The use of alternative double number lines as models of ratio tasks and as models for ratio relations and scaling

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In this paper we draw on ICCAMS project materials that used the double number line (DNL) to develop secondary school students' understanding of multiplicative reasoning. In particular, we look at the use of a DNL, and its alternative version, as a *model of* ratio tasks, as a *model for* developing an understanding of ratio relations, and finally (but only briefly) as a *model for* developing the notion of multiplication as scaling.

Keywords: double number line, ratio, scaling, multiplicative reasoning

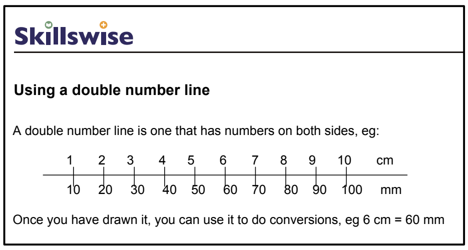
Introduction

Increasing Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS) was a 4½-year research project funded by the Economic and Social Research Council in the UK. Phase 1 consisted of a cross-sectional survey of 11-14 years olds' understandings of algebra and multiplicative reasoning, and their attitudes to mathematics. Phase 2 was a collaborative research study with a group of teachers that aimed to improve students' attainment and attitudes in these two areas (Brown, Hodgen and Küchemann, 2012). This included a design research element (Cobb, Confrey, diSessa, Lehrer and Schauble, 2003) that investigated how cognitive tools influenced student learning. In Phase 3 the work was implemented on a larger scale.

In Phase 2, we developed tasks involving the double number line (DNL). In this paper we discuss some of the insights gained from this. The DNL enables students to develop their understanding - it is more than just a neat tool for solving ratio tasks and is a more subtle and complex model than many curriculum authors suggest.

It is relatively easy to find advocates for the DNL, especially from researchers in the Dutch *Realistic Mathematics Education* (RME) tradition (e.g., van den Heuvel-Panhuizen, 2001). However, substantive research papers on the DNL are rare. We have found some interesting studies (e.g., Moss and Case, 1999; Misailidou and Williams, 2003; Corina, Zhao, Cobb and McClain, 2004; Orrill and Brown, 2012), but often the DNL plays only a small part in the research or the tasks used are not particularly well designed or implemented.

The Double Number Line (DNL) is beginning to appear quite widely in school mathematics curriculum materials, especially those influenced (directly or indirectly) by RME. Materials in the English language that stand out are the *Mathematics in Context* (MiC) project (developed in collaboration with the Wisconsin Center for Educational Research, University of Wisconsin-Madison and the Freudenthal Institute), and a UK project based on this, *Making Sense of Maths*. The DNL can also be found in homespun materials published on the internet, such as this extract (right) from a worksheet on the BBC's Skillswise website. Note here that the DNL is poorly articulated – for example, the zero marks are missing - and the approach is very procedural. Such limitations are not uncommon in materials



involving the DNL. The Common Core State Standards (which have been adopted by the majority of states in the USA) include this reference to the DNL:

> CCSS.Math.Content.6RP.A.3 Use ratio and rate reasoning to solve real-world and mathematical problems, e.g. by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

The mathematics standards for New Zealand also refer to the DNL, though surprisingly perhaps it does not appear in the September 2013 English National Curriculum 'programmes of study', nor in the NCTM Standards in the USA. Interestingly, though, NCTM has a 'representation standard', which is separated into these three components:

> Instructional programs from prekindergarten through grade 12 should enable all students to
>
> • create and use representations to organize, record, and communicate mathematical ideas;
> • select, apply, and translate among mathematical representations to solve problems;
> • use representations to model and interpret physical, social, and mathematical phenomena.

The third component chimes (to some extent) with the RME notion of creating a 'model of' a situation (e.g., Gravemeijer, 1999). RME argues that one should start by introducing students to accessible, perhaps 'real-life', situations which they are able to model in a natural way, and then, over time, students use these models in their own right to develop and formalise mathematical ideas (through 'vertical mathematising'). The models shift from being 'models of' a situation to being 'models for' mathematical ideas.

