Revisiting school mathematics: A key opportunity for learning mathematics-for-teaching

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Learning mathematics-for-teaching (MfT) involves revisiting school mathematics and learning new mathematics. The notion of *revisiting* is operationalised and exemplified within pre-service secondary mathematics teacher education, specifically a financial mathematics course. A framework for MfT was developed consisting of nine interrelated aspects of teachers' mathematical knowledge, and provided an analytic tool for exploring opportunities to learn MfT within the course. These opportunities are exemplified through a revisiting task where pre-service teachers (PSTs) were required to make sense of learners' responses to a compound growth task. The learners' responses provided a springboard for learning other aspects of MfT of compound growth such as essential features, modelling and applications, and knowledge of context. The decompressed nature of the learners' responses opened opportunities for PSTs to reconsider their own knowledge of compound growth, of the mathematics that underpins the formula, the process to obtain the formula and thus the way in which the formula models compound growth in different contexts.

Keywords: revisiting, mathematics-for-teaching, teacher knowledge, compound interest

Introduction

There are compelling arguments that revisiting school mathematics is an important component of learning mathematics-for-teaching (MfT) (e.g. Cooney and Wiegel, 2003; Stacey, 2008; Zazkis, 2011). However, existing literature gives few suggestions as to how this revisiting might be done. Moreover, the calls for revisiting school mathematics tend to assume pre-service teachers (PSTs) had adequate opportunity to learn the mathematical content at school, and now need to deepen and/or "top up" their knowledge with particular attention to conceptual aspects. In South Africa many PSTs have not had the kinds of opportunities to learn mathematics at school that might be assumed in other parts of the world. So for many students revisiting school mathematics is much more than "topping up". In the most extreme cases, it may involve learning for the first time the mathematics they will teach in schools. In this paper we focus on PSTs who are preparing to teach Grades 8-12 (14-18 year olds).

Our paper is drawn from a larger study of teachers' mathematical knowledge for teaching involving 42 PSTs registered for a course on financial mathematics specifically designed for teachers, and taught by the first author. The construct of revisiting emerged through the analysis of the data, and thus reflected the interplay between the theoretical field and the empirical setting (Brown and Dowling, 1998). In this paper we consider revisiting school mathematics as a particular aspect of learning MfT. We elaborate the notion of revisiting and focus on the differences between revisiting school mathematics and learning school mathematics for the first time. The specific instances discussed below emerge from an analysis of analytic narrative vignettes (Erickson, 1986) constructed from classroom episodes in the course, and from PSTs' journal entries. We begin with a brief discussion of a framework for MfT.

A framework for Mathematics-for-teaching

In the larger study, a framework was developed to elaborate the notion of mathematics-for-teaching, drawing together elements of several existing frameworks and typologies for teachers' mathematical knowledge for teaching (e.g. Ball, Bass and Hill, 2004; Ball, Thames and Phelps, 2008; Even, 1990; Even, 1993; Ferrini-Mundy et al., 2006; Huillet, 2007; Kazima and Adler, 2006; Shulman, 1986, 1987; Watson, 2008). The framework consists of nine interrelated aspects of teachers' mathematical knowledge all of which contain elements of mathematics and teaching, but which can be broadly grouped into three clusters:

Aspects that are mainly mathematical: essential features, relationship to other mathematics, mathematical practices, modelling and applications;

Aspects that are mainly pedagogical: basic repertoire of key tasks and examples, different teaching sequences and approaches, explanations, learners' conceptions; and

Contextual knowledge of finance: financial concepts and conventions, socio-economic issues and financial literacy.

This framework provided an analytic tool for exploring opportunities to learn MfT in the course – both school mathematics such as compound interest, and new mathematics such as annuities.

Elaborating the notion of *revisiting*

We define *revisiting* as the re-learning of known mathematical content for the ultimate purpose of teaching. Revisiting assumes we cannot learn everything about an idea at once, but it is not simply repetition for mastery. The purpose of revisiting is to increase one's mathematical knowledge through connections that deepen and broaden knowledge of the concept, paying attention to mathematical aspects that are important for teaching. While such knowledge may be useful in many professions, it is essential for teaching. Zazkis (2011) explored the similarities and differences between learning and re-learning (or revisiting) mathematical content in the context of a mathematics course for pre-service elementary teachers. She argues that re-learning mathematics includes (1) reconstructing and restructuring; (2) shifting from operational to structural conceptions (Sfard, 1991), (3) expanding PSTs' concept image (Vinner and Tall, 1981) by extending the range of examples that PSTs encounter, and (4) connecting a known mathematical concept to a larger class of mathematical entities which may require a redefinition of the concept.

