdivision of the 5-15 line into equally sized fives. This is suggestive, as is the observed correlation between strategy difference and estimation result for Imogen's estimates on the range 0-100 (see Figure 1).

This paper has attempted to show the potential of the current methodology to illuminate children's interactions with number line estimation tasks. The findings discussed are to be followed up in the full multiple case study, to contribute further to understanding the relationships between cognitive representation, estimation strategies and results, and the interaction between task difference and strategy difference.

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