As mentioned above, most curriculum materials seem to focus on the second NCTM component, i.e. where a representation (or 'model of') is used directly as a device that helps students solve problems. So for example, in the MiC book *Models You Can Count On* (Abels, Wijers, Pligge and Hedges, 2006) students are told, "Learning how to use a double number line will help you make precise calculations effortlessly" (p. 43). In the teacher's version of the book (Webb, Hedges, Abels and Pahla, 2006), it is stated

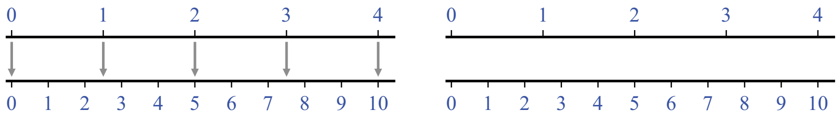
> The operations that students use on a double number line are similar to the operations they learned ... when they used a ratio table. Instead of a double number line, a ratio table could also be used. However, a double number line gives visual support: the numbers are ordered. Note that a double number line can start at zero, but a ratio table cannot (p. 40B).

This statement is highly cryptic, yet it is not elaborated for the teacher-reader. As such, it is likely to convey a procedural view of the DNL: students use the DNL to perform operations. We get a hint about the nature of the DNL from the reference to 'visual support' - but, the suggestions that this means 'the numbers are ordered' and that the DNL 'can start at zero' are rather inadequate (see below). The idea that a ratio table cannot contain 0, 0 seems plain wrong.

For pragmatic reasons, we did not spend as much time as RME would advocate to allow the DNL model to 'emerge'. Rather, our focus was on the first NCTM component, i.e. on using the DNL as a 'model for' exploring mathematical ideas.

The DNL appears most commonly in curriculum materials as a fraction-, decimal- or percentage-bar, with the purpose of comparing fractions (e.g., Which is greater, $\frac{2}{5}$ or $\frac{3}{7}$?) or for finding equivalences between fractions, decimals and percentages. However, it also used more generally for situations involving ratio relations, such as conversions (e.g. of metres to feet on a map scale) and geometric enlargement.

The DNL is essentially a mapping diagram, but one in which the scales on the two, parallel, axes have been adjusted in such a way that the mapping arrows are all parallel. It is most commonly used to represent linear relations, i.e. relations of the form $f(x) = kx$. For this, the zeros on the two scales are aligned and the scales themselves are both linear, as in the example for $f(x) = 2.5x$ below. (The standard version, without the mapping arrows, is shown on the right.)



A linear relation $f(x) = kx$ has the properties $f(p+q) = f(p) + f(q)$ and $f(rp) = rf(p)$. This means that if we have a linear relation that maps 3 onto 7.5, say (as in the DNL above) and we want to find the image of, say, 4, we can do this not just by finding and applying the general multiplier ×2.5, but by using a *rated addition* method such as this:

> if the relation is linear ('in proportion') and 3 maps onto 7.5, then 3÷3 maps onto 7.5÷3, i.e. 1 maps onto 2.5; and then (3+1) maps onto (7.5+2.5), i.e. 4 maps onto 10.

The multiplier method can be said to operate *between* the lines, whereas rated addition operates *along* the lines. The rated addition approach might appear more cumbersome; however, it is often the basis for mental methods and allows us, at least to some degree, to adopt an informal approach using simple relations of our choosing. There is no choice about the between-lines multiplier - unless we are prepared to work 'outside' the given lines, by in effect creating an alternative DNL (we discuss this in depth later). There is considerable evidence, albeit indirect, to support the notion that working along the lines is often more accessible to students than the general multiplier approach. For example, Vergnaud (1983) has found that students are far more likely to establish a relation that is *within* a measure space (what he calls a *scalar* relation) than *between* measure spaces (a *function* relation). We found evidence to support this when we gave these two versions of the *Spicy Soup* item, below, to parallel (but non-representative) samples of mostly Year 8 students (N=77 and N=74 respectively). Notice that the numbers 33 and 25 had been changed round.