Four aspects of revisiting

An important component of the larger study was to describe how revisiting was enacted in the course, and then to identify how it might differ from learning a piece of mathematics for the first time. By studying several classroom episodes of revisiting through the lens of the MfT framework, and identifying commonalities across the sessions, four inter-related aspects were identified: content, goals/purposes, task and activity, and resources. These aspects are illustrated with reference to a compound growth task that PSTs worked on in one of the sessions on school mathematics. However, first it is important to distinguish the original (school level) learner task from the revisiting task for PSTs. The learner task was a test question, typical of Grade 10 level:

> "A computer operator earns R96 000 a year. Her salary increases by 6% per year. What will her salary be after 3 years?"

The solution in the memo indicated learners were expected to apply the formula for compound growth and to substitute the given values. However, a closer reading of the question suggests that the wording is ambiguous and therefore open to different interpretations of the timeframes involved. For example, it is not clear whether the increase should be applied at the beginning or the end of the first year.

The revisiting task contained the test question (without memo) and responses of four learners. PSTs were required to analyse the learners' responses, to identify errors and to suggest ways of helping one particular learner. We now return to the four aspects of revisiting:

Content – Revisiting a piece of mathematics assumes the mathematical content has been encountered previously, that the PST has an overview of the content and is familiar with the terminology, notation and techniques, yet may display some typical misconceptions. In this task PSTs were familiar with these aspects of compound interest/growth. The learner responses had been carefully chosen – none of the learners had used the compound growth formula, and some responses did not involve compound growth. Furthermore the logic and strategies of some learners were not immediately obvious, and their responses included incorrect use of symbols and notation. For example, Learner 1 (figure 1) determined that 6% of 96 000 is R5 760 and added this to R96 000, giving R101 760 for the salary for year 1. The learner then added R5 760 twice more getting R107 520 for year 2 and R113 280 for year 3. Thus the learner modelled the increase at the beginning of the first year, and a simple growth scenario for the following two years since each salary-increase was calculated on the base amount of R96 000. This does not reflect the typical method of calculating salary-increases in the workplace. Note too the use of "100%" as a label rather than a quantity in the third line, and the inefficient strategy to determine 6% of 96 000

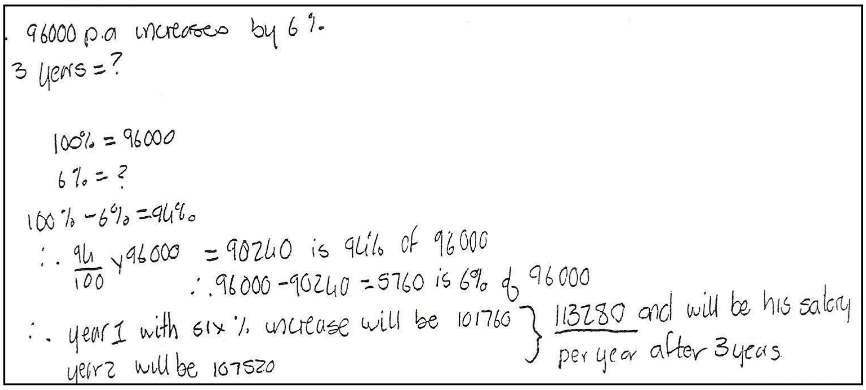

Figure 1. Learner 1's response

Learner 2 (figure 2) made several arithmetic errors, including a conversion of $\frac{6}{100}$ to 0.6, 'losing zeros' when R96 000 became R96, and adding percentages to amounts of money. This learner modelled a simple interest scenario by adding 0.6.

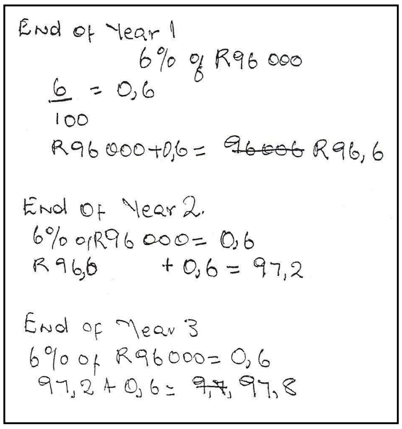

Figure 2. Learner 2’s response

Thus we see how the learners’ responses become the content of the revisiting task together with the original mathematics task.