> Ant is making spicy soup for 11 people. He uses 33 ml of tabasco sauce. Bea is making the same soup for 25 people. How much tabasco sauce should she use?

> Ant is making spicy soup for 11 people. He uses 25 ml of tabasco sauce. Bea is making the same soup for 33 people. How much tabasco sauce should she use?

Both items can be said to involve the multiplicative relations ×3 and ×2.27 (approx). The version where the simpler relation is scalar (11 people and 33 people) was found to be much easier than the parallel version where this relation was functional (11 people and 33 ml), with facilities of 91% and 51% respectively.

In the DNL, each number line usually represents a single measure space. So where these measure spaces are different (e.g. £ and \$, metres and feet, people and sauce), it is likely, that students will work with relations along the lines (as this involves *within* measure space relations), rather than between them, unless, perhaps, the between-lines relation is a very simple multiplier.

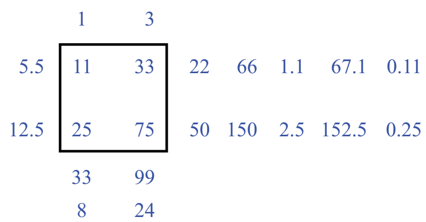
Our purpose in using the DNL was twofold – to explore the nature of ratio relations and to model a particular aspect of multiplication, namely *multiplication as scaling*. Our experience suggests that both uses can be enriching. However, they are far from unproblematic.

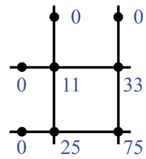
The use of alternative DNLs as models of ratio relationships

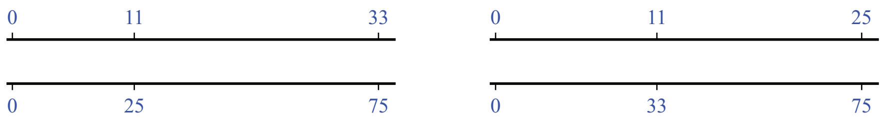
The use of the DNL to solve or analyse ratio tasks is not as straightforward as many curriculum materials seem to suggest. It is often possible to create *two* DNLs for a given task, and they can represent the situation in subtly different ways, or in ways that are hard to interpret.

Imagine we have a table of numbers (right) where there is a ratio relation between the rows, i.e. $\frac{11}{25} = \frac{33}{75}$ (and hence between the columns, i.e. $\frac{11}{33} = \frac{25}{75}$. We can extend the rows with other numbers fitting the $\frac{11}{25}$ relation, and we can extend the columns with other numbers fitting the $\frac{11}{33}$ relation, e.g. like this (near right). And we can express this in a more general and coherent way using a horizontal DNL and a vertical DNL (far right). [The DNLs are drawn again (below), in the usual format.]

11	33
25	75

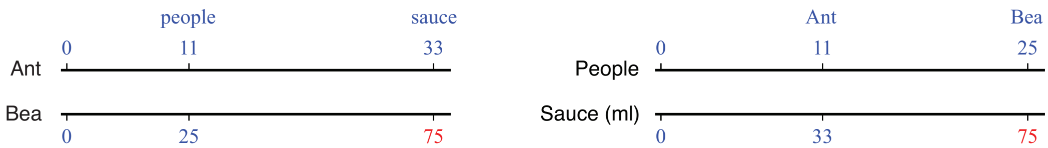






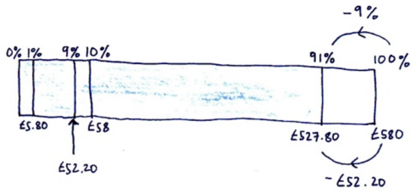
Now imagine that our original numbers arose from a 'real life' ratio context. What might the DNLs mean? Consider a recipe context, e.g. the *Spicy Soup* task discussed earlier and summarised in this ratio table (right). Here the second DNL (shown again, below right) seems to make perfect sense. One line represents numbers of people, the other ml. of sauce. We can easily create other, perfectly meaningful pairs of numbers on this DNL by 'skipping' along the lines, such as 11+11, 33+33 (22 people would need 66 ml) or 11÷3, 33÷3 (1 person needs 3 ml). However, on the first DNL (below, left) the lines seem to be hybrids, representing both people and sauce. It might appear that we can skip nicely from 11, 25 to 22, 50, say, to 33, 75, thereby solving the task, but what does a pair like 22, 50 mean? If it is 22 people and 50 people, how does this fit the story? To resolve this requires quite a high level of abstraction: students will need to blur the distinction between people and sauce, e.g. by thinking of the lines as simply representing 'quantity of ingredients' (if we're happy to accept people in our soup ...). Then 22, 50 could refer to, say, ounces of sugar for Ant's soup and Bea's soup, or respective numbers of tomatoes. However, operating on numbers along the line might still seem rather odd.

	people	sauce (ml)
Ant	11	33
Bea	25	75



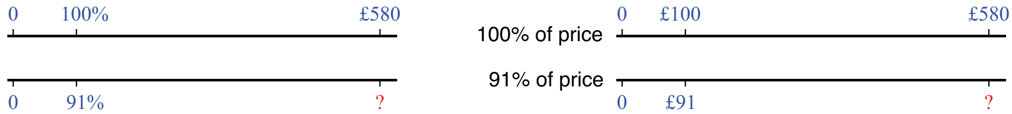
In some contexts it is easier to give a sensible meaning to both DNLs. The booklet *Fair Shares* (Dickinson, Dudzic, Eade, Gough and Hough, 2012: 16), from the RME-inspired series *Making Sense of Maths*, shows how a DNL can be used to find the cost of a £580 computer after a 9% reduction. The DNL is shown below (we have added a paired-down ratio table of the basic information). As can be seen, this

DNL works very well, since it allows students to solve the task using relatively simple moves along the lines.



100	580
91	?

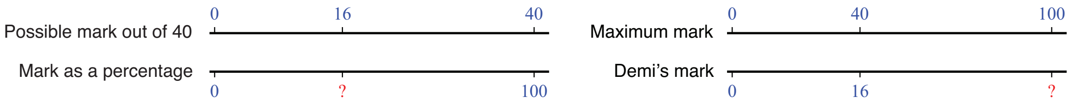
The alternative DNL would look like the one on the left, below. At first sight, this does not appear to work well: as with one of the *Spicy Soup* DNLs, the number lines seem to be hybrids, this time representing percentage (e.g. 100%) and price (e.g. £580) simultaneously. However, with this context it takes less abstraction to smooth this out, e.g. by letting all the numbers represent prices (below, right): the top line could then be thought of as showing the full price of various articles (be they computers or other objects), with the bottom line showing 91% of these prices. It then becomes possible to use a rated addition method along-the-lines in a quite meaningful way: if a £100 computer is reduced to £91, then a £600 computer would be reduced to 6×£91 = £546 and a £20 computer would be reduced to



£91÷5 = £18.20, and so a £580 computer would be reduced to £546 – £18.20 = £527.80.

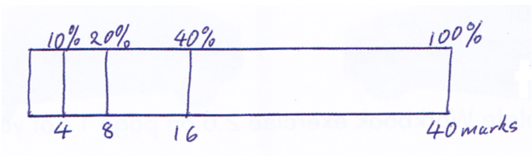
The booklet *Fair Shares* (p19) also includes a task about converting a test result into a percentage, in this case a mark of 16 out 40 achieved by a character 'Demi'. The task can be summarised by this ratio table (below, right). The booklet first tackles the task using a DNL. Again there are two possibilities: we can draw parallel lines through 16 and 40 and ? and 100, or through 16 and ? and 40 and 100. This time both DNLs work perfectly well (both are amenable to an along-the-lines approach) but they model the situation in

16	40
?	100



markedly different ways, as can be seen from the different labels we have given to the lines (below).