Goals – The goals of revisiting are different from the initial encounter. Most obviously when the PSTs first studied the mathematics, they were learning it as school learners; now they are learning it as a requirement to become teachers, and then to teach it to others. One goal is to deepen and broaden their knowledge, for example making links between representations, and links to other aspects of mathematics. In this task they were required to make links between the compound growth formula and the multi-step approaches produced by learners. In terms of the MfT framework, the revisiting task explicitly required PSTs to engage with *learners’ conceptions*. However, the deliberate selection of unanticipated learner responses was also intended to challenge students’ own knowledge of compound growth and use of the formula. Therefore it was anticipated that PSTs would also have to engage with *essential features* and *modelling and applications.* For example, they may need to check whether the formula gives the annual salary earned during the third year or the starting salary for the fourth year.

Task and activity – We draw on the well-known distinction that a *task* is set by the teacher (or teacher educator), while *activity* refers to what students do in response to the task (Christiansen and Walther, 1986). Thus the task is a key element in framing the PST’s mathematical activity. When revisiting mathematics, the task should require more of PSTs than it would of school learners. In this case the task involved dealing with learners’ responses that in turn led to a deeper analysis of the question itself, and the mathematics compressed in the formula.

Resources – Revisiting involves the use of new or additional resources (tools, artefacts and knowledge). The learners’ responses may be considered as resources that are brought to bear on the learner task. The PSTs introduced T_n notation as an additional resource to resolve a concern about timeframes. When they were first introduced to compound interest in Grade 10, they would not yet have been aware of this notation. Furthermore, some PSTs were not familiar with the details of annual salary-increase, and therefore were unable to draw on this assumed everyday knowledge as a resource for interpreting the learner task and the learners’ responses.

What opportunities emerge for learning MfT of compound interest?

While it would be expected that the inclusion of learners’ work would provide opportunity for PSTs to deal with learners’ conceptions, the revisiting task also provided opportunity to engage with other, more mathematical, aspects of the MfT framework, each of which is discussed below.

Learners' conceptions – PSTs had to make sense of learners' responses, which involved recognising a range of different errors, including adding a constant amount each year rather than calculating 6% on the latest salary, and adding amounts of money to percentages. After completing the task, PSTs were required to write a journal entry to reflect on how the learners' responses might impact their future teaching of simple and compound growth. Several students noted that learners had different interpretations of the question and that these should be taken seriously in teaching. They also noticed that learners may not make use of efficient strategies, and that the question could be correctly answered without use of a formula. In her journal entry, Palesa commented "formulas are just the simple ways of getting the answers". I followed up on this comment in an interview with her mid-way through the course.

> Palesa: They kind of all had different approaches to the, the question. But it was different from how would I do it because I would think about the formula. If I want the interest it's *Prt* or what, but they did it in, they put their understanding in the thing like the maths understanding like if *x* of this is how much, and how will I find it? So it was not about the formula of compound interest or the simple interest. They just used their thinking and that's how they find it …

Palesa acknowledged the need to make sense of learners' strategies. She implied that the learners had to think carefully to produce their answers and that their responses should be taken seriously. The learners' responses required the PSTs to work between compressed and decompressed forms of mathematics, a key component of MfT (Ball et al., 2004; Zazkis, 2011).

Essential features – An essential feature (Even, 1990) of working with compound growth is the need for precision in references to time. This does not necessarily mean the task should be unambiguous about timeframes but rather that students should be precise in the ways they work with time. Revisiting compound growth provides opportunity to emphasise the importance of explicit and precise references to time. For example, during class discussion Sizwe raised a concern about the timeframes in the original question.

> Sizwe: … eish, they say, this, this computer operator earns ninety-six thousand per year, so what I don't understand is if you start counting from this year, two thousand and eight (2008) to two thousand and nine (2009), that is the first year, how much is the computer operator earning in that year and after three years, what is "after three years"? What, which year is that?