The booklet goes for the first version (flipped over), which is used in an along-



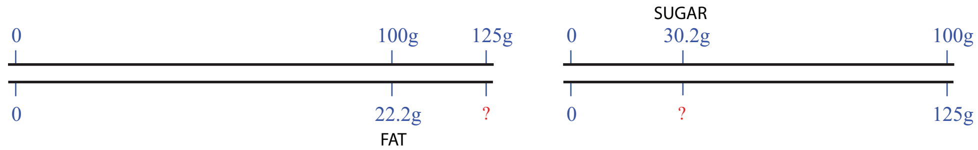
Mark	16	8	4	40
Out of	40	20	10	100

the-lines way (below, left) to arrive at the answer, 40%. The task is then solved again using an extended ratio table (below, right). However, this does not fit the first DNL. It is perhaps unfortunate that the answer, 40 (%), is the same as the total number of marks on the given test. As a consequence, we get the same pairs of corresponding numbers on the DNL as in the ratio table (16,40; 8,20; 4,10; 40,100). However, their meanings are very different. The DNL and ratio table do *not* correspond here - the ratio table matches our second DNL (above). A ratio table that matches the first DNL would look something like this (below, right).

Possible mark out of 40	40	4	16
Mark as a percentage	100	10	40

The first DNL (and corresponding ratio table) models what a range of possible marks on the 40-mark test would be as a percentage, whereas the second DNL (and corresponding ratio table) is taking Demi's specific test result of 16 out of 40 and modelling what her equivalent score would be if the total number of marks was different. Both DNLs (and corresponding ratio tables) are fairly easy to use in this task, i.e. they lend themselves to an informal, rated addition approach. However, students need to be able to switch between the two views of the task, which may not be easy, especially if the existence of two viewpoints is not acknowledged.

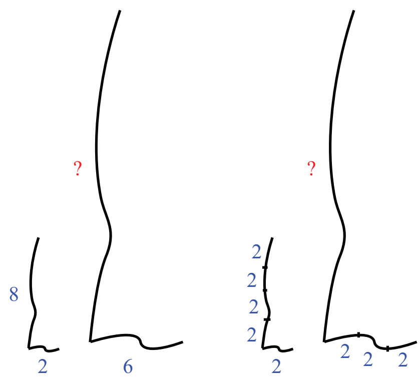
An early draft of the ICCAMS materials included a task about a 125g portion of cheesecake. Students were asked to estimate the amount of fat in the portion, on the basis of a 'nutrition table' which stated that there were 22.2g of fat per 100g of cheesecake. Students tended to solve this informally, using rated addition, in this kind of way: "An extra 25g will contain an extra 5g and-a-bit of fat, making about 28g in all". The obvious way to model this on a DNL would be as shown below, left. However, we wanted to look at other ingredients, e.g. sugar, of which there were 30.2g per 100g. We thus decided to present a DNL like the one below, right.



This latter DNL is very powerful, if it is perceived as expressing the fact that we can map *any* quantity in the 'per 100g' nutrition table onto the 125g portion of cheesecake, by using the single, general, between-the-lines multiplier ×1.25. However, this is far from intuitive and thus, as an early example of the DNL, it caused considerable confusion - among students, teachers on the project, and ourselves. In time, working through this confusion was an enriching experience - and it roundly demonstrated that the DNL is something other than a problem-free device for solving ratio tasks. However some teachers were put off the DNL, as has occurred in other studies (e.g., Orrill and Brown, 2012).

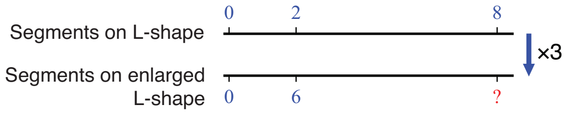
A vital context for a thorough understanding of ratio (although not featured in the *Fair Shares* booklet) is geometric enlargement. There is considerable evidence to suggest that this is a challenging context (e.g., Hart, 1981; Hodgen et al., 2012). A possible reason for this is that enlargement, especially of a curved 2-D shape, does not lend itself well to rated addition. However, in turn this suggests it might lend itself, *relatively well at least*, to using the general multiplicative relationship, which of course in this context is the scale factor.

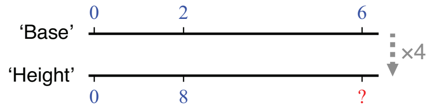
Consider an L-shape with a curved 'base' of 2 units and a curved 'height' of 8 units and imagine it is enlarged (near right) such that the curved base is now 6 units. We can try to construct two kinds of rated addition arguments:



1. The original 'base' fits 3 times into the enlarged 'base', so the enlarged 'height' is 3×8 =24.
2. The original 'base' fits 4 times into the original 'height', so the enlarged 'height' is 4×6=24.

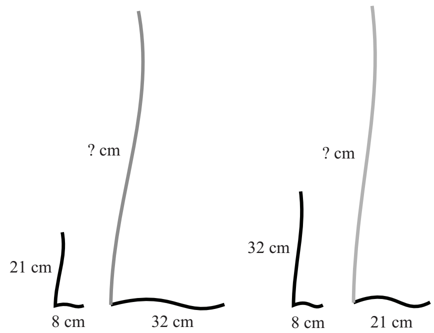
However, neither argument is entirely convincing. Because the segments are curved, the original 'base' clearly does *not* fit into the enlarged 'base' or into the original 'height' (above, far right). The bits are *different shapes*. The true relationship here is that the enlarged 'base' is *the same shape* as the original and that it is *3 times as large*. And, of course, this is a general rule that applies to the whole plane and, specifically, to any corresponding line segments on the original and enlarged shapes. As far as the two potential DNLs for this situation are concerned, this general relation is best expressed by the between-the-lines multiplier on the DNL below, left. (The DNL





below, right models the less compelling between-the-lines relation that for any scale factor the 'height' is 4 times the 'base'.)

We wrote earlier, in reference to Vergnaud's work, that a between-the-lines multiplier expresses a function relation when the lines represent different measure spaces, and that students tend to prefer scalar relations. In the present context, it can be argued that the lines represent the same measure space, so that the between-the-lines multiplier is scalar for an enlargement. Either way, we have evidence [below] that for an enlargement, students are more likely to relate elements *between* an object than *within* an object - in terms of the DNL above, left, this suggests they tend to prefer between-lines rather than along-lines relations.



[We gave parallel samples of mostly Year 8 students an *Enlarged-L* item where they were asked to find the length of the grey line in one or other of these diagrams (right), given that the two Ls were "exactly the same shape". Both items involve the relatively simple multiplier ×4. In the case of the near-right diagram, where this is a between-objects multiplier, the facility was 75% (*N*=73), whereas for the far-right diagram, where ×4 is a within-object multiplier, the facility was only 36% (*N*=74). Note also that both facilities are substantially lower than the corresponding *Spicy Soup* facilities of 91% and 51%.]

Multiplication as scaling

Young children tend to see multiplication additively, i.e. in terms of repeated addition. Even when multiplication involves non-whole numbers, it can be difficult to free oneself from an additive view: it is still possible to construe an expression like 2.3 × 3.7 as '2.3 lots of 3.7'. The area model might be helpful here (e.g, Barmby, Harries, Higgins and Suggate, 2009), though in the UK it tends to be introduced rather hastily and reduced to a rule (such as area = length × breadth) whose meaning students can quickly loose touch with. And area does not really banish an additive perspective: we can still think of the area of a 2.3 cm by 3.7 cm rectangle as being covered by 2.3 rows of 3.7 unit squares, or 3.7 columns of 2.3 unit squares.

A situation where an additive view can be more problematic is *scaling*, as in 'This pumpkin weighs 2.3 kg; that one weighs 3.7 times as much'. Here one could think of 3.7 lots of the smaller pumpkin as being equivalent to the larger pumpkin, but this is not the *same* as the larger pumpkin – it would win you no prizes ... The same thing arises in the case of geometric enlargement of the plane: an additive interpretation of the enlargement of a line segment, say, can give the correct total length, but the result is not congruent to the enlarged segment. This is particularly salient when a segment is curved, as with the L-shapes above. This suggests that geometric enlargement, despite being cognitively demanding, provides a vital context for developing the notion of multiplication as scaling.

In turn, an awareness of the notion of scaling should help students apprehend that the DNL provides models for ratio by means of between-the-lines as well as along-the-line relations and thus help students develop a more abstract, multiplicative understanding of ratio.

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