Sizwe's concern appeared to have been prompted by the learners' interpretations of timeframes which in turn led him to reconsider his assumptions about the timeframes in the question statement. This led to an extended discussion about whether the salary increase should be implemented at the beginning or end of the first year and the meaning of "after three years" – was it measured from the first salary increase or from "this year" (i.e. 2008)? During this discussion students introduced T_n notation to talk about time. Sizwe's difficulties stemmed from the fact that he did not distinguish between the beginning and end of a year, and he appeared to muddle discrete points in time with the intervals between those points. He was not assisted by Jenny who explicitly linked time to the terms of a sequence when she said "so, say term one *is* year zero, then term two *is* year one, term three *is* year two and term four *is* year three". Jenny's repeated use of "is" left the timeframe ambiguous although from her other contributions it was clear that she was referring to the end of each year. This instance of interaction between PSTs reflects how they introduced a new resource (T-notation) in order to negotiate their interpretation of the original task.

Knowledge of context – In the MfT framework this aspect was referred to as *contextual knowledge of finance*. However, in the context of this task, the broader label *knowledge of context* is more appropriate. The learner task assumes knowledge of the salary context. Firstly, it assumes that a salary-increase with a constant rate of increase can be modelled by the compound growth formula. Secondly, it assumes that salary is taken to be the amount earned in a twelve-month period (as opposed to cumulative earnings over several years). The responses from both the learners and the PSTs suggest that knowledge of the context of salary-increase cannot be taken for granted. It might, however, be argued that the learner task resembles a typical text book word problem and does not require knowledge of the context of salary-increase but simply the selection of the correct formula and appropriate substitution. Thus a learner who knows the relevant mathematical concepts and the genre of word problems may complete the learner task successfully with little knowledge of the salary context. However, teachers require knowledge of the salary-increase context, and they may need to draw on this knowledge in helping learners. It seems that knowledge of the salary context becomes even more important when dealing with multi-step approaches, and when unpacking how the compound growth formula models salary-increase. This became visible through the PSTs interactions around timeframes and their interpretations of learners' responses.

Modelling and applications – The learners' and PSTs' responses confirm that the original question contained some level of ambiguity with regard to timeframes. While the single solution in the memo suggests that the ambiguity was unintended, modelling tasks are typically designed with some degree of ambiguity. This requires one to make assumptions explicit as part of the modelling process. For example, one needs to be explicit about whether the 6% increase is assumed to take place at the beginning or the end of the year (which is effectively the salary for the third year). It can be shown that the compound growth formula models (annual) salary-increase, with a fixed rate of increase, *r*, for *n* years. Thus in the formula, $A = P(1+r)^2$, the amount *A* gives the new salary at the end of the second year, where the increase has been effected twice. We can view salary in two subtly different ways. Consider a salary of R96 000. This means that R96 000 is both the amount earned by the *end* of the year *and* the amount on which each month's salary is calculated from the *beginning* of the year. Put another way, it is the "whole" that is subdivided into 12 equal monthly portions. It is thus acted on from the beginning of the year, giving a monthly salary of R8 000. Thus 96 000 is a number linked to the end of a year but also applied throughout the year. Once again when working with decompressed forms of compound growth, one is forced to consider more deeply the relationship between the mathematics and the situation that is being modelled. These aspects are not in focus when learning to apply the compound interest formula in high school.

Conclusion

We have shown how a revisiting task that includes learners' responses to a school mathematics task provides a springboard for learning several aspects of MfT of compound growth such as essential features, modelling and applications, and knowledge of context. We do not claim that the PSTs in the study learned these aspects but we suggest that the revisiting task, and how it played out in the course, *provided the potential* for PSTs to learn aspects of MfT beyond the aspect of learners' conceptions. However, it is not the use of learners' work *per se* that opens up these opportunities. It is dependent on the *kinds* of learner work that is selected. The mathematical demands of making sense of learners' inefficient, multi-step and partially correct responses are far greater than checking whether a learner has substituted correctly into a formula. It is the decompressed nature of the learners' responses that opens opportunities for PSTs to reconsider their own knowledge of compound growth, of the mathematics that underpins the formula, the process to obtain the formula and thus the way in which the formula models compound growth in different contexts. This requires them to shift between operational and structural conceptions of compound interest. Furthermore, working with carefully-selected learner responses may prompt consideration of issues that would not arise when pre-service teachers simply work though school mathematics tasks to produce answers for themselves.

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